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THEORY

OF

DIFFERENTIAL EQUATIONS.

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PART IV.

PARTIAL DIFFERENTIAL EQUATIONS.

BY

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PREFACE.

THESE two volumes, now published as Part IV of the present work, are my final contribution towards the fulfilment of a promise made twenty-one years ago. They are devoted to the theory of partial differential equations.

Though the work thus is completed, no claim is made that every topic of importance has been discussed. In the earlier volumes, indications of omissions from other portions of the whole subject were given and need not now be repeated: here also, there have been definite omissions. Nothing, for instance, is said concerning the researches of Picard and Dini on the method of successive approximations for the construction of an integral which obeys assigned conditions; these investigations limit the variables to real values, and throughout the treatise I have dealt with variables having complex values. Formal questions, such as those which arise out of the application of the theory of groups, are hardly mentioned; here, as in the preceding volumes, I have concerned myself with organic properties, given by applications of the theory of functions, rather than with formal properties. Again, the subject of boundary problems is not dealt with; it appears to me to belong to the theory of functions in its applications to mathematical

physics rather than to the theory of differential equations. In the branches of the subject that are discussed, I have tried to deal as completely as possible with what seems to me to be essential: and I have omitted what are purely formal extensions, to equations of general order, of the properties of equations of the second order when such extensions contain no intrinsic novelty.

In the preparation of the volumes, I have consulted the works of many writers; and references are freely given. My aim has been to make these references relate to the main issues; not a few results, extracted from memoirs, have been used to construct examples; and the name of the author is (I hope) given in every such case. But I have not attempted to select and arrange the references, so that they might make the framework of a history of the subject; had the latter been my purpose, names such as Lagrange, Cauchy, Jacobi, whose work is now the common possession of all writers, would have received more frequent specific references in my pages. It will be seen that Darboux's treatise, *Théorie générale des surfaces*, and Goursat's three volumes, *Leçons sur l'intégration des équations aux dérivées partielles*, have been frequently quoted: I wish to make also a comprehensive acknowledgement of my indebtedness to those works.

The earlier of the two volumes is devoted mainly to equations of the first order. The theory of these equations may be regarded as almost complete, because the actual integration of the equations is made to depend solely upon the solution of difficulties which occur in connection with a system of ordinary equations of the first order.

An introduction to the subject is provided by Cauchy's existence-theorem; it is discussed in the first two chapters. The next chapter is specially concerned with linear equations and linear systems; these admit of a separate and special mode of treatment. The fourth chapter gives an exposition of what, on the whole, I regard as the most effective method of integration for non-linear equations: it contains what is usually called Jacobi's second method, with Mayer's developments. In the succeeding chapter will be found Lagrange's classification of integrals, based upon the process of variation of parameters: but something still remains to be done in this branch of the subject, because even simple examples shew that the customary classes may fail to be entirely comprehensive. The next three chapters are devoted to Cauchy's method of characteristics, alike for two and for any number of independent variables, and to the geometrical associations in the case of two independent variables. Then follows a chapter dealing with Lie's methods, based upon contact-transformations and upon the properties of groups of functions: it was possible to abbreviate this chapter, because Pfaff's problem had already been discussed in the first volume of this work. A chapter has been added dealing with the equations of theoretical dynamics, partly because of their intrinsic connection with partial equations, yet mainly in order to shew the origin of what is usually called Jacobi's first method of integration of partial equations. The concluding chapter of this volume discusses those simultaneous equations of the first order, involving more than one dependent variable, which can be integrated by operations of the same class as those in any of the methods mentioned.

The later of these two volumes is devoted to the consideration of partial equations of the second order and of higher orders, mainly (though not entirely) involving two independent variables. A perusal of the volume will shew that, outside the limits of Cauchy's existence-theorem, knowledge is fragmentary: the inversion of operations of the second order has not yet been discovered and, accordingly, any effective process consists of a succession of operations of the first order.

After a chapter devoted to the discussion of questions connected with the existence of integrals and, in particular, to the discussion of the constitution of a general integral, two chapters are occupied with Laplace's method (and with its developments, due to Darboux) for the integration of the homogeneous linear equations of the second order: the effective success of the method depends upon the vanishing of some invariant, in one or other of two progressively constructed sets of functions involving the coefficients of the original equation. The result raises the question of the form of equations, the primitive of which can be expressed in finite terms: and, to this matter, one chapter is assigned.

In the attempt to integrate any equation of the second order, it is natural to enquire whether an equation of the first order exists which is its complete equivalent: and equations, characterised by this property, will obviously constitute a distinct class. Such, indeed, were the equations of the second order for which integrals (now called intermediate) were first obtained; and one method of their construction is due to Monge. Later, another (and a more direct) method for their construction was given by Boole: but both methods assume that a special form

attaches to the intermediate integral, and the assumption demands that a very restricted form shall be possessed by the original equation. Basing his argument entirely upon an assumed type of integral, Ampère devised another process of integration: his method makes no demand for the existence of an intermediate integral: and the result is often effective when no such integral exists. All these three methods, (and another method of some generality, as given), require the construction of integrable combinations of one (and ultimately the same) set of subsidiary equations, when they are applied to the same original equation. But Ampère's method is applicable also to equations of less restricted form.

It may, however, happen that an equation of the second order is not of the restricted form or, being of that form, does not possess an intermediate integral, or is not amenable to Ampère's method. In that case, a method due to Darboux may be applicable, whereby a compatible equation of the second order (or of some higher order) can be constructed; provided only that a compatible equation of finite order can be obtained, a primitive of the original equation can be derived. To these matters, three chapters are given: they explain the working processes that are effective for the determination of an integral in finite terms, whether by a single equation or a set of equations.

One chapter is devoted to the generalisation of integrals which involve some arbitrary parameters, and another to the discussion of characteristics of equations of the second order. The investigations in both of these chapters are clearly incomplete: they could be continued along lines that lead to the complete classification of integrals of equations of the first order.

In the theory of equations of the first order, much information is given by Lie's general theory of contact-transformations: and an obvious investigation is thereby suggested as to whether there is a corresponding theory for equations of order higher than the first. The question has been considered, and partly solved, by Bäcklund and others: one chapter gives an outline of their work: it is clear that much yet remains to be done in this subject.

In the last three chapters of the volume, some of the preceding methods and theories are extended to equations, which are of order higher than the second or which involve more than two independent variables. Only the simplest extensions are discussed: they could be amplified to any extent: but the result would be merely an accumulation of formal theorems possessing neither individuality nor intrinsic value.

From this brief sketch of the contents of these two volumes, it will be manifest that, in the theory of equations of order higher than the first, there are many gaps and that the theory is far from complete: and even a summary perusal of the volumes will give some indication of these gaps. It is my intention to point out, in a presidential address which will be delivered to the London Mathematical Society next month, some of the more obvious and practicable questions which are waiting for solution. Of these, there is no lack: it is only the workers who are wanted.

On not a few occasions, it has been my privilege to acknowledge the help which has been given to me by the Staff of the University Press. Once more, an opportunity comes to me: and I gladly seize it, to express my indebtedness to them all for the care, the attention, and the consideration, by which they have lightened what to me is never an easy or a simple duty.

So I pass from a task, which has filled the greater part of many years of my life, which has broadened in my view as they passed, and which has suffered interruptions that threatened to end it before its completion. Many of its defects are known to me: after it has gone from me, others will become apparent. Nevertheless, my hope is that my work will ease the labour of those who, coming after me, may desire to possess a systematic account of this branch of pure mathematics.

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TRINITY COLLEGE, CAMBRIDGE.
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CHAPTER I.

INTRODUCTION: TWO EXISTENCE-THEOREMS.

1. THE investigations, which constitute this Part of the present work, are devoted to the consideration of properties of partial differential equations. In text-books which deal with the modes of constructing the integrals of such equations, several processes are given, often with the main purpose of obtaining the integrals in finite terms; but the processes are limited in the scope of their application, because the equations which prove amenable to their action are few in character and not infrequently have been artificially constructed. When these processes either are not applicable or cannot conveniently be completed, no information concerning the solution of the equation would then be obtained; indeed, they offer no guarantee that an integral even exists.

Accordingly, it is desirable to discuss the whole theory of partial differential equations from the foundations and, in the course of that discussion, not only to revise known results but also, so far as may be possible, to place them in their fitting positions in the ordered body of doctrine. Such a discussion was found to be necessary for the proper establishment of results relating to ordinary differential equations. It is even more necessary in the case of partial differential equations, partly because the inversion of simultaneous partial differential operations is more difficult than the inversion of ordinary differential operations, partly because the suggestions as to the character of an integral, as offered by processes of inversion, are less significant for partial equations than for ordinary equations.

2. Two kinds of illustration should suffice for a justification of this last statement.

One mode of attempting to discover the character of the most complete integral of a partial equation would be by generalisation from the case of an ordinary equation.

For an ordinary equation, which has y for its dependent variable and x for its independent variable, the integral is made complete by the assignment of initial values to the variables; that is, y is some function of x and so, when a constant value is assigned to x, the function y and all its derivatives become constants. As the equation is to be satisfied and yet the integral is to be as complete as possible, these constants will be as unrestricted as possible: and therefore it is to be expected that some at least of them will be arbitrary constants. There thus arises a suggestion that the most complete integral will be such that, when some constant value is assigned to x, the function y and some of its derivatives acquire arbitrary constant values. The suggested property has been established under appropriate limitations and conditions.

To extend these results, if possible, to partial differential equations, consider a single partial equation of the first order, having z for its dependent variable and $x_1, \ldots, x_n$ for its independent variables. If an integral exists, that integral must determine z as a function of $x_1, \ldots, x_n$; and so, when an initial value a_n is assigned to x_n, the first derivatives of z with respect to $x_1, \ldots, x_{n-1}$ can be deduced from an assigned expression for z, and then (save in special circumstances) the partial equation determines the first derivative with regard to x_n. By using the equation in combination with the expressions for the first derivatives, the derivatives of higher order can be obtained for the value a_n of x_n; and thus no limitation appears to be imposed on the value of z as an assigned function of $x_1, \ldots, x_{n-1}$, when $x_n = a_n$. If the integral is to be as general as possible, it is reasonable to expect that the assigned function shall be as general as possible. But at this stage, questions arise as to what is the most general function admissible? Is it to be made general by possessing the greatest possible number of arbitrary constants? Can the assigned function be an arbitrary function, subject possibly to limitations imposed by the partial equation? and, if so, must it

be explicit or may it be given implicitly, for example, by means of quadratures which cannot be effected in finite terms? Or are all the modes indicated for securing the generality of the integral admissible, so that there are different kinds of general integrals? and if so, are there any relations among the various integrals? To such questions the argument offers no hint of an answer.

Similarly, when a partial equation of the second order is propounded in the same variables z, x_1, ..., x_n, the extension of the results obtained for ordinary equations suggests that z and $\frac{\partial z}{\partial x_n}$ should acquire assigned values as functions of x_1, ..., x_{n-1}, when $x_n = a_n$. For the values of $\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_{n-1}}$, when $x_n = a_n$, could be deduced from the value of z; and then the values of $\frac{\partial^2 z}{\partial x_r \partial x_s}$, for $r = 1, \ldots, n-1$, and $s = 1, \ldots, n$, could be deduced from the values of $\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}$ already known; and the partial equation would (save in special circumstances) determine the value of $\frac{\partial^2 z}{\partial x_n^2}$. As before, the values thus obtained, when combined with the use of the partial equation, lead to the values of all the derivatives. Thus all the quantities associated with z are known: at the utmost, only special limitations appear to be imposed upon the assigned functions by the process adopted; and therefore it is reasonable to expect that the integral will become the most general possible when the two assigned functions are as general as possible. Again, at this stage, questions arise as to the constitution of the generality of these assigned functions. Is the generality to be secured, by arranging that they shall involve the greatest possible number of arbitrary constants? or by making them independent arbitrary functions of x_1, ..., x_{n-1}? or by associating them with a possibly even more general function of $x_1, \ldots, x_n$ for the particular value a_n of x_n? If the functions are arbitrary, must they be given explicitly or may they be given implicitly as, for example, by uncompleted quadratures? Again, are all the modes admissible as alternatives, so that they lead to different kinds of general integrals? and if so, what relations (if any) subsist among the integrals? As in the former case, the argument offers no hint of answer to the questions.

3. In both the instances that have been briefly considered, the argument offers suggestions and even stirs expectations: that this is the limit of the attention to be paid to it, can perhaps be most simply seen by a particular case. Applied to a couple of simultaneous partial equations determining a couple of dependent variables, it would lead to a suggestion that the most general integral would involve at least two sets of general elements, whatever be their form; yet the integral of the simultaneous equations

$$\left.\begin{aligned}\frac{\partial^2 z_1}{\partial x^2} - a^2\frac{\partial^2 z_1}{\partial y^2} + 2a\frac{\partial z_1}{\partial y} + \frac{\partial z_2}{\partial x} - a\frac{\partial z_2}{\partial y} + z_2 &= f(x, y)\\ \frac{\partial z_1}{\partial x} + a\frac{\partial z_1}{\partial y} - z_1 + z_2 &= g(x, y)\end{aligned}\right\}$$

is

$$\left.\begin{aligned}z_1 &= f(x, y) - g(x, y) - \frac{\partial g(x, y)}{\partial x} + a\frac{\partial g(x, y)}{\partial y}\\ z_2 &= f(x, y) - \frac{\partial f(x, y)}{\partial x} - a\frac{\partial f(x, y)}{\partial y}\\ &\quad + 2a\frac{\partial g(x, y)}{\partial y} + \frac{\partial^2 g(x, y)}{\partial x^2} - a^2\frac{\partial^2 g(x, y)}{\partial y^2}\end{aligned}\right\},$$

a being a constant; manifestly it contains no arbitrary element. In fact, the utmost to be inferred from the argument is that some kinds of equations may possess integrals involving arbitrary elements in their most general forms, and that there may be different kinds of general integrals.

Whether these general integrals include all the integrals of an equation is a matter that demands separate consideration, to be undertaken later in another line of inquiry: and, naturally, a detailed consideration of the generality of integrals must also be undertaken later.

4. Another mode of attempting to discover the character of the most complete integral of a partial equation consists in comparing differential equations, constructed from initial integral equations, with those integral equations: but it is easily seen to be untrustworthy.

Thus if an integral equation

$$\frac{az+b}{cz+d} = \phi(x_1, \ldots, x_n),$$

where $ad - bc = 1$, be propounded, the result of eliminating the constants between the equation and its derivatives leads to the set of partial equations

$$\frac{1}{p_1}\frac{\partial\phi}{\partial x_1} = \frac{1}{p_2}\frac{\partial\phi}{\partial x_2} = \dots = \frac{1}{p_n}\frac{\partial\phi}{\partial x_n},$$

where

$$p_r = \frac{\partial z}{\partial x_r},$$

for $r = 1, \dots, n$. The process cannot be reversed, so as to lead to an inference that the most general integral of the set of partial equations contains three arbitrary essential constants: the inference would be incorrect, for the set of equations is satisfied by

$$f(z) = \phi(x_1, \dots, x_n),$$

where $f(z)$ is any function of z containing any number of arbitrary constants.

Again, if there is given an integral equation

$$f(x_1, \dots, x_n, z, a_1, \dots, a_m) = 0,$$

it is possible to construct a set of partial equations with which the integral equation is consistent, by forming the n derived equations which give the values of $\frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n}$, and then eliminating the m constants $a_1, \dots, a_m$. For the present purpose, the m constants may be assumed to be not reducible to a smaller number, and m may be assumed not greater than n; also, when elimination takes place, the number of resulting equations will be not less than $n + 1 - m$. It will be assumed that the number of such equations in the set is actually $n + 1 - m$; each of them is partial, and of the first order.

If m is equal to n, there is a single partial equation: and the argument suggests that a single partial equation of the first order may possess an integral involving n arbitrary constants: it does not prove this result, for there is nothing to shew that the partial equation is not of a special form, arising from the limitation that it has been deduced from an integral equation of specified form. The argument offers no contribution to the question as to whether, if the integral is possessed by the partial equation, it is the most general integral.

If m is less than n, there is a set of simultaneous partial equations of the first order: and the argument might be held

to suggest that $n+1-m$ simultaneous partial equations of the first order, involving one dependent variable and n independent variables, may possess a common integral involving m constants. The result is, of course, not proved and it is not true in general fact: for, independently of the impossibility of reversing the process of elimination, the $n+1-m$ equations are affected by the form of the original integral and therefore will have relations with one another, while such a set of partial equations postulated initially need not have any relations with one another. Thus the existence of a common integral is even more doubtful than in the case of a single equation: if it exists, no inference as to its generality can be drawn.

5. After these explanations and criticisms, it is manifest that attempts to obtain information as to the solution of partial equations by vague extensions of the knowledge of the solution of ordinary equations must be abandoned. The constructive process, that will be adopted instead of them, consists in the gradual establishment of results, beginning with the proof of the existence of integrals possessing definite assigned characters. The actual construction of the integrals when their existence has once been established, the discussion of the range of their generality, and the possibility of using them in the derivation of integrals of other kinds, all are matters for subsequent investigation.

It will be assumed that, save in special examples, the number of independent variables is n; and they will usually be denoted by $x_1, \ldots, x_n$. The number of dependent variables may be taken as m, the simplest case arising when $m=1$; they will be denoted by $z_1, \ldots, z_m$; and when there is only one variable, it will be denoted by z. For the present purpose, these dependent variables are to be determined by partial differential equations; let the number of such equations in a given set be s, and suppose that the highest derivatives that occur in them are of order μ.

Let derivatives of each of the equations be constructed, of all orders up to those of order κ inclusive. Then the total number of equations in the amplified set is

$$\begin{aligned} &s\left\{1+n+\tfrac{1}{2}n(n+1)+\ldots \text{ to } (\kappa+1) \text{ terms}\right\} \\ &= s\,\frac{(n+1)(n+2)\ldots(n+\kappa)}{1\,.\,2\,.\ldots\kappa} \\ &= sN, \end{aligned}$$

say; and the total number of dependent quantities, being the dependent variables and their derivatives of all orders up to $\mu + \kappa$ inclusive, is (or can be, for some of the dependent quantities may not occur explicitly)

$$\begin{aligned} m\,\{1 + n + \tfrac{1}{2}n(n+1) + \ldots \text{ to } (\kappa + \mu + 1) \text{ terms}\} \\ = m\,\frac{(n+1)(n+2)\ldots(n+\kappa+\mu)}{1\,.\,2\,.\ldots(\kappa+\mu)} \\ = mNK, \end{aligned}$$

where

$$K = \frac{(n+\kappa+1)\ldots(n+\kappa+\mu)}{(\kappa+1)\ldots(\kappa+\mu)}.$$

The factor K is obviously always greater than unity; and therefore if $s < m$, or if $s = m$, the number sN is less than mNK. The number of equations in the amplified set is less than the number of dependent quantities in the amplified aggregate; and therefore it will generally be impossible to eliminate the dependent quantities from among the equations. Were such elimination possible, the results would take the form of relations between the independent variables: and these, of course, do not occur. There is therefore nothing incompatible with the analytical nature of the case, if $s < m$, or if $s = m$.

Next, consider the possible hypothesis that $s > m$. The factor K is greater than unity; but its value decreases as κ increases, and it tends towards unity with large increase of κ. Let κ_1 be the earliest value of κ for which

$$K < \frac{s}{m};$$

then for the value κ_1, and for every value of κ which is greater than κ_1, we have

$$s > mK,$$

and therefore

$$sN > mNK.$$

For such values of κ, the number of equations in the amplified system is greater than the number of dependent quantities in the amplified aggregate. The dependent quantities could then, in general, be eliminated from the amplified system of equations; the results would take the form of relations among the independent variables alone, and such relations cannot occur. Such a conclusion is, in general, not compatible with the nature of the

case: and therefore, in general, s cannot be greater than m. If however the elimination could be performed for any given set of equations, amplified in the manner indicated, the final relations would be evanescent, and the incompatibility would not appear. This last event could occur only if such conditions were satisfied by the original system and consequent conditions were satisfied by the amplified system, as would reduce the number of independent equations in the amplified system so that, at the utmost, it should not be larger than the number of dependent quantities in the amplified aggregate.

Hence, in general, the number of equations in a given system must not be greater than the number of dependent variables involved; but the number of equations may be the greater in particular systems, and the investigation of the necessary and sufficient conditions will be a matter for subsequent discussion.

It is clear without detailed argument that, when s is less than m and when the equations are general, then $m-s$ of the dependent variables can have values assigned (either quite arbitrarily or arbitrarily within proper limits), still leaving as many equations as undetermined dependent variables.

Accordingly, the most general case to be considered for the present is that in which the number of equations is the same as the number of dependent variables.

6. Two properties of such a system of equations may be mentioned; their importance is mainly formal, and only a brief consideration is needed.

The first of the properties can be stated as follows: if a system of m partial equations in m dependent variables involves derivatives of order higher than the first, it can be replaced by an equivalent system of equations containing only derivatives of the first order, the number of independent equations in the new system being the same as the number of dependent variables which it involves.

The property is practically obvious and so hardly requires proof: it can be seen in connection with any particular example. Let there be a single equation, involving derivatives of the second order as the highest: when n new dependent variables are introduced by the equations

$$\frac{\partial z}{\partial x_r} = p_r, \qquad (r = 1, \ldots, n),$$

the given equation can be expressed in a form

$$f\left(x_1, \ldots, x_n, z, p_1, \ldots, p_n, \frac{\partial p_1}{\partial x_1}, \ldots, \frac{\partial p_n}{\partial x_n}\right) = 0,$$

which involves only derivatives of the first order; and the new system now contains $n+1$ equations, involving $n+1$ dependent variables with derivatives of the first order.

It may be added that the main use of the property lies in deducing existence-theorems for equations of order higher than the first from the existence-theorems which soon will be established for systems of equations of the first order.

An extended form of the property enables us, not merely to replace any given system by a system containing only derivatives of the first order, but also to secure that each equation, which in the new system involves derivatives of the first order, is linear in those derivatives. Thus, in the preceding example, additional dependent variables would be introduced by the equations

$$\frac{\partial z}{\partial x_\mu} = p_\mu, \qquad \frac{\partial p_\mu}{\partial x_s} = q_{\mu s},$$

for μ and $s = 1, \ldots, n$: the original equation takes the form of a relation

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n, q_{11}, \ldots, q_{nn}) = 0$$

among the variables free from derivatives; and the derived equations

$$\frac{\partial f}{\partial x_s} + p_s \frac{\partial f}{\partial z} + \sum_{\mu=1}^{n} \frac{\partial f}{\partial p_\mu} q_{\mu s} + \sum_{\lambda, \mu}\sum \frac{\partial f}{\partial q_{\lambda\mu}} \frac{\partial q_{\lambda\mu}}{\partial x_s} = 0$$

are formed for $s = 1, \ldots, n$. All the equations are linear in the derivatives which are of the first order; but it should be noted that the number of equations in the modified system is larger than the number of dependent variables, though the conditions for coexistence are satisfied.

When the number of variables is other than very few, the extended form of the property tends to be cumbrous. It is, however, of definite use, as will be seen later (Chap. XVIII), as part of a method for obtaining integrals of equations of order higher than the first when they possess integrals that are expressible in finite terms.

7. The other of the two properties indicated in § 6 bases the solution of a system of m partial equations in m dependent variables upon the solution of one equation (or of more than one equation) in a single dependent variable in association with algebraic processes: the substituted equation or equations usually (but not universally) involve derivatives of order higher than those which occurred in the original system.

Reverting to the method adopted in § 5, and applying it for the purpose of eliminating $z_2, \ldots, z_m$ and all their derivatives, we should have mN equations in the amplified set, while the number of dependent quantities to be eliminated is $(m-1)NK$. Accordingly, let κ_1 be the least value of κ for which

$$K < \frac{m}{m-1},$$

and therefore

$$(m-1)NK < mN.$$

The dependent quantities, composed of $z_2, \ldots, z_m$ and their derivatives, can be eliminated from the amplified set of equations: the results of the elimination will take the form of one equation or more than one equation involving z_1 and its derivatives, the latter being of order higher than those which occur in the original system. Moreover, by the algebraic processes, all the dependent quantities that are eliminated are expressible in terms of those that survive. Accordingly, when the solution of the equation or equations in z_1 is known, the other dependent quantities can be regarded as known: and then the solution of the original system will have been obtained.

It should be added that this property is not of importance in the general theory: its chief value lies in the fact that it provides a method which sometimes is effective in leading to the solution of particular classes of equations.

PREPARATION FOR CAUCHY'S THEOREM: THE FIRST OF THE SUBSIDIARY EXISTENCE-THEOREMS.

8. We proceed now to the establishment of some positive results, in particular, to the establishment of Cauchy's theorem affirming the existence of integrals of a system of partial equations.

The system of equations contains the same number of dependent variables as of equations; in form, it does not include all possible systems of such a character, but it will be found to include a large selection of important and representative systems; and the integrals will be proved to exist, subject to an aggregate of assigned conditions. For the purpose in view, the method devised by Madame Kowalevsky* will be adopted, whereby the main theorem is approached through two existence-theorems belonging to partial equations of comparatively simple type.

The first of these theorems can be stated as follows:—

Let a set of partial equations be given in the form

$$\frac{\partial z_i}{\partial x_1} = \sum_{j=1}^{m} \sum_{r=2}^{n} G_{ijr} \frac{\partial z_j}{\partial x_r},$$

for values $i = 1, \ldots, m$, being m equations in m dependent variables; the coefficients G_{ijr} are functions of $z_1, \ldots, z_m$ alone. Let $c_1, \ldots, c_m$ be a set of values of $z_1, \ldots, z_m$ respectively, in the vicinity of which each of the functions G_{ijr} is regular; and let $\phi_1, \ldots, \phi_m$ be a set of functions of $x_2, \ldots, x_n$, which acquire the values $c_1, \ldots, c_m$ respectively when $x_2 = a_2, \ldots, x_n = a_n$, which are regular in the vicinity of these values of $x_2, \ldots, x_n$, and which otherwise are arbitrary. Then *a system of integrals of the equations can be determined, which are regular functions of* $x_1, \ldots, x_n$ *in the vicinity of the values* $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$, *and which acquire the values* $\phi_1, \ldots, \phi_m$ *when* $x_1 = a_1$; *moreover, the system of integrals, determined in accordance with these conditions, is the only system of integrals that can be so determined as regular functions.*

9. It is convenient, for the sake of conciseness in the formulæ, to write

$$x_s - a_s = y_s, \quad z_r - c_r = \zeta_r, \quad \phi_r - c_r = \psi_r,$$

for $s = 1, \ldots, n$, and $r = 1, \ldots, m$: and then we have to deal with quantities in the vicinity of zero values of y and ζ.

As the functions G are regular within this vicinity over some finite region, we select a portion of the region defined by the ranges

$$|\zeta_1| \leqslant R, \quad |\zeta_2| \leqslant R, \ \ldots, \ |\zeta_m| \leqslant R;$$

* *Crelle*, t. LXXX (1875), pp. 1—32.

and we denote by M the greatest value among the quantities $|G_{ijr}|$ within this portion of the region selected, M being finite. The functions G can be expressed as power-series; and if we take

$$G_{ijr} = \Sigma\Sigma \ldots a^{ijr}_{s_1 s_2 \ldots} \zeta_1^{s_1} \zeta_2^{s_2} \ldots,$$

where the multiple summation is for all integer values of $s_1, s_2, \ldots$ from zero upwards, simultaneous zeros being included, then*

$$\begin{aligned} \left| a^{ijr}_{s_1 s_2 \ldots} \right| &\leqslant \frac{M}{R^{s_1+s_2+\ldots}} \\ &\leqslant \frac{(s_1+s_2+\ldots)!}{s_1!\, s_2! \ldots} \frac{M}{R^{s_1+s_2+\ldots}}, \end{aligned}$$

a fortiori. Similarly, within a selected region of existence of the functions ψ, defined by the ranges

$$|y_2| \leqslant \rho, \quad |y_3| \leqslant \rho, \ldots, \quad |y_n| \leqslant \rho,$$

we have

$$\psi_\mu = \Sigma\Sigma \ldots c_{\mu\mu_2\mu_3} \ldots y_2^{\mu_2} y_3^{\mu_3} \ldots,$$

where the multiple summation is for all integer values of $\mu_2, \mu_3, \ldots$ from zero upwards, simultaneous zeros being excluded; and then, if N denote the greatest value among the quantities $|\psi_\mu|$ within this region, we have

$$\begin{aligned} \left| c_{\mu\mu_2\mu_3\ldots} \right| &\leqslant \frac{N}{\rho^{\mu_2+\mu_3+\ldots}} \\ &\leqslant \frac{(\mu_2+\mu_3+\ldots)!}{\mu_2!\, \mu_3! \ldots} \frac{N}{\rho^{\mu_2+\mu_3+\ldots}}, \end{aligned}$$

a fortiori.

If functions ζ exist possessing the character required in the theorem, they can be expanded as series of powers of y_1 in the vicinity of the origin; having regard to the value they must acquire when $y_1 = 0$, we can take them in the form

$$\zeta_\mu = \psi_\mu + y_1 \psi_{\mu 1} + y_1^2 \psi_{\mu 2} + \ldots,$$

where (as the functions ζ are to be regular in all their variables) the coefficients $\psi_{\mu 1}, \psi_{\mu 2}, \ldots$ must be regular functions of $y_2, y_3, \ldots$ within the selected region of existence, and they do not involve y_1. These functions ζ, if they exist, are to satisfy the differential equations: we substitute them therein, and compare the coefficients of the various powers of y_1 and, from the fact that the derivatives

* See my *Theory of Functions*, (Second edition), hereafter quoted as *T. F.*, § 22.

with regard to x_1 (and consequently of y_1) occur on the left-hand sides of the equations only, we find relations of the form

$$p\psi_{\mu p} = \Sigma Z,$$

where the number of terms in the summation on the right-hand side is finite. Each term Z is the product of four factors:

(i) a coefficient a from the expansion of the functions G:

(ii) a product of the functions $\psi_{\kappa\lambda}$, the second subscript index λ being less than p:

(iii) a first derivative of one of the functions $\psi_{\kappa\lambda}$, the second subscript index λ being less than p:

(iv) a positive integer.

Now all the functions $\psi_{\kappa\lambda}$ having their second subscript index zero, are the functions $\psi_1, \ldots, \psi_m$; and their expressions as regular functions of $y_2, \ldots, y_n$ are known. Hence the relations

$$p\psi_{\mu p} = \Sigma Z$$

give a formal determination of the functions $\psi_{\mu p}$ in succession; they appear as power-series in $y_2, \ldots, y_n$, the coefficients in which involve the constants a from the expansion of the functions G, the constants c from the expansion of the functions $\psi_1, \ldots, \psi_m$, and positive numerical factors.

When these values are substituted in the y_1-expansions of $\zeta_1, \ldots, \zeta_m$, expressions result which formally satisfy the differential equations. In order that they may possess functional significance, these multiple series must converge; this necessary convergence can be proved as follows.

10. We consider variation in a more restricted range for the quantities ζ, given by

$$|\zeta_1| < \frac{R}{m}, \quad |\zeta_2| < \frac{R}{m}, \ldots, \quad |\zeta_m| < \frac{R}{m},$$

so that

$$|\zeta_1 + \zeta_2 + \ldots + \zeta_m| < R;$$

and we construct a dominant function* G in the form

$$G = \Sigma\Sigma \ldots \frac{(s_1 + s_2 + \ldots)!}{s_1!\, s_2! \ldots} \frac{M}{R^{s_1+s_2+\ldots}} \zeta_1^{s_1} \zeta_2^{s_2} \ldots,$$

* *T. F.*, § 23.

where the multiple summation is for all integer values of $s_1, s_2, \ldots$ from zero upwards, simultaneous zeros being included. Then the modulus of any term in G is at least as great as the modulus of the corresponding term in G_{ijr} having the same combination of the variables. Moreover, in the region of variation considered, G can be expressed in finite terms: we have

$$G = \frac{M}{1 - \frac{1}{R}(\zeta_1 + \zeta_2 + \ldots + \zeta_m)}.$$

We also consider variation in a more restricted range for the quantities $y_2, \ldots, y_n$, given by

$$|y_2| < \frac{\rho}{n-1}, \ \ldots, \ |y_n| < \frac{\rho}{n-1},$$

so that

$$|y_2 + \ldots + y_n| < \rho;$$

and we construct another dominant function ψ in the form

$$\psi = \Sigma\Sigma \ldots \frac{(\mu_2 + \mu_3 + \ldots)!}{\mu_2! \ \mu_3! \ldots} \frac{N}{\rho^{\mu_2 + \mu_3 + \ldots}} y_2^{\mu_2} y_3^{\mu_3} \ldots,$$

where the multiple summation is for all integer values of $\mu_2, \mu_3, \ldots$ from zero upwards, simultaneous zeros being excluded. Then the modulus of any term in ψ is at least as great as the modulus of the corresponding term in ψ_μ having the same combination of the variables. Moreover, in the region of variation considered, ψ can be expressed in finite terms; writing

$$y = y_2 + \ldots + y_n,$$

we have

$$\psi = \frac{Ny}{\rho} \frac{1}{1 - \frac{y}{\rho}}$$

$$= \frac{Ny}{\rho - y},$$

and the range for the variable y is given by

$$|y| < \rho.$$

11. By means of these dominant functions, a dominant system of partial equations

$$\frac{\partial Z_i}{\partial y_1} = \sum_{j=1}^{m} \sum_{r=2}^{n} G \frac{\partial Z_j}{\partial y_r}$$

is constructed; and we assign, as conditions, that each of the quantities Z_i shall acquire the value ψ when $y_1=0$. Taking G and ψ in their expanded forms as power-series, and applying to these equations the process applied to the original system, we obtain equations of precisely the same form as before, to determine the successive coefficients in the expressions for the variables Z as power-series in the variables $y_1, y_2, \ldots, y_n$: and the successive operations for the construction of these coefficients are the same as before. In these new operations, all the terms in the expression for any coefficient are positive; the modulus of each term is at least as great as was the modulus of the corresponding term in the former operations; and therefore, if

$$Z_\mu = \psi + y_1\psi'_{\mu 1} + y_1^2\psi'_{\mu 2} + \ldots,$$

we have

$$| p\psi'_{\mu p} | \geqslant | p\psi_{\mu p} |,$$

that is,

$$| \psi_{\mu p} | \leqslant | \psi'_{\mu p} |.$$

Hence the series for ζ_μ will certainly converge if the series for Z_μ converges.

The values of $Z_1, \ldots, Z_m$, and their consequent expressions as converging series, can be otherwise obtained. Returning to the dominant system and using the finite forms for G and ψ, we have to determine values of $Z_1, \ldots, Z_m$, satisfying the equations

$$\frac{\partial Z_i}{\partial y_1} = \frac{M}{1-\dfrac{1}{R}(Z_1+\ldots+Z_m)} \sum_{j=1}^{m} \sum_{r=2}^{n} \frac{\partial Z_j}{\partial y_r},$$

and such that

$$Z_\mu = \frac{Ny}{\rho - y},$$

when $y_1=0$. Thus Z_μ, a function of all the variables, is a function of the combination of them represented by y, say a function of y alone, when $y_1=0$; and therefore $\dfrac{\partial Z_\mu}{\partial y_r}$, for $r=2, \ldots, n$, is also a function of y alone when $y_1=0$. The differential equations then shew that $\dfrac{\partial Z_\mu}{\partial y_1}$ is a function of y alone when $y_1=0$. Again, differentiating all the equations with regard to y_1, noting that the quantities Z and all their first derivatives are functions of y alone when $y_1=0$, and applying a similar argument, we find that $\dfrac{\partial^2 Z_\mu}{\partial y_1^2}$

is a function of y alone when $y_1 = 0$. Similarly, for all the derivatives in succession. Inserting their forms in

$$Z_\mu = [Z_\mu]_0 + y_1 \left[\frac{\partial Z_\mu}{\partial y_1}\right]_0 + \frac{y_1^2}{2}\left[\frac{\partial^2 Z_\mu}{\partial y_1^2}\right]_0 + \ldots,$$

we see that Z_μ, if it exists, is expressible as a function of y_1 and y.

Now from the equations, we have

$$\frac{\partial Z_1}{\partial y_1} = \frac{\partial Z_2}{\partial y_1} = \ldots = \frac{\partial Z_m}{\partial y_1};$$

and therefore, taking account of the conditions that

$$Z_1 = Z_2 = \ldots = Z_m = \frac{Ny}{\rho - y},$$

when $y_1 = 0$, we have

$$Z_1 = Z_2 = \ldots = Z_m,$$

in general. Denote this common value by Z, which now is a function of y_1 and y; then all the equations in the dominant system are satisfied, provided Z can be determined to satisfy the equation

$$\frac{\partial Z}{\partial y_1} = \frac{M}{1 - m\dfrac{Z}{R}}\, m(n-1)\frac{\partial Z}{\partial y}$$

and is such that it acquires the value $\dfrac{Ny}{\rho - y}$ when $y_1 = 0$. It is easy to verify that the equation is satisfied by a relation

$$\left(1 - m\frac{Z}{R}\right) y + Mm(n-1)\, y_1 = f(Z),$$

where f is any function whatever of Z: and therefore all the requirements will be met if f can be chosen so as to allow Z to acquire the value $\dfrac{Ny}{\rho - y}$ when $y_1 = 0$. For this purpose, the two equations

$$\left(1 - m\frac{u}{R}\right) y = f(u),$$

$$\frac{Ny}{\rho - y} = u,$$

must be the same. The latter gives

$$y = \frac{\rho u}{N + u},$$

which, substituted in the former, gives

$$f(u) = \left(1 - m\frac{u}{R}\right)\frac{\rho u}{N+u},$$

being the appropriate expression of the function f. Thus all the requirements are met by a value or values of Z determined by the integral relation

$$\left(1 - m\frac{Z}{R}\right) y + Mm(n-1)y_1 = \left(1 - m\frac{Z}{R}\right)\frac{\rho Z}{N+Z}.$$

This quadratic equation has two roots. One of the two roots becomes $\frac{R}{m}$ when $y_1 = 0$, and must be discarded as not satisfying the imposed condition when $y_1 = 0$. The other root is given in the form

$$\begin{aligned} 2\frac{m}{R}(\rho - y)Z - \left\{(\rho - y) + m\frac{N}{R}y - Mm(n-1)y_1\right\} \\ = -\left[\left(\rho - y - m\frac{N}{R}y\right)^2 + M^2m^2(n-1)^2 y_1^2 \right. \\ \left. - 2Mm(n-1)y_1\left\{\rho - y + m\frac{N}{R}(2\rho - y)\right\}\right]^{\frac{1}{2}}; \end{aligned}$$

it can be expanded in powers of y_1 in the series

$$Z = \frac{Ny}{\rho - y} + Mm(n-1)y_1\frac{N\rho}{\rho - y}\frac{1}{\rho - y - m\frac{N}{R}y}$$
$$+ \text{higher powers of } y_1.$$

It is clear from this expression for Z obtained in finite form that Z can be expanded in a series of powers of y and y_1, which converges in a non-infinitesimal range round $y = 0$ and $y_1 = 0$. When y is replaced by its value $y_2 + y_3 + \ldots + y_n$, the modified power-series in $y_1, y_2, \ldots, y_n$ still converges in a non-evanescent range round $y_1 = 0$, $y_2 = 0$, $\ldots$, $y_n = 0$; consequently, the quantity Z of the required type does exist.

It therefore follows that the quantities $Z_1, \ldots, Z_m$ exist as determined by the equations in the dominant system; and therefore integrals of the original equations exist satisfying the prescribed conditions.

12. The preceding investigation establishes the existence of integrals which are regular functions of the variables in the

selected region; it is easy to see that they are the only set of integrals which, satisfying the prescribed conditions, are regular functions of the variables. If any other set existed, being regular functions and satisfying the prescribed conditions, they would be expressible in a form

$$\zeta'_\mu = \psi_\mu + y_1 \psi'_{\mu 1} + y_1^2 \psi'_{\mu 2} + \dots;$$

when these are substituted in the differential equations, they would lead to relations

$$p\psi'_{\mu p} = \Sigma Z'$$

for the determination of the coefficients, similar to the relations

$$p\psi_{\mu p} = \Sigma Z.$$

When $p = 1$, no double-suffix function occurs in Z', which then is the same as Z, because the term in ζ'_μ independent of y_1 is the same as the corresponding term in ζ_μ: hence

$$\psi'_{\mu 1} = \psi_{\mu 1}.$$

When $p = 2$, the only double-suffix functions that can occur in Z' are the functions $\psi'_{\lambda 1}$, which have been shewn to be the same as $\psi_{\lambda 1}$; hence, for this value, $Z' = Z$, and therefore

$$\psi'_{\mu 2} = \psi_{\mu 2}.$$

Similarly for all the coefficients in succession: we find

$$\zeta'_\mu = \zeta_\mu,$$

for all the values of μ; and therefore the set of regular integrals obtained, subject to the prescribed conditions, are unique regular integrals.

As an example illustrating the general theorem, we require the integrals of the simultaneous equations

$$\frac{\partial u}{\partial x} = u^2 \frac{\partial u}{\partial y},$$

$$\frac{\partial v}{\partial x} = (u^2 - u + v) \frac{\partial u}{\partial y},$$

such that, when $x = 0$,

$$u = y\,(a_0 + a_1 y + a_2 y^2 + \dots) = R\,(y),$$
$$v = y\,(b_0 + b_1 y + b_2 y^2 + \dots) = S\,(y).$$

The equations are amenable to the ordinary practical methods. The most general integral of the first equation is easily found to be

$$u = f(y + u^2 x),$$

where f is arbitrary so far as the equation is concerned. Also

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}=(v-u)\frac{\partial u}{\partial y}$$

$$=(v-u)\frac{1}{u^2}\frac{\partial u}{\partial x},$$

and therefore

$$v-u=e^{-\frac{1}{u}}g(y),$$

where g is arbitrary so far as the equation is concerned.

To determine these arbitrary functions, the imposed conditions are used. As

$$u=f(y+u^2x),$$

and as $u=R(y)$ when $x=0$, we have

$$R(y)=f(y),$$

and therefore, generally,

$$u=R(y+u^2x).$$

Laplace's theorem in expansion can be used to give the explicit expression for u in terms of x and y: this is

$$u=R(y)+xR^2(y)R'(y)+\frac{x^2}{2}\frac{d}{dy}\{R^4(y)R'(y)\}+\frac{x^3}{6}\frac{d^2}{dy^2}\{R^6(y)R'(y)\}+\dots$$

$$=R(y)\{1+xR(y)T(x, y)\},$$

where $T(x, y)$ is a series of powers of x, the coefficients being functions of y which vanish if $R(y)$ vanishes identically.

Again, as $v=S(y)$ when $x=0$, we have

$$S(y)-R(y)=e^{-\frac{1}{R(y)}}g(y),$$

so that, generally,

$$v-u=e^{\frac{1}{R(y)}-\frac{1}{u}}\{S(y)-R(y)\}.$$

The apparent singularity can be removed; for

$$\frac{1}{R(y)}-\frac{1}{u}=\frac{xT(x, y)}{1+xR(y)T(x, y)},$$

where the function on the right-hand side is regular in the vicinity of $x=0$, $y=0$. Thus the required integrals are

$$u=R(y)\{1+xR(y)T(x, y)\},$$

$$v-u=\{S(y)-R(y)\}\,e^{\frac{xT(x, y)}{1+xR(y)T(x, y)}}.$$

In particular, if the imposed conditions should be that $u=0$ and $v=S(y)$ when $x=0$, then as $R(y)$ vanishes identically, $T(x, y)$ vanishes. The full expressions for the integrals are

$$u=0, \qquad v=S(y);$$

these are easily obtainable directly from the differential equations but not from the integrals which involve the arbitrary functions f and g.

13. It should be observed that, throughout the foregoing proof, there has been a complete restriction to regular functions. The possibility of non-regular functions, satisfying the equations and obeying the prescribed conditions, has nowhere been taken into account and the proof does not shew that it should be rejected as inadmissible; what is established is that a unique set of regular integrals exists. In the statement of the argument, it is practically assumed that the integrals are regular. Thus z_1 is made to acquire the value of a regular function of $x_2, \ldots, x_n$ when $x_1 = a_1$; and this could be secured when z_1 is a uniform function of x_1, even if x_1 is an essential singularity, provided x_1 then be allowed to approach a_1 by an appropriate path*. If however z_1 is made to acquire its value when $x_1 = a_1$, quite independently of the path by which x_1 approaches the position a_1, and so also for the other dependent variables, then it is not difficult to see that the integrals must be regular under the assigned conditions. For the differential equations then make $\frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial z_m}{\partial x_1}$ regular functions of $x_2, \ldots, x_n$, whatever be the x_1-path of approach to a_1; when derivatives of the equations are formed and suitably combined it could be inferred that $\frac{\partial^2 z_1}{\partial x_1^2}, \ldots, \frac{\partial^2 z_m}{\partial x_1^2}$, in like circumstances, become regular functions of $x_2, \ldots, x_n$; and so on, for the derivatives in succession. The inference that $z_1, \ldots, z_m$ are regular functions of $x_1, x_2, \ldots, x_n$ is then immediate.

The assumption made by ignoring the path of approach of x_1 to a_1 may fairly be described as a customary assumption: it does, in effect, exclude the consideration of the possibility that a_1 is an essential singularity of a uniform function, and it may exclude the consideration of other possibilities of deviation from regularity. Yet it is not inconceivable that, in particular instances, such as the stability of a system in a critical condition, the excluded possibilities are of importance†: in such an instance, it might be actually the fact that the variable must approach its value by a specific path and is not permitted an unrestricted approach to the value.

* See *T. F.*, p. 57 (second edition), Ex. 4.

† The same considerations occur in connection with the integrals of an ordinary equation of the first order: see §§ 28—34 in volume II of this work, where (§ 34) the condition given for that case by Fuchs is explained.

THE SECOND OF THE SUBSIDIARY EXISTENCE-THEOREMS.

14. In the preceding theorem, the only variables which occur explicitly in the set of partial equations are the dependent variables: it can, however, be extended so as to allow the explicit occurrence of all the variables. The extended theorem is as follows:—

Let a set of partial equations be given in the form

$$\frac{\partial z_i}{\partial x_1} = \sum_{j=1}^{m} \sum_{r=2}^{n} G_{ijr} \frac{\partial z_j}{\partial x_r} + G_i,$$

for values $i = 1, \ldots, m$, being m equations in m dependent variables; the coefficients G_{ijr} and the quantities G_i are functions of all the variables, dependent and independent. Let $c_1, \ldots, c_m, a_1, \ldots, a_n$ be a set of values of $z_1, \ldots, z_m, x_1, \ldots, x_n$ respectively, in the vicinity of which all the functions G_{ijr} and G_i are regular; and let $\phi_1, \ldots, \phi_m$ be a set of functions of $x_2, \ldots, x_n$, which acquire values $c_1, \ldots, c_m$ respectively when $x_2 = a_2, \ldots, x_n = a_n$, which are regular in the vicinity of these values of $x_2, \ldots, x_n$, and which otherwise are arbitrary. Then *a system of integrals of the equations can be determined, which are regular functions of $x_1, \ldots, x_n$ in the vicinity of the values $x_1 = a_1$, $x_2 = a_2$, $\ldots$, $x_n = a_n$, and which acquire the values $\phi_1, \ldots, \phi_m$ when $x_1 = a_1$; moreover, the system of integrals, determined in accordance with these conditions, is the only system of integrals that can be thus determined as regular functions.*

The establishment of this theorem can be derived from the former theorem in a simple manner. Let n new dependent variables $t_1, \ldots, t_n$ be introduced, defined by equations

$$\frac{\partial t_1}{\partial x_1} = \frac{\partial t_2}{\partial x_2}, \quad \frac{\partial t_2}{\partial x_1} = 0, \quad \frac{\partial t_3}{\partial x_1} = 0, \ldots, \quad \frac{\partial t_n}{\partial x_1} = 0,$$

and by the conditions that, when $x_1 = a_1$,

$$t_1 = a_1, \quad t_2 = x_2, \quad \ldots, \quad t_n = x_n.$$

From the last $n - 1$ of these equations, combined with the imposed conditions, it is clear that

$$t_2 = x_2, \quad t_3 = x_3, \quad \ldots, \quad t_n = x_n,$$

in general. Then

$$\frac{\partial t_1}{\partial x_1} = \frac{\partial t_2}{\partial x_2} = 1,$$

so that

$$t_1 = x_1 + \text{function of } x_2, \ldots, x_n$$
$$= x_1,$$

on applying the imposed condition.

We replace $x_1, \ldots, x_n$ in G_{ijr} and in G_i by $t_1, \ldots, t_n$, and denote the functions resulting after the change by H_{ijr} and H_i. Also, noting the fact that $\frac{\partial t_2}{\partial x_2}$ is unity, we take an amplified system of equations

$$\frac{\partial z_i}{\partial x_1} = \sum_{j=1}^{m} \sum_{r=2}^{n} H_{ijr} \frac{\partial z_j}{\partial x_r} + H_i \frac{\partial t_2}{\partial x_2},$$
$$\frac{\partial t_1}{\partial x_1} = \frac{\partial t_2}{\partial x_2},$$
$$\frac{\partial t_\mu}{\partial x_1} = 0,$$

for $i = 1, \ldots, m$, and $\mu = 2, \ldots, n$; and the imposed conditions are that, when $x_1 = a_1$,

$$z_1 = \phi_1, \quad z_2 = \phi_2, \quad \ldots, \quad z_m = \phi_m,$$
$$t_1 = a_1, \quad t_2 = x_2, \quad \ldots, \quad t_n = x_n.$$

The coefficients in the modified system are functions of the dependent variables; the properties of the modified system, when account is taken of the imposed conditions, are the properties of the systems to which the former theorem applies. Hence, by that former theorem, a set of integrals

$$z_i = \psi_i(x_1, x_2, \ldots, x_n),$$
$$t_\mu = x_\mu,$$

for $i = 1, \ldots, m$ and $\mu = 1, \ldots, n$, exists; the functions ψ_i are regular functions of the variables x, and when $x_1 = a_1$, the functions $\psi_1, \ldots, \psi_m$ reduce to $\phi_1, \ldots, \phi_m$ respectively; and these regular integrals are the only set of regular integrals which satisfy the imposed conditions.

When we substitute $t_1 = x_1$, $t_2 = x_2$, $\ldots$, $t_n = x_n$ in the modified system, we return to the original system: the results just obtained constitute the theorem required.

Note. There is the same kind of limitation as in the former case (§ 13): it is possible that, for reasons connected with essential

singularities such as particular modes of the approach of x_1 to a_1, there may be non-regular integrals of the equations satisfying the imposed conditions.

Ex. 1. Obtain the integral of the equations

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=u-2v-x^2+2x+(u-2v-x^2)\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x}&=(u-2v-x^2)\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\end{aligned}\right\},$$

subject to the initial conditions that, when $x=0$,

$$u=2e^{(y+z)^2}+ae^{-z},\qquad v=e^{(y+z)^2},$$

(where a is a constant), in the form

$$u-2v-x^2=ae^{-z},$$
$$\log v=\{x+y+z+ae^{-z}(1-e^{-x})^2\}^2.$$

(Riquier.)

Ex. 2. Integrate similarly the equations

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=2x+(u-2v-x^2)(1-2t^2)+2t(u-2v-x^2)\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x}&=(x^2+2v-u)t^2+2t(u-2v-x^2)\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\\ \frac{\partial t}{\partial x}&=2t(u-2v-x^2)\frac{\partial t}{\partial y}+\frac{\partial t}{\partial z}\\ \frac{\partial w}{\partial x}&=(x^2+2v-u)t^2+2t(u-2v-x^2)\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}\end{aligned}\right\}$$

subject to the initial conditions that, when $x=0$, the variables u, v, t, w, respectively acquire the values

$$u=2(y^2+z^2)+ae^{-z},\qquad v=y^2+z^2,\qquad t=1,\qquad w=2z.$$

(Riquier.)

Ex. 3. As an example of the general theorem, let it be required to obtain the integral of the equation

$$\frac{\partial z}{\partial x}+\frac{y^2}{y^2-y+x}\frac{\partial z}{\partial y}=0,$$

which acquires the value y when $x=1$.

After the explanations that have been given, it will be of the form

$$z=y-(x-1)y_1+(x-1)^2y_2-\dots;$$

in order that this may satisfy the differential equation, the coefficients $y_1, y_2, \dots$ are determined by the relation

$$ny_n=\frac{y^2}{y^2-y+1}\frac{\partial y_{n-1}}{\partial y}+\frac{y^2}{(y^2-y+1)^2}\frac{\partial y_{n-2}}{\partial y}+\frac{y^2}{(y^2-y+1)^3}\frac{\partial y_{n-3}}{\partial y}+\dots$$
$$\dots+\frac{y^2}{(y^2-y+1)^{n-1}}\frac{\partial y_1}{\partial y}+\frac{y^2}{(y^2-y+1)^n},$$

as may be verified by substituting the expression for z and comparing coefficients of powers of $x-1$. In particular,

$$y_1=\frac{y^2}{y^2-y+1},$$

$$2y_2=\frac{y^2}{y^2-y+1}\frac{d}{dy}\left(\frac{y^2}{y^2-y+1}\right)+\frac{y^2}{(y^2-y+1)^2},$$

and so on.

The equation is amenable to the ordinary practical method. The most general integral is found to be

$$(x-y)\,e^{\frac{1}{y}}=f(z),$$

where f is an arbitrary function, to be rendered definite by means of the assigned condition, which is that z must acquire the value y when $x=1$. Hence

$$(1-y)\,e^{\frac{1}{y}}=f(y),$$

and therefore

$$(1-z)\,e^{\frac{1}{z}}=f(z),$$

so that the required integral is given by the equation

$$(1-z)\,e^{\frac{1}{z}}=(x-y)\,e^{\frac{1}{y}}.$$

But in connection with this equation, it must be specified as that value of z which acquires the value y when $x=1$; it is not enough to take any root of the equation for the integral, because (when $x=1$) there is an infinitude of values of z as functions of y, and only one of these is actually equal to y. In fact, the finite form of the equation, though it includes the required integral, does not give a unique expression for z.

Note. It sometimes is convenient* to associate an ordinary equation

$$\frac{dy}{dx}=f(x,\ y)$$

with a partial differential equation

$$\frac{\partial z}{\partial x}+f(x,\ y)\frac{\partial z}{\partial y}=0.$$

It is known that integrals of the respective equations exist. Taking the equation just discussed, so as to have

$$f(x,\ y)=\frac{y^2}{y^2-y+x},$$

the only regular integral of the ordinary equation, acquiring a value zero when $x=1$, is given by

$$y=0.$$

* Picard, *Traité d'Analyse*, t. II, ch. XI, § 15.

Now taking the partial equation and supposing an integral of the ordinary equation, assumed to exist in the same regular field of variation, to be substituted in z, we have

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}=0,$$

that is, $z=A$, a constant, for that substitution: and the constant A manifestly can be made zero. Conversely,

$$z=0$$

clearly gives the regular integral of the ordinary equation in the form

$$y=0.$$

We can also, by quadratures, obtain the complete primitive of the ordinary equation in the form

$$(x-y)\,e^{\frac{1}{y}}=c,$$

where c is an arbitrary constant. This cannot be obtained from the regular integral of the partial equation, by taking

$$z=A,$$

for any value of A, if we take unlimited variation of y; because $y=0$ is an essential singularity for one equation and an ordinary point for the other. But it can be obtained from the non-regular integral

$$(x-y)\,e^{\frac{1}{y}}=f(z),$$

by taking $z=A$: the appropriate value of c is

$$c=f(A).$$

CHAPTER II.

Cauchy's Theorem.

The main results in this chapter are associated with theorems establishing the existence, under assigned conditions, of integrals of systems of partial equations, the number of equations in a system being the same as the number of dependent variables: they are conveniently described as Cauchy's Theorem, because they have their origin in Cauchy's investigations* on the subject. The method adopted is based upon the memoir of Madame Kowalevsky, quoted in the preceding chapter (p. 11); reference may also be made to the expositions given by Jordan, *Cours d'Analyse*, t. III (1896), ch. III, § 1, and by Goursat, *Leçons sur l'intégration des équations aux derivées partielles du premier ordre*, (1891), ch. I.

15. The existence-theorems established in the preceding chapter can be applied to equations, and to systems of equations, of representative types; and to such applications we now proceed. But some passing remarks must be made upon the limitations that have been imposed. All the equations are linear in the derivatives of the dependent variables: this character, if not initially possessed, frequently (though not universally) can be secured by appropriate transformations. All the coefficients of the derivatives in the equations have been assumed to be regular functions of the independent variables (and, in the case of the earlier theorem, of the dependent variables) within the fields of variation considered: no result as to the character, or even the existence, of integrals has been obtained when there is any deviation from the postulated regularity. The imposed initial conditions are of a similar type, because they require the assumption, as values, of arbitrary functions of a regular character for a

* *Œuvres de Cauchy*, 1re Sér., t. VII, p. 17, and elsewhere. These memoirs were published in the *Comptes Rendus* in 1842; his earliest researches on the subject date back to 1819.

chosen value of a particular variable: but no result is established if these assigned arbitrary functions are not regular, in any form of deviation from regularity. It may be possible (and frequently it is possible) to transform the equations in such a manner that another particular variable may be selected as the pivot of initial conditions, with the appropriate modification as to the arguments of the functions in the assigned initial conditions; and the existence-theorems establish no relation between the integrals proved to exist in the respective cases. Moreover, the integrals considered are functions which are regular within the fields of variation; the limitation to uniformity, instead of to regularity so as to exclude essential singularities, is (for the almost complete part) excluded from discussion in the present state of knowledge.

Even within these restrictions, the existence-theorems already proved have a wide range of important applications; some of these applications will now be taken in succession.

Cauchy's Theorem for Equations of the First Order.

16. We begin with the simplest case, being that of a single equation of the first order in one dependent variable and two independent variables; taking the latter to be x and y, and denoting the first derivatives of z with regard to these variables by p and q respectively, we may consider the equation in the form

$$f(x, y, z, p, q) = 0,$$

where f will be taken to be regular in its arguments: and we shall assume that the equation is irreducible. Let $x = a$, $y = b$, $z = c$, $p = \lambda$, $q = \mu$, be a set of values satisfying the equation $f = 0$; then unless the quantity $\frac{\partial f}{\partial p}$ vanishes for these values, the equation can be resolved so as to express p in terms of the remaining arguments in a form

$$p - \lambda = \bar{g}(x - a, y - b, z - c, q - \mu),$$

say

$$p = g(x, y, z, q),$$

where g is a regular analytic function of its arguments. Now, as $\frac{\partial f}{\partial p}$ usually involves the variables that occur (or some of them), it

usually is possible to choose initial values so that $\dfrac{\partial f}{\partial p}$ does not vanish: and then the analytic resolution of the original equation is possible. But it may happen that values of z (if any) and of its derivatives which satisfy the original equation $f=0$, that is, which make f vanish consistently with a relation between z, x, y and with derivatives from that integral relation, also make $\dfrac{\partial f}{\partial p}$ vanish similarly: the suggested resolution of the original equation is then impossible, so that p could not be expressed as a regular analytic function of x, y, z, q.

17. As a first alternative, we assume that the resolution with regard to p is possible in the form

$$p = g(x, y, z, q),$$

where g is regular in the vicinity of $x=a$, $y=b$, $z=c$, $q=\mu$. We can apply the existence-theorems, already established, to prove that *an integral z of the equation exists, having the properties*:—

(i) *it is a regular function of x and y within fields of variation round $x=a$ and $y=b$ given by*

$$|x-a| \leqslant r, \quad |y-b| \leqslant r,$$

where r is not infinitesimal:

(ii) *when $x=a$, the integral z reduces to $\phi(y)$, where $\phi(y)$ is a regular function of y within the field $|y-b| \leqslant r$, acquiring the value c when $y=b$, and otherwise arbitrary;*

(iii) *the integral z, as determined by these conditions, is unique as a regular integral.*

In order to deduce this result from the former theorems, we consider a system

$$\left.\begin{aligned}
&\frac{\partial z}{\partial x} = p \\
&\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} \\
&\frac{\partial p}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}p + \frac{\partial g}{\partial q}\frac{\partial p}{\partial y}
\end{aligned}\right\},$$

regarding it as a system in three dependent variables z, p, q. Applying the second of the existence-theorems (§ 14), we infer

that integrals of this system of equations exist which are regular in the vicinity of $x = a$, $y = b$, and, when $x = a$, are such that

$$z = \phi(y),$$

$$q = \frac{d\phi(y)}{dy},$$

$$p = g\left\{a,\ y,\ \phi(y),\ \frac{d\phi(y)}{dy}\right\},$$

where $\phi(y)$ is the foregoing regular function of y acquiring the value c when $y = b$, and $\dfrac{d\phi(y)}{dy}$ is therefore also regular, acquiring the value μ when $y = b$; moreover, this set of regular integrals is unique. Let the set of integrals of the system, thus known to exist, be denoted by

$$z = Z(x, y), \quad p = P(x, y), \quad q = Q(x, y);$$

we proceed to prove that $z = Z(x, y)$ satisfies the original equation so that, owing to its other properties, it is the announced integral.

As these quantities Z, P, Q satisfy the amplified system of equations, we have

$$\frac{\partial Z}{\partial x} = P$$

from the first of those equations, so that

$$\frac{\partial z}{\partial x} = p.$$

Again, we have

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

from the second of those equations, so that

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}\right),$$

and therefore

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y} - Q\right) = 0.$$

Hence $\dfrac{\partial Z}{\partial y} - Q$ is a function of y only, and its value is the same whatever value be assigned to x. When $x = a$, we have

$$\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y}\phi(y) = \frac{d\phi(y)}{dy}$$

from the assigned value of Z, and

$$Q = \frac{d\phi(y)}{dy}$$

from the assigned value of Q; hence, when $x = a$, the value of $\frac{\partial Z}{\partial y} - Q$ is zero, and therefore

$$\frac{\partial Z}{\partial y} - Q = 0$$

generally, that is,

$$\frac{\partial z}{\partial y} = q.$$

Again, denoting $P - g(x, y, Z, Q)$ by u, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial P}{\partial x} - \frac{\partial g}{\partial x} - \frac{\partial g}{\partial Z}\frac{\partial Z}{\partial x} - \frac{\partial g}{\partial Q}\frac{\partial Q}{\partial x} \\ &= \frac{\partial P}{\partial x} - \frac{\partial g}{\partial x} - \frac{\partial g}{\partial Z}P - \frac{\partial g}{\partial Q}\frac{\partial P}{\partial y} \\ &= 0,\end{aligned}$$

by the third equation of the system. Thus u is independent of x; when $x = a$, its value is

$$g\left\{a,\ y,\ \phi(y),\ \frac{d\phi(y)}{dy}\right\} - g\left\{a,\ y,\ \phi(y),\ \frac{d\phi(y)}{dy}\right\}$$

on inserting the values acquired by Z, Q, P when $x = a$: this is zero, and therefore $u = 0$ generally, that is,

$$P = g(x, y, Z, Q),$$

and therefore

$$p = g(x, y, z, q).$$

We thus have

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad p = g(x, y, z, q),$$

in association with $z = Z(x, y)$, that is, $z = Z(x, y)$ satisfies the equation

$$\frac{\partial z}{\partial x} = g\left(x, y, z, \frac{\partial z}{\partial y}\right),$$

which is the original equation. Owing to its other properties, by which it obeys the assigned conditions, $z = Z(x, y)$ is the integral required.

18. Passing now to the other alternative, under which the equation $f=0$ cannot be resolved with regard to p because the magnitude $\frac{\partial f}{\partial p}$ vanishes for values of the variables that make f vanish, we consider the resolubility of the equation $f=0$ with regard to q. This resolution will be possible unless the magnitude $\frac{\partial f}{\partial q}$ vanishes for values of the variables that make f vanish; when it is possible, the resolved form will be

$$q = h(x, y, z, p),$$

where h is a regular function of its arguments, in the vicinity of values (say) $x=a$, $y=b$, $z=c$. *An integral of the equation exists having the properties:—*

(i) *it is a regular function of x and y within fields of variation round $x=a$ and $y=b$ given by*

$$|x-a| \leqslant r, \quad |y-b| \leqslant r,$$

where r is not infinitesimal:

(ii) *when $y=b$, the integral z reduces to $\psi(x)$, where $\psi(x)$ is a regular function of x within the field $|x-a| \leqslant r$, acquiring a value c when $y=b$, and otherwise arbitrary:*

(iii) *the integral z, as determined by these conditions, is unique as a regular integral.*

The proof of this proposition is similar to the proof of the proposition in the case when the equation $f=0$ was resolved with regard to p; it will not be set out in detail.

19. Combining these results, it follows that an irreducible equation $f=0$ possesses a regular integral with assigned conditions if it is resoluble with regard to p, that it possesses another regular integral with other assigned conditions if it is resoluble with regard to q, and that each of these integrals is unique under its conditions. These integrals have been obtained from equations

$$p = g(x, y, z, q), \quad q = h(x, y, z, p),$$

respectively, which arise from the resolution of $f=0$ in the respective cases: but they do not generally represent the whole of the equation $f=0$, for if f were of degree m in p and n in q, there would generally be m equations of the former type and n of

the latter, distinct from one another in their respective sets. Each such equation determines a unique regular integral under the assigned conditions, which may be made the same for each equation in the set. If for the m equations, the respective regular integrals are

$$z = \theta_1(x, y), \quad z = \theta_2(x, y), \quad \ldots, \quad z = \theta_m(x, y),$$

then clearly the equation

$$\{z - \theta_1(x, y)\}\{z - \theta_2(x, y)\} \ldots \{z - \theta_m(x, y)\} = 0$$

gives the integrals of the equation

$$f(x, y, z, p, q) = 0,$$

supposed of degree m in p and resoluble with regard to p, such that when $x = a$, z assumes the assigned functional value $\phi(y)$. Similarly, if

$$f(x, y, z, p, q) = 0$$

be of degree n in q and be resoluble with regard to q, an equation

$$\{z - \vartheta_1(x, y)\}\{z - \vartheta_2(x, y)\} \ldots \{z - \vartheta_n(x, y)\} = 0$$

gives the integrals of the equation such that, when $y = b$, z assumes the assigned functional value $\psi(x)$.

But it may happen that the equation

$$f(x, y, z, p, q) = 0$$

is not resoluble with regard either to p or to q, that is to say, it may happen that the magnitudes $\frac{\partial f}{\partial p}$ and $\frac{\partial f}{\partial q}$ vanish for values of the variables which make f vanish. The existence-theorem cannot then be applied, and so it provides no information as regards integrals of the equation. We must then investigate independently the character of those integrals (if any) of the equation

$$f(x, y, z, p, q) = 0,$$

which at the same time are such as to satisfy the equations

$$\frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0.$$

This discussion will come later.

20. The initial conditions imposed upon the integrals, in the cases where existence has been established, are associated with particular values of the variable x or of the variable y: a more

general form can be given to the theorems, by a change in the independent variables. Let these be changed from x and y to X and Y, where

$$X = X(x, y), \quad Y = Y(x, y),$$

and denote by P and Q the derivatives of z with regard to X and Y respectively. Then if the transformed equation is resoluble with regard to P, it *possesses an integral* z (which therefore is an integral of the original equation) *characterised by the following properties:*

(i) *it is a regular function of x and y within domains that are not infinitesimal:*

(ii) *when $X(x, y) = a$, the integral acquires a value $\theta(x, y)$, which is a regular function within the domains considered, which is not expressible in terms of X alone, and which otherwise is arbitrary:*

(iii) *the regular integral thus determined is unique* for the branch of the equation given by the resolution with regard to P.

Note 1. When the equation $f(x, y, z, p, q) = 0$ is resoluble with regard both to p and to q, regular integrals are obtained each of which is unique under the initial conditions imposed. Such integrals are, in general, independent of one another; if an integral possessed by the equation resolved with respect to p proved to be the same as the integral possessed by the equation resolved with respect to q, there must be relations between the two sets of initial conditions.

Note 2. In each set of initial conditions, a single function occurs which, within certain very broad limitations, is arbitrary: subject to the associated conditions, this arbitrary function determines a regular integral uniquely. We may therefore expect that, when classes of integrals of partial equations of the first order are being discussed, one class will emerge characterised by the occurrence of a single arbitrary function.

This result will be found to be a special case of a more general result.

Note 3. The equation $f(x, y, z, p, q) = 0$ has been described as irreducible; the property has been tacitly used, though explicit

reference to the irreducibility has not been made after the first statement.

The reason for assuming the property is practically obvious. If $f(x, y, z, p, q)$ can be expressed as the product of independent regular factors, say F and G, and if, in considering integrals of $f=0$, we begin with the integrals of $F=0$, we have

$$\frac{\partial f}{\partial p}=G\frac{\partial F}{\partial p}, \quad \frac{\partial f}{\partial q}=G\frac{\partial F}{\partial q},$$

so that, as G is not zero, the critical quantities for the resolution of the equation are $\frac{\partial F}{\partial p}$ and $\frac{\partial F}{\partial q}$. We thus, in effect, do consider separately the integrals of $F=0$ and $G=0$; and therefore no generality is lost by assuming the equation as irreducible in this case.

21. We shall frequently have recourse to geometrical illustrations, particularly in the case of equations involving one dependent variable and two independent variables. Such illustrations limit the range of variation of the variables to real quantities; they will, however, be found an occasionally convenient method of statement.

Thus consider the equation $f(x, y, z, p, q)=0$: an integral is a relation between x, y, and z, and this can conveniently be interpreted as the equation of a surface. We have seen that, under conditions which do not need restatement for the present purpose, there is an integral such that, when $x=a$, the integral acquires a value $\phi(y)$. But

$$x=a, \quad z=\phi(y),$$

are the equations of a plane curve, as arbitrary as is the function $\phi(y)$. Hence a surface can be drawn that will satisfy the partial equation and will pass through a plane curve which (within certain large limitations) can be taken arbitrarily.

Similarly, as regards the modified result of § 20: the equations

$$X(x, y)=a, \quad z=\theta(x, y),$$

are the equations of a skew curve; and therefore a surface can be drawn that will satisfy the partial equation and will pass through a skew curve which (within certain large limitations) can be taken arbitrarily.

22. A precisely similar application of the general existence-theorems in the last chapter can be made when the differential equation of the first order involves n independent variables: it will therefore be sufficient to state the results.

Denoting the independent variables by $x_1, \ldots, x_n$, and the first derivatives of z by $p_1, \ldots, p_n$ as usual, we take the equation in the form

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0,$$

and we assume it to be irreducible: and we have the following results.

Except for such values of the variables (if any) as make $\dfrac{\partial f}{\partial p_r}$ vanish at the same time as f, the equation can be resolved with regard to p_r; and if $x_1 = a_1, \ldots, x_n = a_n$, $z = c$, $p_1 = \lambda_1, \ldots, p_n = \lambda_n$, be an ordinary set of values for the equation $f = 0$, so that f is regular in their vicinity, then the resolved expression for p_r is regular in the vicinity of those values. Let ϕ_r denote a function of $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$, which is regular in the vicinity of $x_1 = a_1, \ldots, x_n = a_n$, which at $a_1, \ldots, a_n$ acquires the value c_r, and which is otherwise arbitrary. Then an integral of the resolved equation exists, determined by the conditions

(i) it is a regular function of the variables within fields of variation given by

$$|x_1 - a_1| \leqslant \rho, \quad \ldots, \quad |x_n - a_n| \leqslant \rho,$$

where ρ is not infinitesimal;

(ii) when $x_r = a_r$, the integral acquires the value ϕ_r.

Moreover, the integral of the resolved equation, as determined by these conditions, is unique.

If the original equation is of degree μ in p_r, there are μ resolved equations equivalent to $f = 0$ save when $\dfrac{\partial f}{\partial p_r}$ vanishes with f; each such resolved equation determines a unique integral, subject to the imposed conditions; if these be $\zeta_1, \ldots, \zeta_\mu$, then the equation

$$(z - \zeta_1) \ldots (z - \zeta_\mu) = 0$$

can be regarded as providing the integral of $f = 0$, subject to the imposed conditions.

The resolution of the equation $f=0$ is possible with regard to each of the n quantities p in turn, except only when $\frac{\partial f}{\partial p}$ vanishes with f; and each such solution leads, under corresponding imposed conditions similar to those used for the resolution with regard to p_r, to similar integrals of the resolved equations and to a corresponding integral of $f=0$, unique under the imposed conditions.

Hence, by resolving with regard to one or other of the derivatives $p_1, \ldots, p_n$, we establish the existence of integrals of the equation, uniquely determined by imposed conditions which, within certain large limitations, involve an arbitrary functional element.

This establishment of the existence of integrals of the equation $f=0$ is effective except in the single conjunction that all the quantities

$$\frac{\partial f}{\partial p_1}, \quad \frac{\partial f}{\partial p_2}, \quad \ldots, \quad \frac{\partial f}{\partial p_n},$$

vanish for values of the variables which make $f=0$: in that conjunction, if it can occur, the existence-theorems cannot be applied. There will therefore remain, as a subject for separate consideration, the discussion of the integrals (if any) of the equation

$$f=0,$$

which simultaneously satisfy the equations

$$\frac{\partial f}{\partial p_1}=0, \quad \ldots, \quad \frac{\partial f}{\partial p_n}=0.$$

As before, we can deduce the existence of integrals which are such that, when some relation

$$\mu(x_1, \ldots, x_n)=0$$

is satisfied, z acquires a value $\phi(x_1, \ldots, x_n)$, where μ and ϕ are regular functions, and ϕ is not expressible in terms of μ alone: the general condition, necessary for the existence of the integral, is that the quantity

$$\frac{\partial f}{\partial p_1}\frac{\partial \mu}{\partial x_1}+\frac{\partial f}{\partial p_2}\frac{\partial \mu}{\partial x_2}+\ldots+\frac{\partial f}{\partial p_n}\frac{\partial \mu}{\partial x_n}$$

shall not vanish in virtue of $f=0$.

CAUCHY'S THEOREM FOR EQUATIONS OF THE SECOND ORDER.

23. The equation, next in simplicity, to which the existence-theorems can be applied, is an irreducible equation of the second order in one dependent variable and two independent variables. Denoting the second derivatives of z with regard to x and y by r, s, t, as usual, we may take the equation in the form

$$f(x,\ y,\ z,\ p,\ q,\ r,\ s,\ t) = 0,$$

where f will be assumed to be a regular function of its arguments. Let a, b, c, λ, μ, α, β, γ be a set of values of the arguments of f in the vicinity of which f is regular; then, unless $\dfrac{\partial f}{\partial r}$ vanishes for those values, the equation can be resolved so as to express r in terms of the remaining quantities by an equation

$$r - \alpha = \bar{g}(x - a,\ y - b,\ z - c,\ p - \lambda,\ q - \mu,\ s - \beta,\ t - \gamma),$$

say

$$r = g(x,\ y,\ z,\ p,\ q,\ s,\ t),$$

where g is a regular analytic function. Now, as $\dfrac{\partial f}{\partial r}$ usually involves at least some of the variables, it usually is possible to choose initial values so that $\dfrac{\partial f}{\partial r}$ does not vanish; and then the resolution of the original equation can be effected. But it might happen that values of z (if any) and of its derivatives, which make f vanish, also make $\dfrac{\partial f}{\partial r}$ vanish: the resolution of the original equation with regard to r could not be effected, and we should have to proceed otherwise.

When the resolution is possible, the general theorems can be applied to establish the existence of *an integral z having the properties:*

(i) *it is a regular function of x and y within fields of variation round a and b, given by*

$$|\,x - a\,| \leqslant \rho, \quad |\,y - b\,| \leqslant \rho,$$

where ρ is not infinitesimal;

(ii) *when $x=a$, then z reduces to $\phi_0(y)$ and $\dfrac{\partial z}{\partial x}$ reduces to $\phi_1(y)$, where $\phi_0(y)$ and $\phi_1(y)$ are regular functions of y within the domain $|y-b| \leqslant \rho$, acquiring the values c and λ respectively when $y=b$, and are otherwise arbitrary;*

(iii) *the integral z as determined by these conditions is unique.*

The mode of establishment is similar to that in the case of the equation of the first order, and so the exposition will be brief. We consider a system of equations

$$\frac{\partial z}{\partial x}=p\ ,$$

$$\frac{\partial p}{\partial x}=r\ ,$$

$$\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y},$$

$$\frac{\partial s}{\partial x}=\frac{\partial r}{\partial y},$$

$$\frac{\partial t}{\partial x}=\frac{\partial s}{\partial y},$$

$$\frac{\partial r}{\partial x}=\frac{\partial g}{\partial x}+\frac{\partial g}{\partial z}p+\frac{\partial g}{\partial p}r+\frac{\partial g}{\partial q}\frac{\partial p}{\partial y}+\frac{\partial g}{\partial s}\frac{\partial r}{\partial y}+\frac{\partial g}{\partial t}\frac{\partial s}{\partial y},$$

of the same character as in the general existence-theorems; and we regard the system as involving six dependent variables z, p, q, r, s, t. When the former results are applied, we infer the existence of integrals of this system of equations, characterised by the properties:

(i) they are regular functions of x and y within the fields of variation

$$|x-a| \leqslant \rho, \quad |y-b| \leqslant \rho,$$

(ii) when $x=a$, then

$$z=\phi_0(y),$$

$$p=\phi_1(y),$$

$$q=\frac{d\phi_0(y)}{dy},$$

$$s=\frac{d\phi_1(y)}{dy},$$

$$t = \frac{d^2\phi_0(y)}{dy^2},$$

$$r = g\left\{a,\ y,\ \phi_0(y),\ \phi_1(y),\ \frac{d\phi_0(y)}{dy},\ \frac{d\phi_1(y)}{dy},\ \frac{d^2\phi_0(y)}{dy^2}\right\},$$

where $\phi_0(y)$ and $\phi_1(y)$ are the foregoing regular functions:

(iii) the set of integrals determined by these conditions is unique.

Let the set of integrals thus determined be

$$z = Z(x,\ y),\quad p = P(x,\ y),\quad q = Q(x,\ y),$$
$$r = R(x,\ y),\quad s = S(x,\ y),\quad t = T(x,\ y);$$

then $z = Z(x, y)$ *is the announced integral of the original resolved equation.*

The proof is simple, on the same lines as before. From the first of the equations, we have

$$\frac{\partial Z}{\partial x} = P,$$

and therefore

$$\frac{\partial z}{\partial x} = p.$$

Similarly

$$r = \frac{\partial p}{\partial x} = \frac{\partial^2 z}{\partial x^2}.$$

Again, the third equation gives

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$
$$= \frac{\partial^2 Z}{\partial x \partial y},$$

so that

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y} - Q\right) = 0.$$

Thus $\frac{\partial Z}{\partial y} - Q$ is independent of x: inserting the values of Z and Q when $x = a$, we find the value to be zero, so that

$$\frac{\partial Z}{\partial y} = Q,$$

and therefore

$$\frac{\partial z}{\partial y} = q.$$

The fourth equation gives

$$\frac{\partial S}{\partial x} = \frac{\partial R}{\partial y}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right),$$

and therefore $S - \dfrac{\partial P}{\partial y}$ is independent of x: its value is zero when $x = a$, and so

$$S = \frac{\partial P}{\partial y},$$

that is,

$$s = \frac{\partial p}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x \partial y}.$$

Similarly from the fifth equation, we have $T - \dfrac{\partial^2 Z}{\partial y^2}$ independent of x: its value is zero when $x = a$, so that

$$T = \frac{\partial^2 Z}{\partial y^2},$$

that is,

$$t = \frac{\partial^2 z}{\partial y^2}.$$

Lastly, writing

$$v = R - g(x, y, Z, P, Q, S, T),$$

the sixth equation shews that v is independent of x: its value is zero when $x = a$, and so $v = 0$ generally. Thus

$$R - g(x, y, Z, P, Q, S, T) = 0,$$

that is,

$$r - g(x, y, z, p, q, s, t) = 0,$$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$, in association with $z = Z(x, y)$.

Owing to the other properties, by which it obeys the assigned conditions, $z = Z(x, y)$ is the integral of the original differential equation, having the prescribed character.

Thus the existence of an integral of the equation

$$f(x, y, z, p, q, r, s, t) = 0$$

is established, save in the case when $\dfrac{\partial f}{\partial r}$ vanishes for values (if any) of the variables simultaneously with f. When $\dfrac{\partial f}{\partial r}$ does not thus vanish, the original equation is analytically resoluble: there is one such integral, subject to the imposed conditions, for each resolved branch of the equation: and if ζ_1, ζ_2, ... be these integrals, then

$$(z-\zeta_1)(z-\zeta_2)\ldots=0$$

provides an integral of the original equation.

Similarly, if the equation is resolved with regard to t—and this will be possible except for such values (if any) of the variables as make $\dfrac{\partial f}{\partial t}$ vanish simultaneously with f—and if the resolved form is

$$t=h(x, y, z, p, q, r, s),$$

where h is a regular function of its arguments, then *an integral z* exists, characterised by the properties:

(i) *it is a regular function of x and y within fields of variation round a and b, given by*

$$|x-a| \leqslant \rho, \quad |y-b| \leqslant \rho,$$

where ρ is not infinitesimal;

(ii) *when $y=b$, then z reduces to $\psi_0(x)$ and $\dfrac{\partial z}{\partial y}$ reduces to $\psi_1(x)$, where $\psi_0(x)$ and $\psi_1(x)$ are regular functions of x within the domain $|x-a| \leqslant \rho$, acquiring the values c and μ when $x=a$, and are otherwise arbitrary;*

(iii) *the integral z, determined by these conditions, is unique.*

The proof is similar to that of the earlier proposition and so need not be expounded.

24. It may happen that there are values of the variables for which the equation is not resoluble with regard either to r or to t, so that $\dfrac{\partial f}{\partial r}$, $\dfrac{\partial f}{\partial t}$, f vanish simultaneously for such values: yet for these values the equation $f=0$ may be resoluble with regard to s. In that case, we change the independent variables from x and y to

x' and y'; denoting the derivatives with regard to the new variables by p', q', r', s', t', we have

$$\left.\begin{aligned} r &= r'\left(\frac{\partial x'}{\partial x}\right)^2 + 2s'\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} + t'\left(\frac{\partial y'}{\partial x}\right)^2 + p'\frac{\partial^2 x'}{\partial x^2} + q'\frac{\partial^2 y'}{\partial x^2} \\ s &= r'\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \ldots\ldots \\ t &= r'\left(\frac{\partial x'}{\partial y}\right)^2 + \ldots\ldots \end{aligned}\right\}.$$

When these are substituted, the new equation will be resoluble with regard to r' except for values (if any) of the variables which make $\frac{\partial f}{\partial r'}$ vanish, that is, which make

$$\frac{\partial f}{\partial r}\left(\frac{\partial x'}{\partial x}\right)^2 + \frac{\partial f}{\partial s}\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial t}\left(\frac{\partial x'}{\partial y}\right)^2$$

vanish. In the present case, we can choose x' so that it shall involve x and y; and therefore, even though $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial t}$ vanish, the foregoing quantity will not vanish unless $\frac{\partial f}{\partial s}$ vanishes. When $\frac{\partial f}{\partial s}$ does not vanish, the equation can be resolved with respect to r': and an integral of the equation exists, uniquely determined by conditions similar to those in former cases.

This transformation, moreover, shews that the initial conditions can be modified in all the preceding cases: they can be associated with an initial value $x' = a'$, of course with the appropriate modifications, that is, they can be associated with an initial relation

$$\theta(x, y) = a',$$

where θ is a regular function.

The existence of an integral of the equation

$$f(x, y, z, p, q, r, s, t) = 0$$

is thus established except for such values (if any) of the variables as make the equations

$$\frac{\partial f}{\partial r} = 0, \quad \frac{\partial f}{\partial s} = 0, \quad \frac{\partial f}{\partial t} = 0$$

satisfied simultaneously with $f = 0$. If this be possible, the existence-theorem does not apply: and there must be an independent

discussion of these integrals, if any. This discussion will be deferred.

In the case of equations of the first order, say in two independent variables only, we are accustomed to the existence of integrals such that f, $\frac{\partial f}{\partial p}$, $\frac{\partial f}{\partial q}$ vanish simultaneously. Without anticipating the discussion of the corresponding question for equations of the second order, it is to be remarked that the four equations

$$f=0, \quad \frac{\partial f}{\partial r}=0, \quad \frac{\partial f}{\partial s}=0, \quad \frac{\partial f}{\partial t}=0,$$

may coexist without rendering the elimination of r, s, t possible. An example is furnished by the equation

$$(rx^2+2sxy+ty^2-2px-2qy+2z)^2-a^4(r^2+2s^2+t^2)$$
$$=\frac{4a^4}{a^4-(x^2+y^2)^2}(px+qy-z)^2.$$

Cauchy's Theorem in general.

25. We now proceed to apply the existence-theorems, in order to establish the existence of integrals of the system of equations

$$\frac{\partial^{r_1} z_1}{\partial x_1^{r_1}}=Z_1, \quad \frac{\partial^{r_2} z_2}{\partial x_1^{r_2}}=Z_2, \quad \ldots, \quad \frac{\partial^{r_m} z_m}{\partial x_1^{r_m}}=Z_m,$$

where the quantities $Z_1, Z_2, \ldots, Z_m$ are regular functions of the independent variables $x_1, \ldots, x_n$, of the dependent variables $z_1, \ldots, z_m$, and of the derivatives of the latter of all orders up to (and including) $r_1, \ldots, r_m$ respectively, except only those derivatives which appear on the left-hand sides of the equations.

Let $a_1, \ldots, a_n$ be values of $x_1, \ldots, x_n$ within the field of regular existence of the quantities $Z_1, \ldots, Z_m$; and let a number of functions of $x_2, \ldots, x_n$ be chosen, which are regular in the vicinity of $a_2, \ldots, a_n$, and (subject to certain limitations upon their coefficients about to be stated) which are otherwise arbitrary. These functions will be denoted by $\phi_{\lambda\mu}$, for

$$\lambda=1, \quad \mu=0, 1, \ldots, r_1-1;$$
$$\lambda=2, \quad \mu=0, 1, \ldots, r_2-1;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lambda=m, \quad \mu=0, 1, \ldots, r_m-1.$$

Then *a system of integrals* $z_1, \ldots, z_m$ *of the equations exists, characterised by the properties:*

(i) *they are regular functions of* $x_1, \ldots, x_n$ *in fields of variation given by*

$$|x_1 - a_1| \leqslant \rho, \quad |x_2 - a_2| \leqslant \rho, \quad \ldots, \quad |x_n - a_n| \leqslant \rho,$$

where ρ *is not infinitesimal;*

(ii) *when* $x_1 = a_1$, *the values acquired by the integrals and by their derivatives are given by the relations*

$$\frac{\partial^{\mu} z_{\lambda}}{\partial x_1^{\mu}} = \phi_{\lambda\mu},$$

for the various values of λ *and* μ, *it being assumed that the values of the functions* $\phi_{\lambda\mu}$, *when* $x_2 = a_2, \ldots, x_n = a_n$, *are values of the derivatives of* $z_1, \ldots, z_m$ *within the field of regular existence of the functions* $Z_1, \ldots, Z_m$;

(iii) *the system of integrals, thus determined, is unique.*

In order to establish this result, we merely generalise the method applied in the preceding special cases of the theorem. We introduce a number of auxiliary variables

$$\frac{\partial^{r+s+t+\cdots} z_{\lambda}}{\partial x_1^{\,r} \partial x_2^{\,s} \partial x_3^{\,t} \cdots} = p_{\lambda rst\ldots},$$

assigning as initial conditions that, when $x_1 = a_1$, the values they shall assume are given by the relations

$$p_{\lambda r_{\lambda} 00\ldots} = [Z_{\lambda}],$$

$$p_{\lambda r 00\ldots} = \phi_{\lambda r},$$

when $r < r_{\lambda}$, and $[Z_{\lambda}]$ is the value of Z_{λ} when $x_1 = a_1$; and we construct the system of equations

$$\frac{\partial p_{\lambda r_{\lambda} 00\ldots}}{\partial x_1} = \frac{\partial Z_{\lambda}}{\partial x_1} + \sum_{s=1}^{m} \frac{\partial Z_{\lambda}}{\partial z_s} p_{s10\ldots} + \cdots,$$

$$\frac{\partial p_{\lambda r 00\ldots}}{\partial x_1} = p_{\lambda,\, r+1,\, 0,\, 0,\, \ldots},$$

$$\frac{\partial p_{\lambda rst\ldots}}{\partial x_1} = \frac{\partial p_{\lambda,\, r+1,\, s-1,\, t\ldots}}{\partial x_2},$$

these holding for $r < r_{\lambda}$, $r + s + t + \ldots < r_{\lambda}$, $s > 0$, and for all values of λ; the right-hand side of the first equation is the complete

derivative of Z_λ with respect to x_1, the derivatives of the included arguments being modified by the rest of the equations in this amplified system.

The substituted system of equations conforms to the type specified in the general existence-theorems, which accordingly apply. The system possesses a set of integrals having the properties:

(i) they are regular functions of the variables $x_1, \ldots, x_n$ within the specified fields of variation:

(ii) when $x_1 = a_1$, the various dependent variables acquire the respective assigned values:

(iii) the integrals, thus determined, are unique.

Let the values of z, which occur in this set of integrals, be

$$z_r = \psi_r(x_1, \ldots, x_n),$$

for $r = 1, \ldots, m$: *these values constitute the announced set of integrals of the original system of equations.*

The method of proceeding is the same as for the simple cases. Thus we have

$$\frac{\partial z_\lambda}{\partial x_1} = p_{\lambda 100\ldots},$$

direct from equations of the substituted system. Again, we have

$$\frac{\partial p_{\lambda 0100\ldots}}{\partial x_1} = \frac{\partial p_{\lambda 100\ldots}}{\partial x_2}$$

$$= \frac{\partial^2 z_\lambda}{\partial x_1 \partial x_2}.$$

Thus $p_{\lambda 0100\ldots} - \dfrac{\partial z_\lambda}{\partial x_2}$ is independent of x_1; its value is zero when $x_1 = a_1$, and therefore is zero generally, that is,

$$\frac{\partial z_\lambda}{\partial x_2} = p_{\lambda 0100\ldots}.$$

And so on, step by step: the last step gives

$$\frac{\partial}{\partial x_1}(p_{\lambda r_\lambda 00\ldots} - Z_\lambda) = 0,$$

so that $p_{\lambda r_\lambda 00\ldots} - Z_\lambda$ is independent of x_1: its value is zero when $x_1 = a_1$, owing to the assigned conditions; and therefore the value is zero generally, that is,

$$p_{\lambda r_\lambda 00\ldots} = Z_\lambda,$$

and therefore we have

$$\frac{\partial^{r_\lambda} z_\lambda}{\partial x_1^{r_\lambda}} = Z_\lambda,$$

equations which, for all the values of λ, constitute the original system. These are equations satisfied by

$$z_\lambda = \psi_\lambda (x_1, \ldots, x_n), \qquad (\lambda = 1, \ldots, m),$$

which accordingly are the integrals of that original system. The properties, which they possess as integrals of the substituted system, both as regards regular character, values acquired when $x_1 = a_1$, and uniqueness, shew that they obey the conditions imposed in connection with the original system.

A simple illustration is provided by the differential equation of a vibrating plane membrane, which is

$$h^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) = \frac{\partial^2 z}{\partial t^2},$$

where h^2 is a constant: an integral of this equation is uniquely determined by the condition of being a regular function of x, y, t, and by the conditions that, when $t=0$,

$$z=f(x, y), \quad \frac{\partial z}{\partial t}=g(x, y).$$

By the nature of the case, the boundary of the membrane is fixed; hence, along the boundary, z and $\frac{\partial z}{\partial t}$ are always zero, so that the regular functions $f(x, y)$ and $g(x, y)$ have their otherwise arbitrary character restricted by this general condition attached to the particular problem. But it follows from the general theorem that, if an integral can be obtained, in any manner, satisfying the imposed conditions, it is the unique integral, subject to those conditions.

For example, let the membrane be rectangular in form, having its sides equal to a and b: let the equations of the sides be $y=0$, $y=b$, $x=0$, $x=a$, so that f and g must vanish for any one of these four relations. Now an integral of the equation is given by

$$z=(\alpha \cos ct+\beta \sin ct) \sin \lambda x \sin \mu y,$$

provided

$$h^2 (\lambda^2+\mu^2)=c^2:$$

and this integral will vanish on the rectangular boundary if

$$\sin \lambda a=0, \qquad \sin \mu b=0.$$

The latter will be satisfied by taking

$$\lambda=\frac{l\pi}{a}, \qquad \mu=\frac{m\pi}{b},$$

where l and m are integers; then

$$c^2 = h^2\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right),$$

and the integral is

$$z = (\alpha \cos ct + \beta \sin ct) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b}.$$

Clearly, the sum of any number of such integrals is also an integral: so that we have an integral given by

$$Z = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (\alpha_{lm} \cos c_{lm}t + \beta_{lm} \sin c_{lm}t) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b}.$$

This quantity Z vanishes on the boundary: if, then, the coefficients α_{lm}, β_{lm} can be determined so as to satisfy the imposed conditions, we shall have the required integral. Now, when $t=0$,

$$Z = \sum_{l=1} \sum_{m=1} \alpha_{lm} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b},$$

$$\frac{\partial Z}{\partial t} = \sum_{l=1} \sum_{m=1} c_{lm}\beta_{lm} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b};$$

and these should be equal to $f(x, y)$, $g(x, y)$, which accordingly impose limitations upon the character of the regular functions. The conditions will be satisfied if

$$\alpha_{lm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b}\, dx\, dy,$$

$$\beta_{lm} = \frac{4}{abc_{lm}} \int_0^a \int_0^b g(x, y) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b}\, dx\, dy.$$

The required integral is uniquely given by the expression

$$z = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (\alpha_{lm} \cos c_{lm}t + \beta_{lm} \sin c_{lm}t) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b},$$

with the foregoing values for the coefficients α and β.

Note. It will be noticed that the existence-theorem provides for the introduction of a number of functions which, within certain very wide limitations, are arbitrary functions of all the variables but one, or are arbitrary functions of all the variables subject to an assigned relation among the variables. In the case of the system of equations considered in this section, the number of these functions is

$$r_1 + r_2 + \ldots + r_m,$$

being the sum of the orders of the highest derivatives that occur.

In particular, if there be only a single dependent variable and a single equation of order r, *the number of arbitrary functions* provided by the theorem for the precise determination of the

integral *is* r, *the same as the order of the equation.* Special illustrations have been furnished by an equation of the first order and by an equation of the second order.

26. In the establishment of the theorem as to the existence of integrals of the equations

$$\frac{\partial^{r_1} z_1}{\partial x_1^{r_1}} = Z_1, \ \ldots, \ \frac{\partial^{r_m} z_m}{\partial x_1^{r_m}} = Z_m,$$

it was assumed that no derivatives of z_1 of order higher than r_1 occur, and similarly for the derivatives of the other dependent variables. This limitation is important: it is actually necessary in order that the convergence of the series (and therefore the functional significance of the integrals) may be established.

The importance of the condition may be illustrated by a single example *. Consider the equation

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial x},$$

which belongs to the system when associated with imposed conditions to be satisfied for an assigned value of y; but the limitation is not obeyed when the imposed conditions are to be satisfied for an assigned value of x. To see the effect of the limitation, let it be required to obtain an integral of the equation which shall acquire a value $P(y)$ when $x=0$, $P(y)$ being an analytic function of y, regular in the vicinity of $y=0$. A formal solution is manifestly given by

$$z = \sum_{n=0} \frac{x^n}{n!} \frac{d^{2n} P}{dy^{2n}}.$$

The convergence of the series cannot be established: indeed, the series in general is not a converging series. To make the series more precise, let

$$P(y) = \frac{1}{1-y},$$

which satisfies all the conditions: then

$$\begin{aligned} z &= \sum_{n=0} \frac{x^n}{(1-y)^{2n+1}} \frac{2n!}{n!} \\ &= \frac{1}{1-y} \sum_{n=0} \frac{2n!}{n!\,(1-y)^{2n}} x^n. \end{aligned}$$

Now it is known † that ρ, the radius of convergence of a converging series $\Sigma a_m x^m$, is given by

$$\frac{1}{\rho} = \operatorname*{Lim}_{m=\infty} | a_m^{\frac{1}{m}} |;$$

* Kowalevsky, *Crelle*, t. LXXX (1875), p. 22.

† *T. F.*, § 26.

hence, if ρ be the radius of convergence of the series for z regarded as a power-series in x, we have

$$\frac{1}{\rho}=\frac{1}{|1-y|^2}\operatorname*{Lim}_{m=\infty}\left(\frac{2m\,!}{m\,!}\right)^{\frac{1}{m}}$$

$$=\frac{1}{|1-y|^2}\operatorname*{Lim}_{m=\infty}\frac{4m}{e}2^{\frac{1}{2m}},$$

approximately, by the use of Stirling's theorem; thus ρ is zero, whatever finite value be possessed by y. In other words, there is no region of convergence for z in the case of the assumed form of $P(y)$.

The question thus suggests itself: what are the limitations upon $P(y)$ that the series for z should converge? To answer it, we take the equation as an instance of the equations in § 23: the theorem shews that a regular integral exists determined uniquely by the conditions that, when $y=0$,

$$z=Q(x)=\sum_{n=0} c_n x^n,$$

$$\frac{\partial z}{\partial y}=R(x)=\sum_{n=0} k_n x^n,$$

where $Q(x)$ and $R(x)$ are regular functions. The formal expression of this integral is easily found to be

$$z=\sum_{m=0}\frac{y^{2m}}{2m\,!}\frac{d^mQ(x)}{dx^m}+\sum_{m=0}\frac{y^{2m+1}}{(2m+1)\,!}\frac{d^mR(x)}{dx^m}.$$

Hence, when $x=0$, the value of z is given by

$$\sum_{m=0}\frac{m\,!}{2m\,!}c_m y^{2m}+\sum_{m=0}\frac{m\,!}{(2m+1)\,!}k_m y^{2m+1};$$

if the integral is to be given by the former process, this must be the value of $P(y)$ in the assigned initial conditions.

Let r denote the radius of convergence of the power-series $Q(x)$ and $R(x)$ simultaneously: then, because $\sum_{n=0} c_n x^n$ and $\sum_{n=0} k_n x^n$ are converging series when $|x|<r$, a finite quantity G exists such that

$$|c_n|<\frac{G}{r^n},\qquad |k_n|<\frac{G}{r^n},$$

so that we may take

$$c_n=\frac{Gu}{r^n},\qquad k_n=\frac{Gv}{r^n},$$

where $|u|<1$, $|v|<1$, while u and v are not zero. If ρ denote the radius of convergence of the series of powers of y, then

$$\frac{1}{\rho}=\operatorname*{Lim}_{m=\infty}\left|\frac{m\,!}{2m\,!}\frac{Gu}{r^m}\right|^{\frac{1}{m}}$$

$$=0,$$

or the power-series must converge over the whole plane. Consequently, the only functions admissible as values of $P(y)$ in the earlier investigation are

those which are regular over the whole plane and, when expressed as power-series, converge over the whole plane in a manner comparable with the series

$$\sum_{m=0} \frac{m\,!}{2m\,!} y^{2m} + \sum_{m=0} \frac{m\,!}{(2m+1)\,!} y^{2m+1}.$$

Thus possible values of $P(y)$ are given by

$$P(y) = e^y,$$
$$P(y) = J_0(y),$$
$$P(y) = J_n(y),$$

when (in the last example) the real part of n is positive.

27. The equations, in § 25, though not of a completely general character, constitute a very extensive class; and they are even more extensive than their explicit form indicates, because of the possibilities of transformation.

Suppose that, in a given system of m equations, the order of the highest derivative of z_λ is r_λ, for $\lambda = 1, \ldots, m$; then, by transformation of the independent variables, it is usually possible to secure the explicit occurrence of the derivatives

$$\frac{\partial^{r_1} z_1}{\partial x_1^{r_1}}, \ldots, \frac{\partial^{r_m} z_m}{\partial x_1^{r_m}},$$

that is, of the highest derivatives with regard to one and the same variable. If all these occur, no change is needed; if any are absent, we change the variables by relations

$$x_s' = a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n,$$

for $s = 1, \ldots, n$, the constant coefficients a being at our disposal provided their determinant is kept different from zero. Suppose that the required derivative of z_1 has not occurred in the original equations, but that there is a derivative $\dfrac{\partial^{r_1} z_1}{\partial x_1^s \partial x_2^t \partial x_3^u \ldots}$, where $s + t + u + \ldots = r_1$; then, after the transformation, the derivative $\dfrac{\partial^{r_1} z_1}{\partial x_1'^{r_1}}$ will occur unless $a_{11}^s a_{12}^t a_{13}^u \ldots$ vanishes. We can always secure that this negative provision is satisfied; hence the m equations can be transformed so that the required derivatives occur explicitly.

But this result is not sufficient to secure the form of the equations adopted for the existence-theorem; it must further be possible to resolve the m equations with respect to the m selected

derivatives. When the resolution is possible, the resolved equations are of the form

$$\frac{\partial^{r_1} z_1}{\partial x_1^{r_1}} = Z_1, \ \ldots, \ \frac{\partial^{r_m} z_m}{\partial x_1^{r_m}} = Z_m.$$

When the resolution with regard to the selected derivatives is not possible, and is equally not possible with regard to every set of similarly selected derivatives, the equations do not belong to the class considered.

As an instance shewing that the form cannot be regarded as one to which all equations of the type considered are reducible, take the equations

$$P_1 \frac{\partial z_1}{\partial x_1} + P_2 \frac{\partial z_2}{\partial x_1} + P_3 \frac{\partial z_3}{\partial x_1} = Z_1,$$

$$Q_1 \frac{\partial z_1}{\partial x_2} + Q_2 \frac{\partial z_2}{\partial x_2} + Q_3 \frac{\partial z_3}{\partial x_2} = Z_2,$$

$$R_1 \frac{\partial z_1}{\partial x_3} + R_2 \frac{\partial z_2}{\partial x_3} + R_3 \frac{\partial z_3}{\partial x_3} = Z_3;$$

when the transformation

$$x_s' = a_{s1} x_1 + a_{s2} x_2 + a_{s3} x_3$$

(for $s = 1, 2, 3$) is effected, they have the form

$$a_{11}\left(P_1 \frac{\partial z_1}{\partial x_1'} + P_2 \frac{\partial z_2}{\partial x_1'} + P_3 \frac{\partial z_3}{\partial x_1'}\right) + \ldots = Z_1,$$

$$a_{12}\left(Q_1 \frac{\partial z_1}{\partial x_1'} + Q_2 \frac{\partial z_2}{\partial x_1'} + Q_3 \frac{\partial z_3}{\partial x_1'}\right) + \ldots = Z_2,$$

$$a_{13}\left(R_1 \frac{\partial z_1}{\partial x_1'} + R_2 \frac{\partial z_2}{\partial x_1'} + R_3 \frac{\partial z_3}{\partial x_1'}\right) + \ldots = Z_3.$$

The equations can be resolved for $\frac{\partial z_1}{\partial x_1'}, \frac{\partial z_2}{\partial x_1'}, \frac{\partial z_3}{\partial x_1'}$ (and therefore would be reducible to the selected general form) if

$$a_{11} a_{12} a_{13} \begin{vmatrix} P_1, & P_2, & P_3 \\ Q_1, & Q_2, & Q_3 \\ R_1, & R_2, & R_3 \end{vmatrix}$$

is not zero. But it might very well happen that the determinant of the coefficients P, Q, R should vanish identically; the resolution would then be impossible. In that case, it is equally impossible

to resolve the equations with regard to $\frac{\partial z_1}{\partial x_2'}$, $\frac{\partial z_2}{\partial x_2'}$, $\frac{\partial z_3}{\partial x_2'}$, and also with regard to the remaining three derivatives, and so the existence-theorem cannot be applied; but then it is also necessary that the relations

$$\left\| \begin{matrix} P_1, & P_2, & P_3, & Z_1 \\ Q_1, & Q_2, & Q_3, & Z_2 \\ R_1, & R_2, & R_3, & Z_3 \end{matrix} \right\| = 0$$

be satisfied.

Hence the form of equations retained in § 25 is not a completely inclusive normal form; but, as already stated, it includes a very extensive class of equations*.

Ex. Consider the equations

$$\left.\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= \frac{\partial X}{\partial x} \\ a'\frac{\partial u}{\partial y} + b'\frac{\partial v}{\partial y} &= \frac{\partial Y}{\partial y} \end{aligned}\right\},$$

where a's and b's are constants, X is a function of x alone, Y is a function of y alone.

Effecting the transformations, we easily find that the system can be changed so as to have the normal form selected, provided $ab' - a'b$ is not zero. Assuming this proviso satisfied, the existence-theorem applies and the integrals certainly exist: they are most easily obtainable by quadrature from the original equations in the form

$$\left.\begin{aligned} au + bv &= X + f(y) \\ a'u + b'v &= Y + g(x) \end{aligned}\right\},$$

where f and g are arbitrary functions. To determine f and g in connection with assigned initial conditions, we take the existence-theorem for the transformed equations: it would assign values $\phi(\gamma x + \delta y)$ and $\psi(\gamma x + \delta y)$ to u and v respectively when $ax + \beta y$ is constant, say λ, where $a\delta - \beta\gamma$ is not zero. Thus

$$a\phi\left(\frac{\gamma\lambda}{a} + \frac{a\delta - \beta\gamma}{a}y\right) + b\psi\left(\frac{\gamma\lambda}{a} + \frac{a\delta - \beta\gamma}{a}y\right) = X\left(\frac{\lambda - \beta y}{a}\right) + f(y),$$

$$a'\phi\left(\frac{\delta\lambda}{\beta} - \frac{a\delta - \beta\gamma}{\beta}x\right) + b'\psi\left(\frac{\delta\lambda}{\beta} - \frac{a\delta - \beta\gamma}{\beta}x\right) = Y\left(\frac{\lambda - ax}{\beta}\right) + g(x),$$

which determine f and g.

* For a fuller discussion of this matter, see Bourlet, *Ann. de l'Éc. Norm. Sup.*, 3me Sér., t. VIII (1891), supplément. The example that follows is taken from this memoir.

But if $ab'-a'b=0$, the resolution is not possible: and the existence-theorem does not apply. The quadrature is still possible; and we find

$$au+bv=X+f(y),$$
$$a'u+b'v=Y+g(x),$$

where $f(y)$ is arbitrary so far as the first equation is concerned, and $g(x)$ is arbitrary so far as the second equation is concerned. Assuming for purposes of illustration that no one of the constants a, b, a', b' vanishes, we have

$$b'\{X+f(y)\}=b\{Y+g(x)\};$$

hence, as there is no relation between the variables x and y, we must have

$$g(x)=\frac{b'}{b}X-b'c,$$

$$f(y)=\frac{b}{b'}Y-bc,$$

where c is a constant. The two integral equations are now equivalent to one only; hence they do not precisely determine the two quantities u and v. One of these can be taken at will, say

$$v=\theta(x, y);$$

and then

$$u=-\frac{b}{a}\theta(x, y)+\frac{X}{a}+\frac{Y}{a'}-\frac{bc}{a},$$

which accordingly are integral equations in the case when $ab'-a'b=0$.

Other classes of Equations.

28. The preceding forms of equations are thus not universally inclusive; and, in recent years, investigations have been made on general differential systems, so as to establish the existence of integrals under assigned conditions associated with wider classes of equations. These investigations are mainly due to Méray, Riquier, Bourlet, Tresse, and Delassus*: their formal complication is elaborate. There are two main issues in this development of the theory; one is the construction of canonical forms, the other is the establishment of the existence of integrals of the systems of equations, the expression of which involves arbitrary constants or arbitrary functions. And we have seen, by a particular example,

* Many references will be found in von Weber's article on partial differential equations in the *Encyclopädie der mathematischen Wissenschaften*, vol. II, pp. 299, 300. In addition to these, four memoirs by Riquier may be mentioned; they are to be found in the *Acta Math.*, t. XXIII (1900), pp. 203—332, *ib.*, t. XXV (1902), pp. 297—358, *Ann. de l'Éc. Norm. Sup.*, 3me Sér., t. XVIII (1901), pp. 421—472, *ib.*, t. XX (1903), pp. 27—73.

(§ 3), that cases may occur in which integrals certainly exist and cannot contain any arbitrary element whatever.

For such investigations, we refer to the memoirs of the authors quoted; and we shall therefore enter into no further detail as to the existence of integrals of systems of equations in number equal to the number of dependent variables. There still remain, for our consideration, the discussion of the integrals (if any) of an equation or a system of equations in the vicinity of values of the variables where the functions concerned are not regular, and the discussion of the integrals (if any) of systems of equations in which the number of dependent variables is less than the number of equations. To the former, very little space will be devoted as the subject is hardly begun: it certainly seems to have claimed no attention from investigators. The latter is of the utmost importance, particularly in the case when there is only one dependent variable; it will be undertaken in a succeeding chapter.

CHAPTER III.

LINEAR EQUATIONS AND COMPLETE LINEAR SYSTEMS.

FOR the materials of this chapter, reference may be made to the authorities quoted in Part I, ch. II, of this work, in particular, to Mayer's memoir, *Math. Ann.* t. V (1872), pp. 448—470, and also to chapter II of Goursat's *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre.* The chapter is devoted to linear equations, either single or in simultaneous systems.

Single equations and systems of simultaneous equations, which are homogeneous and linear in the differential elements of the variables, have already been discussed. The discussion of exact equations and exact systems of this type is given in the first two chapters of volume I of this work: the remainder of that volume is devoted to the discussion of inexact equations (Pfaff's problem) and of inexact systems.

THE LINEAR EQUATION $\sum_{i=1}^{n} X_i p_i = 0$.

29. We proceed to a more detailed consideration of equations of the first order. Cauchy's theorem establishes the existence of integrals having a considerable degree of generality: but it does not prove that the integrals have the widest degree of generality possible or that they include all integrals by the appropriate specification of the arbitrary elements; and the only method which it provides for the actual construction of the integrals leads to expressions in power-series. It should be added, however, that (save for special classes of equations) the method provided in the proof of the existence-theorem is the only universal mode of constructing the integral: but for those special classes of equations simpler methods can be devised for the construction of the integrals, while further information can be obtained as to their relative generality and their classification. In all that follows, we are

concerned rather with the theory in general than with the practical solution of particular equations as expounded in text-books*.

The simplest equations of all are those which are linear in the derivatives; among them, the simplest is the equation

$$X_1 p_1 + X_2 p_2 + \ldots + X_n p_n = 0,$$

when the coefficients $X_1, \ldots, X_n$ are functions of the variables $x_1, \ldots, x_n$ but do not involve the dependent variable z. It will be seen later that every linear equation can be expressed in this form.

As usual, we associate with the partial equation the system of ordinary equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n};$$

by the theory of such equations, their integral equivalent consists of $n-1$ independent equations in a form

$$u_r(x_1, \ldots, x_n) = c_r, \qquad (r = 1, \ldots, n-1).$$

Taking any one of these integral equations, we have

$$\frac{\partial u_r}{\partial x_1} dx_1 + \ldots + \frac{\partial u_r}{\partial x_n} dx_n = 0,$$

concurrently with the ordinary differential equations; hence

$$X_1 \frac{\partial u_r}{\partial x_1} + \ldots + X_n \frac{\partial u_r}{\partial x_n} = 0,$$

again an integral equation. Now there cannot be an integral equation independent of the set

$$u_1 = c_1, \; u_2 = c_2, \; \ldots, \; u_{n-1} = c_{n-1};$$

so that the new equation is not independent of this set. But it does not involve any of the quantities c; hence, though the equation holds, it does not hold in virtue of the integral set. It therefore can only be an identity; so that the equation

$$X_1 \frac{\partial u_r}{\partial x_1} + \ldots + X_n \frac{\partial u_r}{\partial x_n} = 0$$

is satisfied identically. Consequently, when we put

$$z = u_r$$

* For instance, much of chapter IX in my *Treatise on Differential Equations*, (3d. edn. 1903), will be taken for granted.

in the original partial equation, the latter is satisfied identically: and therefore $z = u_r$ is an integral of the partial equation.

Moreover, this holds for all values of r; and therefore there are $n-1$ functionally distinct integrals of the equation. But there are not more than $n-1$ distinct integrals; that is, every integral can be expressed in terms of these. Let any integral be denoted by

$$z = f(x_1, \ldots, x_n);$$

then the equation

$$X_1 \frac{\partial f}{\partial x_1} + \ldots + X_n \frac{\partial f}{\partial x_n} = 0$$

is identically satisfied. The equations

$$X_1 \frac{\partial u_r}{\partial x_1} + \ldots + X_n \frac{\partial u_r}{\partial x_n} = 0,$$

for $r = 1, \ldots, n-1$, are identically satisfied: and the quantities $X_1, \ldots, X_n$ do not all vanish: hence

$$J = J\left(\frac{f, u_1, \ldots, u_{n-1}}{x_1, x_2, \ldots, x_n}\right) = 0.$$

The quantities u are functionally distinct, so that J does not vanish through an aggregate of vanishing first minors. It cannot vanish in virtue of $z = f$, for it does not involve z. It must therefore vanish identically; and therefore some relation must exist among the quantities $f, u_1, \ldots, u_{n-1}$, the relation involving f because $u_1, \ldots, u_{n-1}$ are functionally distinct: let it be

$$f = \phi(u_1, \ldots, u_{n-1}).$$

Hence *the equation possesses exactly $n-1$ functionally independent integrals.*

If f denote the most general integral of the equation, then ϕ must be the most general function possible: the requirement is satisfied by making ϕ a completely arbitrary function of its arguments. Hence *if $u_1, \ldots, u_{n-1}$ be a set of functionally independent integrals, the most general integral of the equation is given by*

$$z = \phi(u_1, \ldots, u_{n-1}),$$

where ϕ is a completely arbitrary function of its arguments.

The arbitrary function ϕ, and the functionally distinct integrals, can be determined so as to satisfy assigned initial conditions and therefore so as to yield the integral established by Cauchy's

theorem. Let $a_1, \ldots, a_n$ be values of $x_1, \ldots, x_n$ in the vicinity of which all the quantities $X_1, \ldots, X_n$ are regular; and suppose that some one of these quantities, say X_1, does not vanish for those values, an assumption that can always be justified by an appropriate choice of $a_1, \ldots, a_n$. The general initial conditions will be that the integral z is to acquire a value $f(x_2, \ldots, x_n)$, when $x_1 = a_1$, the function f being regular in the fields of variation considered.

The appropriate arguments can easily be constructed. Let an integral v_{r-1} be obtained to satisfy the partial differential equation, subject to the condition that it shall acquire a value x_r, when $x_1 = a_1$; its value is

$$v_{r-1} = x_r + (x_1 - a_1) P_r (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n),$$

where P_r is a regular function of $x_1, \ldots, x_n$ in the vicinity of the initial values. Taking this result for $r = 2, \ldots, n$, we have $v_1, \ldots, v_{n-1}$ as $n - 1$ functionally distinct integrals; and then

$$z = f(v_1, \ldots, v_{n-1})$$

is clearly the integral of the equation which acquires the assigned value $f(x_2, \ldots, x_n)$, when $x_1 = a_1$.

The appropriate arguments can also be constructed from the associated ordinary equations.

COROLLARY. After the preceding analysis, we can state the existence-theorem, in a different but equivalent form, as follows.

If $a_1, \ldots, a_n$ are values of $x_1, \ldots, x_n$, in the vicinity of which all the coefficients X' in the equation

$$p_1 + X_2' p_2 + X_3' p_3 + \ldots + X_n' p_n = 0$$

are regular, the equation possesses $n - 1$ functionally distinct integrals, which are regular in the selected region and which reduce to $x_2, \ldots, x_n$ respectively, when $x_1 = a_1$; and if these integrals be $z_1, \ldots, z_{n-1}$, any integral of the equation can be expressed in the form $z = f(z_1, \ldots, z_{n-1})$ by appropriate choice of f.

Ex. 1. Required an integral of the equation

$$x_1 p_1 + x_2 p_2 + x_3 p_3 = 0,$$

which shall acquire the value $\theta(x_2, x_3)$, when $x_1 = a_1$.

To obtain an integral v_1 which shall acquire the value x_2, when $x_1 = a_1$, we take

$$v_1 = x_2 + (x_1 - a_1) \phi_1 + (x_1 - a_1)^2 \phi_2 + \ldots ;$$

and we find

$$\phi_1 = -\frac{x_2}{a_1}, \qquad \phi_2 = \frac{x_2}{a_1^2}, \ldots$$

so that

$$v_1 = x_2 \left\{1 - \frac{x_1 - a_1}{a_1} + \left(\frac{x_1 - a_1}{a_1}\right)^2 - \ldots\right\}$$

$$= \frac{a_1 x_2}{x_1}.$$

We proceed similarly to obtain an integral v_2 which shall acquire the value x_3, when $x_1 = a_1$; we find

$$v_2 = \frac{a_1 x_3}{x_1}.$$

The required integral is clearly

$$z = \theta(v_1, v_2)$$

$$= \theta\left(\frac{a_1 x_2}{x_1}, \frac{a_1 x_3}{x_1}\right).$$

If we proceed from the associated ordinary equations, we need two integrals of the equations

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3}:$$

these can be taken in the form

$$u_1 = \frac{x_2}{x_1}, \qquad u_2 = \frac{x_3}{x_1}.$$

We then require the form of ϕ such that $\phi(u_1, u_2)$ becomes $\theta(x_2, x_3)$, when $x_1 = a_1$: hence

$$\phi\left(\frac{x_2}{a_1}, \frac{x_3}{a_1}\right) = \theta(x_2, x_3),$$

and therefore

$$\phi(u_1, u_2) = \theta(a_1 u_1, a_1 u_2),$$

that is, the integral is

$$z = \phi(u_1, u_2)$$

$$= \theta(a_1 u_1, a_1 u_2),$$

as before.

Ex. 2. Three given functions u, v, w of x, y, z are such that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0;$$

and three other functions ξ, η, ζ of the same variables are defined by the relations

$$u = \frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}, \qquad v = \frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}, \qquad w = \frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}.$$

Prove that the most general values of ξ, η, ζ are

$$\xi = \frac{\partial F}{\partial x} + G\frac{\partial H}{\partial x}, \qquad \eta = \frac{\partial F}{\partial y} + G\frac{\partial H}{\partial y}, \qquad \zeta = \frac{\partial F}{\partial z} + G\frac{\partial H}{\partial z},$$

where G and H are integrals of the equation

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0,$$

and F is an arbitrary function of x, y, z.

THE LINEAR EQUATION $\sum_{i=1}^{n} X_i p_i = Z$.

30. Next, we consider the linear equation

$$X_1 p_1 + \ldots + X_n p_n = Z,$$

where $X_1, \ldots, X_n, Z$ are functions of the variables $x_1, \ldots, x_n, z$. We shall assume that any factor, which is common to $X_1, \ldots, X_n, Z$, has been removed; it will therefore be unnecessary to take account of a value of z which simultaneously satisfies the equations

$$X_1 = 0, \ \ldots, \ X_n = 0, \quad Z = 0,$$

the differential equation being then satisfied without regard to the derivatives of z.

With the linear equation, we associate the set of ordinary equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = \frac{dz}{Z}.$$

Now whether $X_1, \ldots, X_n, Z$ be uniform or not, we shall assume that there are values of the variables in the vicinity of which $X_1, \ldots, X_n, Z$ behave regularly; and then, from the theory of ordinary equations, we know that the foregoing set possesses n functionally distinct integrals. Let these be

$$\phi_1 (x_1, \ldots, x_n, z) = c_1, \ \ldots, \ \phi_n (x_1, \ldots, x_n, z) = c_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

In the first place, *any equation*

$$\phi_r = c_r$$

gives an integral of the original equation if it involves z explicitly. As it is an integral of the ordinary equations, the relation

$$\frac{\partial \phi_r}{\partial x_1} dx_1 + \ldots + \frac{\partial \phi_r}{\partial x_n} dx_n + \frac{\partial \phi_r}{\partial z} dz = 0$$

is consistent with those equations; and therefore

$$X_1 \frac{\partial \phi_r}{\partial x_1} + \ldots + X_n \frac{\partial \phi_r}{\partial x_n} + Z \frac{\partial \phi_r}{\partial z} = 0.$$

Now this is a relation between the variables: it clearly is not satisfied in virtue of $\phi_r = c_r$; and therefore it is satisfied identically.

Taking $\phi_r = c_r$ as an equation giving a value (or values) of z, the derivatives are given by

$$\frac{\partial \phi_r}{\partial x_m} + \frac{\partial \phi_r}{\partial z} p_m = 0.$$

When these values of $\dfrac{\partial \phi_r}{\partial x_m}$, for $m = 1, \ldots, n$, are substituted in the foregoing equation that is identically satisfied, it becomes

$$\frac{\partial \phi_r}{\partial z}(Z - X_1 p_1 - \ldots - X_n p_n) = 0.$$

Now ϕ_r contains z, so that $\dfrac{\partial \phi_r}{\partial z}$ is not identically zero: and $\dfrac{\partial \phi_r}{\partial z}$ does not vanish because of the equation $\phi_r = c_r$, for it does not contain c_r: hence $\dfrac{\partial \phi_r}{\partial z}$ is different from zero. Accordingly, the equation

$$Z - X_1 p_1 - \ldots - X_n p_n = 0$$

is satisfied: or the equation $\phi_r = c_r$, when ϕ_r involves z explicitly, provides an integral of the partial equation.

The same is true for each of the equations $\phi = c$, provided each particular function ϕ involves z. Now some of the quantities ϕ must involve z, even though each of them may not: for otherwise $\dfrac{\partial \phi_r}{\partial z}$ would vanish for each value of r, and the equations

$$X_1 \frac{\partial \phi_r}{\partial x_1} + \ldots + X_n \frac{\partial \phi_r}{\partial x_n} = 0$$

would be satisfied identically, for $r = 1, \ldots, n$: we should then have

$$J\left(\frac{\phi_1, \ldots, \phi_n}{x_1, \ldots, x_n}\right) = 0$$

satisfied, but not in virtue of $\phi_1 = c_1, \ldots, \phi_n = c_n$: it must be satisfied identically and therefore, as the functions $\phi_1, \ldots, \phi_n$ do not (under the present hypothesis) involve z, there would be a functional relation between them, contrary to the fact that they are functionally independent. Hence, through the integral system of the ordinary equations, we find an integral or integrals of the partial equation.

In the second place, *let* $f(\phi_1, \ldots, \phi_n)$ *denote any arbitrary function of the quantities* ϕ, *and suppose that the equation*

$$f(\phi_1, \ldots, \phi_n) = 0$$

determines a value or values of z: then $f=0$ provides an integral of the differential equation. For the equations

$$X_1 \frac{\partial \phi_r}{\partial x_1} + \ldots + X_n \frac{\partial \phi_r}{\partial x_n} + Z \frac{\partial \phi_r}{\partial z} = 0,$$

for $r = 1, \ldots, n$, are satisfied identically; as f is arbitrary, not all the quantities $\frac{\partial f}{\partial \phi_1}, \ldots, \frac{\partial f}{\partial \phi_n}$ vanish; and therefore, on multiplying by $\frac{\partial f}{\partial \phi_r}$ and adding for all the values of r, the equation

$$X_1 \sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial x_1} + \ldots + X_n \sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial x_n} + Z \sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial z} = 0$$

is satisfied identically. Now the derivatives of z, as determined by $f=0$, are given by

$$\sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial x_m} + p_m \sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial z} = 0,$$

for all the values of m: when these are used, the identical equation becomes

$$(Z - X_1 p_1 - \ldots - X_n p_n) \sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial z} = 0.$$

Now as f contains z, the quantity

$$\sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial z}$$

does not vanish identically; and it does not vanish in virtue of $f=0$, when f is perfectly arbitrary: hence

$$Z - X_1 p_1 - \ldots - X_n p_n = 0,$$

or the equation is satisfied. When the values of $p_1, \ldots, p_n$ of z are determined by $f=0$, the equation is seen above to be identically satisfied: hence $f=0$ provides an integral of the equation.

Of course, there may be special forms of f such that the equation $f=0$ does not determine z: and there may be special forms of f, such that $\sum_{r=1}^{n} \frac{\partial f}{\partial \phi_r} \frac{\partial \phi_r}{\partial z}$ vanishes in virtue of $f=0$. In what precedes, we are concerned with quite arbitrary forms of f.

31. The integral thus obtained is clearly of a very general character. A question arises as to whether it is completely inclusive of all integrals: that is, can the functional form f be so chosen as to provide any integral that the partial equation possesses? Let such an integral be known to occur in a form

$$\psi(z, x_1, \ldots, x_n) = 0;$$

and let $z = c$, $x_1 = a_1, \ldots, x_n = a_n$ be a simultaneous set of values of the variables, in the vicinity of which ψ is a regular function. Also suppose that these values are such that, in their vicinity, the functions $\phi_1, \ldots, \phi_n$ are regular, these providing integrals of the partial equation as before explained. Now the n quantities $\phi_1, \ldots, \phi_n$ are functionally independent of one another; and therefore not all the determinants

$$\left\|\begin{array}{cccc} \dfrac{\partial\phi_1}{\partial z}, & \dfrac{\partial\phi_1}{\partial x_1}, & \ldots, & \dfrac{\partial\phi_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\phi_n}{\partial z}, & \dfrac{\partial\phi_n}{\partial x_1}, & \ldots, & \dfrac{\partial\phi_n}{\partial x_n} \end{array}\right\|$$

vanish identically. Two typical cases will suffice for the general discussion: for the first, it will be assumed that

$$\frac{\partial(\phi_1, \ \ldots, \ \phi_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

does not vanish identically; for the second, it will be assumed that

$$\frac{\partial(\phi_1, \ \ldots, \ \phi_n)}{\partial(z, x_2, \ldots, x_n)}$$

does not vanish identically.

32. In the first place, when the Jacobian of $\phi_1, \ldots, \phi_n$ with regard to $x_1, \ldots, x_n$ does not vanish identically, it is possible to choose the set of values $c, a_1, \ldots, a_n$ for $z, x_1, \ldots, x_n$, so that the Jacobian does not vanish or become infinite for them, unless the set constitute a singularity or other non-regular place for one or more of the quantities ϕ. Assuming this choice made, we then can resolve the n equations

$$\phi_r = \phi_r(z, x_1, \ldots, x_n), \qquad (r = 1, \ldots, n),$$

so as to express $x_1, \ldots, x_n$ as regular functions of $z, \phi_1, \ldots, \phi_n$, in forms

$$x_m = \xi_m(z, \phi_1, \ldots, \phi_n),$$

for $m=1, \ldots, n$. Let these values for the n variables be substituted in $\psi=0$, so that

$$\begin{aligned}\psi = 0 &= \psi(z, \xi_1, \ldots, \xi_m)\\ &= \chi(z, \phi_1, \ldots, \phi_n);\end{aligned}$$

and now the given integral can be taken in the form $\chi=0$. To obtain the derivatives, we have

$$\frac{\partial \chi}{\partial z} p_m + \sum_{r=1}^{n} \frac{\partial \chi}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_m} + \frac{\partial \phi_r}{\partial z} p_m\right) = 0.$$

Multiply by X_m and add for the values $m=1, \ldots, n$: then

$$\frac{\partial \chi}{\partial z} \sum_{m=1}^{n} X_m p_m + \sum_{r=1}^{n} \frac{\partial \chi}{\partial \phi_r}\left\{\left(\sum_{m=1}^{n} X_m \frac{\partial \phi_r}{\partial x_m}\right) + \frac{\partial \phi_r}{\partial z}\left(\sum_{m=1}^{n} X_m p_m\right)\right\} = 0.$$

Now for the integral under consideration, we have

$$\sum_{m=1}^{n} X_m p_m = Z;$$

and we also know that the equation

$$\sum_{m=1}^{n} X_m \frac{\partial \phi_r}{\partial x_m} + Z \frac{\partial \phi_r}{\partial z} = 0$$

is satisfied identically for all values of r. Moreover, in the vicinities concerned, all the functions are regular, so that the quantities $\frac{\partial \chi}{\partial \phi_r}$ are finite in the fields of variation retained. When these relations are used, the above equation becomes

$$Z \frac{\partial \chi}{\partial z} = 0,$$

and this equation must be satisfied in association with $\chi=0$. This requirement may be met in three ways.

It may happen that $\frac{\partial \chi}{\partial z}$ vanishes identically: then z does not occur explicitly in χ, and the expression of χ then gives

$$\psi = \chi(\phi_1, \ldots, \phi_n),$$

that is, a form of function in the integral $f(\phi_1, \ldots, \phi_n)$ has been obtained so that the general integral becomes the given integral.

It may happen that $\frac{\partial \chi}{\partial z}$ vanishes, not indeed identically but only in virtue of $\chi=0$. Then z occurs explicitly in χ; and the

form of the arbitrary function cannot be determined so that the general integral becomes the given integral.

It may happen that $\frac{\partial \chi}{\partial z}$ does not vanish. The condition can only be satisfied, if $Z=0$: and this must hold in association with $\chi=0$. Again, z occurs in χ; thus, once more, the form of the arbitrary function cannot be determined so that the general integral becomes the given integral.

Of these three alternatives, it is clear that the last belongs to a special set: as the integral is given by $Z=0$, we must have

$$\frac{\partial Z}{\partial x_m} + p_m \frac{\partial Z}{\partial z} = 0 ;$$

and then the equation

$$X_1 \frac{\partial Z}{\partial x_1} + \ldots + X_m \frac{\partial Z}{\partial x_m} = 0$$

must be satisfied, concurrently with $Z=0$. Moreover, as $\phi_1 = c_1$, $\ldots$, $\phi_n = c_n$ are a set of n independent integrals of the system of ordinary equations

$$\frac{dx_1}{X_1} = \ldots = \frac{dx_n}{X_n} = \frac{dz}{Z},$$

we have

$$\frac{Z}{J\left(\dfrac{\phi_1, \ldots, \phi_n}{x_1, x_2, \ldots, x_n}\right)} = \frac{(-1)^{r-1} X_r}{J_r\left(\dfrac{\phi_1, \ldots, \phi_n}{z, x_1, \ldots, x_n}\right)},$$

where x_r is omitted from the deriving variables in J_r, and $r=1, \ldots, n$ in turn; hence as $Z=0$ for the integral under consideration, X_r must vanish for the value of z unless J_r should vanish for the value. We have assumed that not all the quantities $Z, X_1, \ldots, X_m$ vanish for the same value of z.

The second alternative may belong to a less special set: it will be illustrated by examples. The first alternative provides the most general case.

Integrals, which arise under the second alternative or under the third alternative, may be called *special* integrals*.

* Sometimes they are called *singular*. This term, however, is better reserved for a class of integrals belonging exceptionally to equations of a degree higher than the first in the derivatives.

33. In the second place, when the Jacobian of $\phi_1, \ldots, \phi_n$ with regard to $z, x_2, \ldots, x_n$ does not vanish identically, to take only a typical case when the Jacobian of those quantities with regard to $x_1, x_2, \ldots, x_n$ does vanish identically, it is possible to choose the set of values $c, a_1, \ldots, a_n$, so that the Jacobian does not vanish or become infinite for them unless they constitute a singularity or other non-regular place of one or more of the quantities ϕ. Assuming this done, we can then resolve the n equations

$$\phi_r = \phi_r(z, x_1, \ldots, x_n)$$

so as to express the variables $z, x_2, \ldots, x_n$ in terms of $x_1, \phi_1, \ldots, \phi_n$ in forms

$$z = \zeta(x_1, \phi_1, \ldots, \phi_n),$$
$$x_r = \eta_r(x_1, \phi_1, \ldots, \phi_n),$$

for $r = 2, \ldots, n$. When these values are substituted for $z, x_2, \ldots, x_n$ in the equation $\psi = 0$ which provides the given integral, it takes the form

$$\psi = 0 = \psi(z, x_1, \ldots, x_n)$$
$$= \theta(x_1, \phi_1, \ldots, \phi_n);$$

and the given integral can now be taken in the form $\theta = 0$. The derivatives of z are given by the n relations

$$\frac{\partial\theta}{\partial x_1} + \sum_{r=1}^{n} \frac{\partial\theta}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_1} + \frac{\partial\phi_r}{\partial z} p_1\right) = 0,$$

$$\sum_{r=1}^{n} \frac{\partial\theta}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_m} + \frac{\partial\phi_r}{\partial z} p_m\right) = 0,$$

for $m = 2, \ldots, n$. Multiplying these by X_1 and by X_m respectively, and adding for the various values of m, we have

$$X_1 \frac{\partial\theta}{\partial x_1} + \sum_{r=1}^{n} \frac{\partial\theta}{\partial\phi_r}\left\{\left(\sum_{m=1}^{n} X_m \frac{\partial\phi_r}{\partial x_m}\right) + \frac{\partial\phi_r}{\partial z} \sum_{m=1}^{n} (X_m p_m)\right\} = 0.$$

For the integral under consideration, we have

$$\sum_{m=1}^{n} X_m p_m = Z;$$

and we know that the relation

$$\sum_{m=1}^{n} X_m \frac{\partial\phi_r}{\partial x_m} + Z \frac{\partial\phi_r}{\partial z} = 0$$

is satisfied identically. Moreover, all the functions are regular in all the vicinities concerned, so that all the quantities $\frac{\partial\theta}{\partial\phi_r}$, for

$r = 1, \ldots, n$, are finite in the fields of variation retained. When these equations are used, the above equation becomes

$$X_1 \frac{\partial \theta}{\partial x_1} = 0;$$

and it must be satisfied in association with $\theta = 0$. This requirement can, as in the preceding discussion, be met in three ways.

It may happen that $\frac{\partial \theta}{\partial x_1}$ vanishes identically; then x_1 does not occur explicitly in θ, and the expression of θ gives

$$\psi = \theta(\phi_1, \ldots, \phi_n),$$

that is, a form of function has been obtained for $f(\phi_1, \ldots, \phi_n)$ so that $f = 0$ has become the given integral $\psi = 0$.

Or it may happen that $\frac{\partial \theta}{\partial x_1}$ vanishes, not indeed identically but only in virtue of $\theta = 0$. Then x_1 occurs explicitly in θ; the form of the arbitrary function f in the general integral cannot be determined so as to particularise the general integral into the given integral.

Or it may happen that $\frac{\partial \theta}{\partial x_1}$ does not vanish. The condition can then only be satisfied if $X_1 = 0$; and this must hold in association with $\theta = 0$. Again, the variable x_1 occurs explicitly in θ: thus, once more, the form of the arbitrary function f in the general integral cannot be determined so as to make the general integral become the given integral.

The three alternatives are similar to those in the former discussion; integrals, that arise in connection with the second or the third of the alternatives, will be called *special*, as before.

34. Gathering together these results, we can summarise them as follows:—

Let $\psi(z, x_1, \ldots, x_n) = 0$ provide an integral of the partial differential equation

$$X_1 p_1 + \ldots + X_n p_n = Z,$$

and let $f(\phi_1, \ldots, \phi_n) = 0$ denote its most general integral, f being an arbitrary function; then the functional form of f can be chosen so that $f(\phi_1, \ldots, \phi_n)$ becomes ψ, unless ψ is of the type of integral called special, or unless the value of z provided by $\psi = 0$ constitutes

a singularity or other non-regular place for one or more of the quantities ϕ.

It thus appears that the general integral for the linear non-homogeneous equation, in which the dependent variable occurs explicitly, is not so completely inclusive as is the general integral for the linear homogeneous equation, in which the dependent variable does not occur explicitly.

Instances of the principal portion of the theorem are so frequent that none need be adduced here: a few examples will be given to illustrate the special integrals and other exceptions.

Ex. 1. Consider the equation

$$xp+yq=z.$$

Two integrals of the associated equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

can be taken in the form

$$\phi_1=\frac{z}{x}, \qquad \phi_2=\frac{z}{y};$$

and the most general integral is given by

$$f(\phi_1, \phi_2)=0.$$

It is easy to verify that

$$\psi=z-\frac{x^2}{y}=0$$

provides an integral of the equation. Expressing ψ in terms of ϕ_1, ϕ_2, and z, we find

$$\psi=z-z\frac{\phi_2}{\phi_1^2}=\psi',$$

so that

$$\frac{\partial\psi'}{\partial z}=1-\frac{\phi_2}{\phi_1^2};$$

thus $\frac{\partial\psi'}{\partial z}$ does not vanish identically but only in virtue of $\psi'=0$, and then only in virtue of the factor $1-\frac{\phi_2}{\phi_1^2}$ in ψ'. Thus the integral given by $\psi=0$ is a special integral; for the form of f in $f(\phi_1, \phi_2)$ cannot be chosen so as to make $f(\phi_1, \phi_2)$ become ψ.

It should be noted that $f(\phi_1, \phi_2)$ can be chosen, in a form $\phi_1^2-\phi_2$, so as to vanish for the integral provided by $\psi=0$: but it does not follow (and it is not the fact) that f can be chosen so that $f(\phi_1, \phi_2)$ becomes ψ.

Ex. 2. Consider the equation

$$\left(z-x_1\frac{x_2^2}{x_3}\right)p_1+x_2p_2+x_3p_3=z.$$

Integrals of the associated ordinary equations

$$\frac{dx_1}{z - x_1 \dfrac{x_2^2}{x_3}} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dz}{z}$$

may be taken in the form

$$\phi_1 = \left(1 - \frac{x_1 x_2^2}{z x_3}\right) e^{\frac{x_2^2}{x_3}}, \qquad \phi_2 = \frac{x_2}{z}, \qquad \phi_3 = \frac{x_3}{z};$$

and the general integral is

$$f(\phi_1, \phi_2, \phi_3) = 0,$$

where f is an arbitrary function.

It is easy to verify that

$$\psi = z - x_1 \frac{x_2^2}{x_3} = 0$$

provides an integral of the equation; but the functional form f cannot be chosen so that $f(\phi_1, \phi_2, \phi_3)$ becomes ψ. In fact, we have

$$\psi = z\phi_1 e^{-\frac{z\phi_2^2}{\phi_3}} = \psi',$$

so that

$$\frac{\partial \psi'}{\partial z} = \phi_1 e^{-\frac{z\phi_2^2}{\phi_3}} \left(1 - z\frac{\phi_2^2}{\phi_3}\right);$$

and $\dfrac{\partial \psi'}{\partial z}$ does not vanish identically. Taking the value of z given by ψ and substituting it in ϕ_1, we find $\phi_1 = 0$: so that $\dfrac{\partial \psi'}{\partial z}$ vanishes in virtue of this result, that is, in virtue of $\psi' = 0$. The integral $\psi = 0$ is a special integral.

If, instead of expressing x_1, x_2, x_3 in terms of the quantities $z, \phi_1, \phi_2, \phi_3$ with a view to the transformation of ψ, we express z, x_2, x_3 in terms of $x_1, \phi_1, \phi_2, \phi_3$, we find

$$\psi = \frac{\phi_3}{\phi_2^2}\left(1 - x_1 \frac{\phi_2^2}{\phi_3}\right) \log\left(\frac{\phi_1 \phi_3}{\phi_3 - x_1 \phi_2^2}\right) = \psi';$$

and then the requisite condition is

$$X_1 \frac{\partial \psi'}{\partial x_1} = 0,$$

in association with $\psi' = 0$. Now $\dfrac{\partial \psi'}{\partial x_1}$ does not vanish identically, nor does it vanish in virtue of $\psi' = 0$; we must therefore have $X_1 = 0$ in association with $\psi' = 0$. This is satisfied: and therefore, as before, $\psi = 0$ provides a special integral of the equation.

Ex. 3. Consider the equation

$$xp + 2yq = 2\left(z - \frac{x^2}{y}\right)^2.$$

The associated ordinary equations are

$$\frac{dx}{x}=\frac{dy}{2y}=\frac{dz}{2\left(z-\frac{x^2}{y}\right)^2},$$

of which two independent integrals are given by

$$\phi_1=\frac{x^2}{y}, \qquad \phi_2=ye^{\frac{1}{z-\frac{x^2}{y}}}.$$

The most general integral of the partial equation is

$$f(\phi_1, \phi_2)=0,$$

where f is an arbitrary function.

It is easy to verify that

$$\psi=z-\frac{x^2}{y}=0$$

provides an integral of the equation: but the functional form f cannot be chosen so as to make $f(\phi_1, \phi_2)$ become ψ. Proceeding as in the general exposition, we have

$$\psi=z-\phi_1=\psi',$$

so that $\frac{\partial\psi'}{\partial z}=1$ and cannot vanish, shewing that f cannot be chosen for the purpose. But the quantity Z of the general investigation vanishes for the value of z given by $\psi=0$.

It will be noted that ψ' does not involve ϕ_1: the special integral is a singularity of ϕ_2.

Ex. 4. Consider the equation*

$$\{1+(z-x-y)^{\frac{1}{2}}\}p+q=2.$$

The integrals of the ordinary equations

$$\frac{dx}{1+(z-x-y)^{\frac{1}{2}}}=\frac{dy}{1}=\frac{dz}{2}$$

can be taken in the form

$$\phi_1=2y-z,$$

$$\phi_2=y+2(z-x-y)^{\frac{1}{2}};$$

and the general integral is

$$f(\phi_1, \phi_2)=0.$$

It is easy to verify that

$$\psi=z-x-y=0$$

provides an integral of the equation; it is clear that no form of f can be found which will make the general function $f(\phi_1, \phi_2)$ become ψ. The integral provided by $\psi=0$ is a special integral; and manifestly any set of values, satisfying $\psi=0$ and chosen as initial values, constitute a branch-place of the quantity ϕ_2 and of the coefficient of p in the equation.

As this coefficient is not regular in the vicinity, Cauchy's theorem does not apply.

* This example is given by Chrystal, *Trans. R. S. E.*, t. xxxvi (1892), p. 557.

35. The discussion of the integrals of the equation

$$X_1 p_1 + \ldots + X_n p_n = Z$$

can be associated with the discussion of the integrals of the equation, which is without the quantity Z and any explicit occurrence of z, by means of a simple transformation. Let the integral be given by the equation

$$u = u(z, x_1, \ldots, x_n) = 0,$$

where, in the circumstances, u involves z; then we have

$$\frac{\partial u}{\partial z} p_m + \frac{\partial u}{\partial x_m} = 0.$$

Now $\frac{\partial u}{\partial z}$ does not vanish identically, and we shall assume* that it does not vanish in consequence of $u = 0$; hence we may resolve these equations for $p_1, \ldots, p_n$. Substituting in the original equation, we have

$$X_1 \frac{\partial u}{\partial x_1} + \ldots + X_n \frac{\partial u}{\partial x_n} + Z \frac{\partial u}{\partial z} = 0,$$

and this must be satisfied identically when a value of z given by $u = 0$ is inserted: in other words, the modified equation is satisfied, not identically but only simultaneously with $u = 0$. The modified equation is of the earlier type: the coefficients of the derivatives involve only the independent variables but not the dependent variable u. Of this modified equation, let

$$u = \theta(z, x_1, \ldots, x_n)$$

be an integral; then obviously $u = 0$ will give an integral of the original equation. But the fact that $\theta(z, x_1, \ldots, x_n)$ is an integral of the modified equation means, as was seen before, that when this value of u is substituted the equation is satisfied identically. This limitation is additional to the earlier requirement, which was only that the equation should be satisfied simultaneously with $u = 0$; it was not necessary that the equation should be satisfied identically. We cannot therefore infer from the argument that any integral of the original equation can thus be obtained from an integral of the

* The significance of the assumption, and the limitation which it imposes, would need to be examined if the character of the integrals were being determined solely by the present argument.

modified equation; and it is clear that any integral so obtained is a special case of an integral given by

$$\theta(z, x_1, \ldots, x_n) - a = 0,$$

where a is an arbitrary constant*.

Ex. As an example, consider the equation

$$(x^2 + 2xy)\frac{\partial z}{\partial x} - z^2\frac{\partial z}{\partial y} = y^2.$$

It clearly is satisfied by a value of z given by the equation

$$x + y + z = 0.$$

But effecting the transformation indicated, viz. taking

$$u = u(z, x, y) = 0,$$

so that u is a new variable, we have

$$(x^2 + 2xy)\frac{\partial u}{\partial x} - z^2\frac{\partial u}{\partial y} + y^2\frac{\partial u}{\partial z} = 0.$$

Any integral of this equation, when substituted, is known (by our earlier argument) to make the equation satisfied identically. If we take

$$u = x + y + z,$$

the equation is not satisfied identically; it can only be satisfied for this value of u simultaneously with $u = 0$; but $u = x + y + z$ is not an integral of the new equation.

On the other hand, the original equation is satisfied by a value of z given by the equation

$$y^3 + z^3 = a,$$

where a is a constant: and

$$u = y^3 + z^3$$

is an integral of the modified equation. Thus the first integral is not given, the second integral is given, by the method.

The distinction between the two cases can be expressed simply by a reference to the theory of continuous groups. Let

$$X(\theta) = (x^2 + 2xy)\frac{\partial \theta}{\partial x} - z^2\frac{\partial \theta}{\partial y} + y^2\frac{\partial \theta}{\partial z}$$

be an infinitesimal transformation.

We have

$$X(y^3 + z^3) = 0;$$

the quantity $y^3 + z^3$ is an *invariant* for the given infinitesimal transformation.

We have

$$X(x + y + z) = (x + y + z)(x + y - z),$$

* The limitation was, I believe, first pointed out by Goursat, in § 16 of the work quoted on p. 55.

so that $x+y+z$ is not an invariant for the infinitesimal transformation: but when we have

$$x+y+z=0,$$

then, in virtue of that equation,

$$X(x+y+z)=0;$$

the equation $x+y+z=0$ is an *invariant equation* for the transformation.

36. It remains to associate Cauchy's theorem with the equation; for this purpose, we have to obtain an integral which, when $x_1=a_1$, reduces to

$$z=g(x_2, \ldots, x_n),$$

where g is a function, which is regular in the domains of the values $x_2=a_2, \ldots, x_n=a_n$, and otherwise is arbitrary.

Choosing a_1 so that X_1 does not vanish there, the integrals of the associated ordinary equations

$$dx_2=\frac{X_2}{X_1}dx_1, \quad dx_3=\frac{X_3}{X_1}dx_1, \quad \ldots, \quad dx_n=\frac{X_n}{X_1}dx_1, \quad dz=\frac{Z}{X_1}dx_1$$

can be obtained, subject to assigned conditions that $x_2=a_2, \ldots, x_n=a_n$, $z=g(a_2, \ldots, a_n)=c$, when $x_1=a_1$; and they have the form

$$\begin{aligned} u_1 &= z \;\; + (x_1-a_1)\,v_1 = c\;, \\ u_2 &= x_2 + (x_1-a_1)\,v_2 = a_2\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n &= x_n + (x_1-a_1)\,v_n = a_n\,, \end{aligned}$$

where $v_1, \ldots, v_n$ are regular functions of the variables $x_1, \ldots, x_n, z$. Now the general integral is

$$f(u_1, u_2, \ldots, u_n)=0;$$

or, changing the form of the arbitrary function, we may take

$$u_1=F(u_2, \ldots, u_n)$$

as the integral, where F also is arbitrary. When $x_1=a_1$, this equation becomes

$$z=F(x_2, \ldots, x_n);$$

but the value of z when $x_1=a_1$, is to be $g(x_2, \ldots, x_n)$: and therefore when the arbitrary function is chosen so that

$$F(x_2, \ldots, x_n)=g(x_2, \ldots, x_n),$$

and consequently

$$F(u_2, \ldots, u_n)=g(u_2, \ldots, u_n),$$

we have an integral

$$u_1 = g(u_2, \ldots, u_n),$$

which is the integral in Cauchy's theorem.

Ex. Required the integral of

$$xp + yq = z,$$

which, when $x = a$, is such that $z = \frac{y^2}{4c}$.

Two integrals of the ordinary equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

are taken such that, when $x = a$, we have $y = b$, and $z = \frac{b^2}{4c}$; these are easily seen to be

$$u_1 = \frac{az}{x} = \frac{b^2}{4c},$$

$$u_2 = \frac{ay}{x} = b.$$

Thus the general integral of the equation can be taken in the form

$$u_1 = f(u_2),$$

where f is arbitrary. When $x = a$, this equation becomes

$$z = f(y),$$

so that, for the required integral,

$$f(y) = \frac{y^2}{4c};$$

and therefore

$$f(u_2) = \frac{u_2^2}{4c}.$$

Hence the required integral is given by the equation

$$u_1 = \frac{u_2^2}{4c},$$

that is,

$$z = \frac{ay^2}{4cx}.$$

If, instead of taking Cauchy's theorem in its simplest form as associated with an initial value $x_1 = a_1$, we require an integral which, when a relation of the form

$$f(z, x_1, \ldots, x_n) = 0$$

exists among the variables, shall be given by the equation

$$g(z, x_1, \ldots, x_n) = 0,$$

effectively what is required is the determination of the arbitrary functional form F in

$$F(\phi_1, \ldots, \phi_n) = 0,$$

so that the equation may be satisfied without any other relation solely in virtue of $f=0$, $g=0$.

As $f=0$ and $g=0$ are two relations between $n+1$ quantities, $n-1$ of these can be regarded as independent: or we may regard all the $n+1$ variables as expressible in terms of $n-1$ independent quantities. Taking the latter mode of representing them, let their expressions be

$$z = \psi\ (\xi_1, \ldots, \xi_{n-1}),$$
$$x_r = \psi_r(\xi_1, \ldots, \xi_{n-1}),$$

for $r=1, \ldots, n$. When these are substituted in the quantities $\phi_1, \ldots, \phi_n$, we have

$$\begin{aligned}\phi_m &= \phi_m(z, x_1, \ldots, x_n)\\ &= \phi_m(\psi, \psi_1, \ldots, \psi_n)\\ &= \bar{\phi}_m(\xi_1, \ldots, \xi_{n-1}) = \bar{\phi}_m \text{ say,}\end{aligned}$$

for $m=1, \ldots, n$; and these n relations, expressing $\phi_1, \ldots, \phi_n$ in terms of $n-1$ quantities, are satisfied concurrently with the relations $f=0$, $g=0$. Among these n relations, let the $n-1$ quantities $\xi_1, \ldots, \xi_{n-1}$ be eliminated, and let the result of the elimination be

$$G(\bar{\phi}_1, \ldots, \bar{\phi}_n) = 0.$$

Now when $f=0$ and $g=0$, we have ϕ_m degenerating to $\bar{\phi}_m$; and the general integral becomes

$$F(\bar{\phi}_1, \ldots, \bar{\phi}_n) = 0,$$

which coexists with $f=0$ and $g=0$, but, as now it involves only the quantities $\xi_1, \ldots, \xi_{n-1}$, it is satisfied by itself and not in virtue of $f=0$, $g=0$. We thus have

$$F(\bar{\phi}_1, \ldots, \bar{\phi}_n) = G(\bar{\phi}_1, \ldots, \bar{\phi}_n),$$

and therefore also

$$F(\phi_1, \ldots, \phi_n) = G(\phi_1, \ldots, \phi_n).$$

Hence the required integral is given by the equation

$$G(\phi_1, \ldots, \phi_n) = 0.$$

Ex. In examples, the details sometimes are developed in a different way. Let it be required to find a surface, satisfying the equation

$$xp+yq=z,$$

and passing through the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad lx+my+nz=1.$$

The curve can be expressed in the form

$$x=a\lambda, \quad y=b\mu, \quad z=c\nu,$$

where

$$\lambda^2+\mu^2+\nu^2=1,$$
$$al\lambda+bm\mu+cn\nu=1.$$

Two integrals of the associated ordinary equations are

$$u=\frac{z}{x}, \quad v=\frac{z}{y};$$

hence, along the curve, we have

$$\bar{u}=\frac{c}{a}\frac{\nu}{\lambda}, \quad \bar{v}=\frac{c}{b}\frac{\nu}{\mu},$$

so that

$$\lambda=\frac{c}{a}\frac{\nu}{\bar{u}}, \quad \mu=\frac{c}{b}\frac{\nu}{\bar{v}},$$

whence

$$\nu^2\left(1+\frac{c^2}{a^2}\frac{1}{\bar{u}^2}+\frac{c^2}{b^2}\frac{1}{\bar{v}^2}\right)=1,$$
$$\nu\left(cn+\frac{cl}{\bar{u}}+\frac{cm}{\bar{v}}\right)=1,$$

and therefore

$$1+\frac{c^2}{a^2}\frac{1}{\bar{u}^2}+\frac{c^2}{b^2}\frac{1}{\bar{v}^2}=c^2\left(\frac{l}{\bar{u}}+\frac{m}{\bar{v}}+n\right)^2.$$

This equation corresponds to the equation $G(\bar{\phi}_1, \ldots, \bar{\phi}_n)=0$ in the preceding discussion. In the present case, the required integral is accordingly given by

$$1+\frac{c^2}{a^2u^2}+\frac{c^2}{b^2v^2}=c^2\left(\frac{l}{u}+\frac{m}{v}+n\right)^2;$$

inserting the values of u and v, the equation of the required surface is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(lx+my+nz)^2.$$

Complete linear Systems that are homogeneous.

37. Before passing to the discussion of the most general equation of the first order and of degree higher than the first, it is convenient to deal with a system of simultaneous linear equations involving one dependent variable. If the dependent variable occurs explicitly, the equations can be changed, by a

transformation as in § 35, so that the new dependent variable does not occur explicitly; the number of independent variables is thus increased by unity. In their transformed expression, the equations are homogeneous in the derivatives; they may be written

$$\left.\begin{aligned} A_1(u) &= a_{11}\frac{\partial u}{\partial x_1} + a_{21}\frac{\partial u}{\partial x_2} + \ldots + a_{s1}\frac{\partial u}{\partial x_s} = 0 \\ A_2(u) &= a_{12}\frac{\partial u}{\partial x_1} + a_{22}\frac{\partial u}{\partial x_2} + \ldots + a_{s2}\frac{\partial u}{\partial x_s} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_\mu(u) &= a_{1\mu}\frac{\partial u}{\partial x_1} + a_{2\mu}\frac{\partial u}{\partial x_2} + \ldots + a_{s\mu}\frac{\partial u}{\partial x_s} = 0 \end{aligned}\right\},$$

where the coefficients a_{mn}, for m and $n = 1, \ldots, s$, are functions of the independent variables $x_1, \ldots, x_s$ alone. We have to investigate the conditions under which an integral (if any) can be possessed by the system; we have also to devise means for the construction of an integral when it is possessed.

Equations of this type have already, in Part I (§§ 38—41) of this work, received some consideration; but there they arose as a class, associated with equations linear and homogeneous in differential elements in the variables, and the limitations imposed upon them were derived from the originating equations. Their importance, not least owing to their frequent occurrence in various theories, justifies an independent treatment; the earlier discussion will render some abbreviation possible.

The μ equations in the propounded system are said to be independent, when no linear relation of the form

$$\xi_1 A_1(u) + \ldots + \xi_\mu A_\mu(u) = 0$$

exists, the quantities $\xi_1, \ldots, \xi_\mu$ being functions of $x_1, \ldots, x_s$, and the quantities $A_1(u), \ldots, A_\mu(u)$ being merely the linear combinations of the derivatives of u. If, however, such a relation or relations should exist, then one or more than one of the equations $A(u) = 0$ would be dependent on the others: an integral of those others would satisfy the dependent equation or equations; and so the dependent equations could be ignored for the present purpose. Accordingly, we shall assume that the equations in the system are independent.

It is clear that, if a linear system in s independent variables contains s equations or more than s equations, the equations can only be satisfied by having

$$\frac{\partial u}{\partial x_1} = 0, \quad \frac{\partial u}{\partial x_2} = 0, \quad \ldots, \quad \frac{\partial u}{\partial x_s} = 0;$$

and then

$$u = \text{constant}$$

is obviously the only integral of the system. Having disposed of systems for which $\mu \geqslant s$, we shall now assume $\mu < s$.

The μ equations are independent; but it may be necessary to associate other equations with them, arising as consequences of their coexistence or as conditions of their coexistence. It is clear that, if the equations

$$A_m(u) = 0, \quad A_n(u) = 0$$

possess a common integral, it makes the left-hand sides vanish identically; and therefore the equations

$$A_m(A_n u) = 0, \quad A_n(A_m u) = 0,$$

and so also

$$A_m(A_n u) - A_n(A_m u) = 0,$$

are satisfied for that common integral; that is, the last equation coexists with $A_m(u) = 0$ and $A_n(u) = 0$, when the two latter are members of a linear system. But the new equation is found also to be linear in the first derivatives of u: for the coefficient of $\dfrac{\partial^2 u}{\partial x_k \partial x_l}$ in $A_m(A_n u)$ is $a_{km} a_{ln} + a_{lm} a_{kn}$, when k and l are different, and is $a_{lm} a_{ln}$, when k is the same as l; and the coefficient of $\dfrac{\partial^2 u}{\partial x_k \partial x_l}$ in $A_n(A_m u)$ is $a_{ln} a_{km} + a_{kn} a_{lm}$, when k and l are different, and is $a_{ln} a_{lm}$, when k is the same as l: thus the derivatives of u of the second order disappear, and only derivatives of the first order remain. The equation is

$$\begin{aligned} 0 &= A_m(A_n u) - A_n(A_m u) \\ &= \sum_{r=1}^{s} \{A_m(a_{rn}) - A_n(a_{rm})\} \frac{\partial u}{\partial x_r}. \end{aligned}$$

Now this equation may be evanescent, because the coefficient of each of the derivatives of u vanishes. Or it may be satisfied in virtue of the original set, as a linear combination of them; it then is not a new independent equation, and consequently it need not

be taken into further account. Or it may be not evanescent, and not a linear combination of the original equations, and yet it must be satisfied; then it is a new equation, and it must be associated with the system.

Similarly for any pair of equations in the system. Suppose that, by taking all possible pairs, r new equations are obtained so that there is a system

$$A_1(u)=0,\ \ldots,\ A_\mu(u)=0,\quad A_{\mu+1}(u)=0,\ \ldots,\ A_{\mu+r}(u)=0.$$

Again we must take all possible pairs; clearly it will be sufficient to take each of the first μ with each of the last r, and all possible pairs of the last r; all new equations are to be retained. And so on, until the process either provides no new equation or until the number of equations has come to be s. The latter case has been dealt with. When the former case occurs, the number of equations being less than s, the system at that stage is called a *complete linear system.* Manifestly, when there is only one dependent variable and there are several linear equations, we have to deal with complete linear systems. Moreover, the only systems of this type that require consideration are those in which the number of independent equations is less than the number of independent variables.

38. Two properties, possessed by complete linear systems, lead to simplification in the analysis: they must be established.

In the first place, *when a complete system is replaced by another, which is its algebraic equivalent, the new system is complete.* Let a system

$$A_1(u)=0,\ \ldots,\ A_\mu(u)=0,$$

supposed complete, be replaced by a system

$$B_1(u)=0,\ \ldots,\ B_\mu(u)=0,$$

where

$$B_m(u)=\sum_{n=1}^{\mu}\xi_{mn}A_n(u),$$

for $m=1,\ \ldots,\ \mu$, and the quantities ξ_{mn} are functions of the variables $x_1,\ \ldots,\ x_s$ such that their determinant does not vanish.

It is clear that the quantities $A_n(u)$ are expressible as linear combinations of the quantities $B_m(u)$; so that, algebraically, the two systems of equations are equivalent to one another.

To decide whether the new system is complete or not, we construct the quantities $B_m(B_n u) - B_n(B_m u)$; and we have

$$\begin{aligned}
B_m(B_n u) &- B_n(B_m u) \\
&= \sum_{r=1}^{\mu} \left[\xi_{mr} A_r \left\{ \sum_{i=1}^{\mu} \xi_{ni} A_i(u) \right\} \right] - \sum_{i=1}^{\mu} \left[\xi_{ni} A_i \left\{ \sum_{r=1}^{\mu} \xi_{mr} A_r(u) \right\} \right] \\
&= \sum_{r=1}^{\mu} \sum_{i=1}^{\mu} \xi_{mr} \xi_{ni} A_r(A_i u) + \sum_{r=1}^{\mu} \sum_{i=1}^{\mu} \xi_{mr} A_r(\xi_{ni}) A_i(u) \\
&\quad - \sum_{r=1}^{\mu} \sum_{i=1}^{\mu} \xi_{mr} \xi_{ni} A_i(A_r u) - \sum_{r=1}^{\mu} \sum_{i=1}^{\mu} \xi_{ni} A_i(\xi_{mr}) A_r(u).
\end{aligned}$$

Combining the first summations in the two lines, we have $A_i(A_r u) - A_r(A_i u)$ as the coefficient of $\xi_{mr} \xi_{ni}$; this quantity is a linear combination of the quantities $A_1(u), \ldots, A_\mu(u)$, because the system is complete: hence these two summations give a linear combination of the quantities $A(u)$. Each of the other two summations is actually a linear combination of these quantities; hence the whole expression for $B_m(B_n u) - B_n(B_m u)$ is a linear combination of the quantities $A(u)$. Each of the quantities $A(u)$ is a linear combination of the quantities $B(u)$; when the values are substituted, we find that $B_m(B_n u) - B_n(B_m u)$ is a linear combination of the quantities $B(u)$. As this holds for all values of m and n, it follows that the system of equations $B_1(u) = 0, \ldots, B_\mu(u) = 0$ is complete.

In the second place, *a complete system remains complete for any transformation of the independent variables.* Let these variables be transformed by the relations

$$x_r' = f_r(x_1, \ldots, x_s),$$

for $r = 1, \ldots, s$, the functions $f_1, \ldots, f_s$ being independent of one another. Then

$$\frac{\partial u}{\partial x_r} = \frac{\partial u}{\partial x_1'} \frac{\partial f_1}{\partial x_r} + \frac{\partial u}{\partial x_2'} \frac{\partial f_2}{\partial x_r} + \ldots + \frac{\partial u}{\partial x_s'} \frac{\partial f_s}{\partial x_r},$$

for all values of r; substituting in $A_n(u)$ for the quantities $\dfrac{\partial u}{\partial x_r}$, we have

$$A_n(u) = A_n'(u),$$

and $A_n'(u)$ is homogeneous and linear in the derivatives

$$\frac{\partial u}{\partial x_1'}, \ldots, \frac{\partial u}{\partial x_s'}.$$

As there is no linear relation among the quantities $A_n(u)$, there can be none among the quantities $A_n'(u)$: the equations $A'(u)=0$ are independent. Further, the operation A_n is replaced by A_n', having the modified coefficients: thus

$$A_m(A_n u) = A_m(A_n' u) = A_m'(A_n' u),$$
$$A_n(A_m u) = A_n(A_m' u) = A_n'(A_m' u),$$

and therefore

$$\begin{aligned} A_m'(A_n' u) - A_n'(A_m' u) &= A_m(A_n u) - A_n(A_m u) \\ &= \text{linear combination of } A_1(u), \ldots, A_\mu(u), \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; A_1'(u), \ldots, A_\mu'(u), \end{aligned}$$

for all values of m and n. Hence the system of equations $A_1'(u)=0, \ldots, A_\mu'(u)=0$ is complete.

39. The first of these properties is used to express a complete linear system in a canonical form: the second of them will be used in the establishment of the existence-theorem.

As regards the expression in a canonical form, let a complete linear system of m equations be given, involving one dependent variable u implicitly through its derivatives and $m+n$ independent variables $x_1, \ldots, x_{m+n}$. As the m equations are independent of one another, they can be resolved algebraically so as to express m of the derivatives of u, say $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}$, linearly in terms of the remainder; let their expression be

$$B_t(u) = \frac{\partial u}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u}{\partial x_s} = 0,$$

for $t = 1, \ldots, m$.

The system was complete in its earlier expression: hence, by the preceding property, it remains complete in the changed expression; consequently

$$B_i(B_j u) - B_j(B_i u) = \sum_{k=1}^{m} \xi_k B_k(u),$$

where the quantities ξ do not involve u or its derivatives. The left-hand side of this relation is

$$\sum_{s=m+1}^{m+n} \{B_i(U_{sj}) - B_j(U_{si})\} \frac{\partial u}{\partial x_s},$$

and it does not contain any of the m derivatives $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}$: whereas the right-hand side does contain a derivative $\frac{\partial u}{\partial x_t}$ unless ξ_t is zero. Hence, in order that the relation may be satisfied, each of the quantities $\xi_1, \ldots, \xi_m$ is zero; and it then becomes

$$\sum_{s=m+1}^{m+n} \{B_i(U_{sj}) - B_j(U_{si})\} \frac{\partial u}{\partial x_s} = 0.$$

Now the system is complete, so that no equation of this type is to be associated with it which is not satisfied in virtue of $B_1(u) = 0$, $\ldots$, $B_m(u) = 0$; consequently, this equation must be evanescent for all values of i and j, and therefore

$$B_i(U_{sj}) - B_j(U_{si}) = 0.$$

This relation involves the independent variables only; hence it must be satisfied identically, for all values of i, j, and s.

Conversely, if this relation be satisfied for all values of i, j, and s, then we have

$$B_i(B_j u) - B_j(B_i u) = 0;$$

and the system of equations $B_1(u) = 0, \ldots, B_m(u) = 0$ is evidently complete. Hence we have the formal result:—

A complete linear system of m equations, involving one dependent variable u and m + n independent variables $x_1, \ldots, x_{m+n}$, and such that only derivatives of u occur, the equations being homogeneous in those derivatives, can be expressed in the form

$$B_t(u) = \frac{\partial u}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u}{\partial x_s} = 0,$$

for $t = 1, \ldots, m$; and the conditions, necessary and sufficient to secure that the system should be complete, are the aggregate of the $\frac{1}{2}m(m-1)n$ relations

$$B_i(U_{sj}) - B_j(U_{si}) = 0,$$

for the values of s, and for the combinations of i and j: each of these relations must be satisfied identically.

In consequence of the conditions, the equation

$$B_i(B_j u) - B_j(B_i u) = 0$$

is satisfied identically, for all values of i and j. A set of equations possessing this property is frequently said to be *in involution*.

A complete linear system, expressed in the above form, is sometimes called a *Jacobian system*.

40. The preceding investigation gives the formal conditions for the coexistence of the equations: it gives no information as to the integral or integrals (if any) of those equations. An existence-theorem, similar to those in the preceding chapters, is as follows:

Let $a_1, \ldots, a_{m+n}$ be a set of values of the independent variables in the vicinity of which all the coefficients U, in the complete Jacobian system

$$B_t(u) = \frac{\partial u}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u}{\partial x_s} = 0, \qquad (t = 1, \ldots, m),$$

are regular functions; then the system possesses n functionally distinct integrals, which are regular functions in the vicinity of the selected values and which reduce respectively to values $x_{m+1}, \ldots, x_{m+n}$, when $x_1 = a_1, x_2 = a_2, \ldots, x_m = a_m$.

The theorem has been established* when $m = 1$. The inductive method will be used for the general case; and we shall prove that it is true for a Jacobian system of m equations in $m + n$ independent variables, if it is true for a Jacobian system of $m - 1$ equations in $m + n - 1$ independent variables.

Accordingly, we make the latter supposition that the theorem is true for a complete Jacobian system of $m - 1$ equations in $m + n - 1$ variables. For brevity, we make $a_1 = 0, \ldots, a_{m+n} = 0$: all that would be necessary to secure this result would be to take $y_\mu = x_\mu - a_\mu$.

The equation

$$B_1(u) = \frac{\partial u}{\partial x_1} + \sum_{s=m+1}^{m+n} U_{s1} \frac{\partial u}{\partial x_s} = 0$$

possesses $m + n - 1$ functionally independent integrals, which are regular functions of the variables in finite fields of variation round $0, \ldots, 0$, and which acquire values $x_2, x_3, \ldots, x_{m+n}$ respectively at that place; this is a theorem already proved (§ 29). Of these integrals, $m - 1$ clearly are given by

$$u = x_2, x_3, \ldots, x_m$$

respectively; let the remainder be denoted by $u = y_{m+1}, \ldots, y_{m+n}$ respectively, where

$$y_{m+s} = x_{m+s} + x_1 R_{m+s}, \qquad (s = 1, \ldots, n),$$

* In § 29, Corollary.

R_{m+s} denoting a regular function of the variables $x_1, \ldots, x_{m+n}$ in the assigned vicinity. Reversing these equations so as to express $x_{m+1}, \ldots, x_{m+n}$, we have

$$x_{m+s} = y_{m+s} + x_1 P_{m+s},$$

where P_{m+s} is a regular function of the variables $x_1, \ldots, x_m, y_{m+1}, \ldots, y_{m+n}$ in the vicinity of $0, \ldots, 0$.

Now let the independent variables be changed from $x_1, x_2, \ldots, x_{m+n}$ to $x_1, x_2, \ldots, x_m, y_{m+1}, \ldots, y_{m+n}$; we know, from the property established in § 38, that the new system of equations is complete. Also let the result of the transformation on any integral u be denoted by v. The effect of the transformation upon $B_1(u) = 0$ can be obtained at once: as its $m + n - 1$ functionally independent integrals now are $x_2, \ldots, x_m, y_{m+1}, \ldots, y_{m+n}$, which are the aggregate of independent variables other than x_1, we have

$$B_1(v) = \frac{\partial v}{\partial x_1} = 0.$$

For the other equations, we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i} + \sum_{s=1}^{n} \frac{\partial v}{\partial y_{m+s}} \frac{\partial y_{m+s}}{\partial x_i},$$

for $i = 2, \ldots, m$, and

$$\frac{\partial u}{\partial x_{m+j}} = \sum_{s=1}^{n} \frac{\partial v}{\partial y_{m+s}} \frac{\partial y_{m+s}}{\partial x_{m+j}},$$

for $j = 1, \ldots, n$; hence the equation $B_t(u) = 0$ becomes

$$B_t(v) = \frac{\partial v}{\partial x_t} + \sum_{s=m+1}^{m+n} V_{st} \frac{\partial v}{\partial y_s} = 0,$$

with new coefficients V_{st}.

The properties of these coefficients could be deduced from those of the coefficients U_{st}: they are most simply deduced by the use of the known property that the new system $B_1(v) = 0, \ldots, B_m(v) = 0$ is complete. On account of this property possessed by a Jacobian system (it will be noticed that the new system has the form of a Jacobian system), we have

$$B_1(V_{sj}) - B_j(V_{s1}) = 0,$$

for all values of s and j. Now all the coefficients V_{s1} are zero, and B_1 is $\dfrac{\partial}{\partial x_1}$; hence the foregoing condition is

$$\frac{\partial V_{sj}}{\partial x_1} = 0,$$

that is, the coefficients V_{sj} do not involve x_1. We also have

$$B_i(V_{sj}) - B_j(V_{si}) = 0,$$

for all values of i and j in pairs combined from $2, \ldots, m$, and for $s = m+1, \ldots, m+n$. The modified Jacobian system is

$$B_1(v) = \frac{\partial v}{\partial x_1} = 0,$$

$$B_t(v) = \frac{\partial v}{\partial x_t} + \sum_{s=m+1}^{n} V_{st}\frac{\partial v}{\partial y_s} = 0,$$

for $t = 2, \ldots, m$.

Now the last $m-1$ equations constitute a complete Jacobian system, for the necessary and sufficient conditions

$$B_i(V_{sj}) - B_j(V_{si}) = 0$$

are satisfied; and they are a system in $m+n-1$ independent variables $x_2, \ldots, x_m, y_{m+1}, \ldots, y_{m+n}$, the variable x_1 not occurring. Owing to $B_1(v) = 0$, it follows that an integral of the system of m equations cannot involve x_1 in the modified set of variables: consequently, every integral of the system of m equations in the $m+n$ independent variables is an integral of the system of $m-1$ equations in $m+n-1$ independent variables, and conversely.

The coefficients V_{st} in the Jacobian system of $m-1$ equations are regular functions of the variables in the vicinity of $x_2, \ldots, x_m, y_{m+1}, \ldots, y_{m+n} = 0, \ldots, 0$; for they are polynomial combinations of the coefficients U_{st} and of the derivatives of $y_{m+1}, \ldots, y_{m+n}$ with respect to the original variables, all of which are regular in the assigned vicinity. By the hypothesis adopted for the systems of $m-1$ equations, the Jacobian system of $m-1$ equations in the $m+n-1$ variables possesses n functionally independent integrals which are regular functions of the variables in the domain considered and which reduce respectively to $y_{m+1}, \ldots, y_{m+n}$, when $x_2 = 0, \ldots, x_m = 0$; let these integrals be

$$v_s = y_{m+s} + \phi_{m+s}, \qquad (s = 1, \ldots, n),$$

where ϕ_{m+s} is a regular function of the variables which vanishes, when $x_2 = 0, \ldots, x_m = 0$. It is clear that no one of the quantities $v_1, \ldots, v_n$ contains x_1, so that each of them satisfies

$$B_1(v) = \frac{\partial v}{\partial x_1} = 0.$$

Consequently, they are integrals of the Jacobian system of m equations.

Moreover, these integrals satisfy the assigned conditions; for we have

$$\begin{aligned} v_s &= y_{m+s} + \phi_{m+s} \\ &= x_{m+s} + x_1 R_{m+s} + \phi_{m+s}, \end{aligned}$$

so that as ϕ_{m+s} is still a regular function vanishing when $x_2 = 0$, ..., $x_m = 0$, the integral v_s reduces to x_{m+s}, when we revert to the original variables and we make $x_1 = 0$, $x_2 = 0$, ..., $x_m = 0$.

The theorem is thus true for a complete Jacobian system of m equations in $m+n$ variables, if it is true for such a system of $m-1$ equations in $m+n-1$ variables. It is known to be true for a single equation in any number of variables: hence it is true generally.

The existence of n functionally independent integrals has thus been established. When $m = 1$, it is known that an equation in $n+1$ independent variables possesses n, and not more than n, such integrals; the course of the preceding argument then shews that *a complete Jacobian system of m equations in $m+n$ variables possesses n, and not more than n, functionally independent integrals.*

41. The set of integrals, determined in association with the assigned conditions of § 40 and reducing to $x_{m+1}, \ldots, x_{m+n}$ for assigned values of $x_1, \ldots, x_m$, is sometimes called a *fundamental system* for the assigned vicinity.

As in the case of a single equation, it can be proved that any integral can be expressed in terms of any set of n functionally independent integrals: and, in particular, the expression in terms of the members of a fundamental system is simple.

To prove the first of these statements, let $u_1, \ldots, u_n$ denote a set of functionally independent integrals of a Jacobian system of m equations in $m+n$ independent variables; so that, with the preceding notation for the system, the equations

$$B_t(u_r) = \frac{\partial u_r}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u_r}{\partial x_s} = 0,$$

for $r = 1, \ldots, n$, and $t = 1, \ldots, m$, are satisfied. Moreover, they are satisfied identically, because the quantities $u_1, \ldots, u_n$ do not occur explicitly.

Now, let

$$u = \psi(x_1, \ldots, x_{m+n})$$

denote any integral of the Jacobian system: then the equations

$$\frac{\partial \psi}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial \psi}{\partial x_s} = 0,$$

for $t = 1, \ldots, m$, are satisfied; and they are satisfied identically, because ψ does not occur explicitly. Hence all the determinants

$$\left\| \begin{array}{cccc} \frac{\partial \psi}{\partial x_1}, \ldots, & \frac{\partial \psi}{\partial x_m}, & \frac{\partial \psi}{\partial x_{m+1}}, \ldots, & \frac{\partial \psi}{\partial x_{m+n}} \\ \frac{\partial u_1}{\partial x_1}, \ldots, & \frac{\partial u_1}{\partial x_m}, & \frac{\partial u_1}{\partial x_{m+1}}, \ldots, & \frac{\partial u_1}{\partial x_{m+n}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1}, \ldots, & \frac{\partial u_n}{\partial x_m}, & \frac{\partial u_n}{\partial x_{m+1}}, \ldots, & \frac{\partial u_n}{\partial x_{m+n}} \end{array} \right\|$$

vanish, in association with the integral

$$u = \psi(x_1, \ldots, x_{m+n});$$

but u does not occur explicitly so that, as the determinants cannot vanish in virtue of that integral equation, they must vanish identically. Accordingly, some functional relation must exist among the quantities $\psi, u_1, \ldots, u_n$; it cannot exist among $u_1, \ldots, u_n$ alone, for these are independent; and therefore it must involve ψ. Hence it may be taken in the form

$$\psi = f(u_1, \ldots, u_n),$$

where f is some function of its arguments.

We thus have a verification of the proposition that the number of functionally independent integrals in a Jacobian system of m equations in $m+n$ variables is exactly n.

The form of the function f is not difficult to obtain when ψ is given, if we take a fundamental system of integrals for the set $u_1, \ldots, u_n$. In particular, let the latter be $v_1, \ldots, v_n$, where v_s (for $s = 1, \ldots, n$) assumes the value x_{m+s}, when $x_1 = a_1, \ldots, x_m = a_m$; it is required to determine the functional form g, such that

$$\psi(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) = g(v_1, \ldots, v_n).$$

Let $x_1 = a_1, \ldots, x_m = a_m$; this equation becomes

$$\psi(a_1, \ldots, a_m, x_{m+1}, \ldots, x_{m+n}) = g(x_{m+1}, \ldots, x_{m+n}).$$

This is true for all variables independent of one another; and therefore

$$\psi(a_1, \ldots, a_m, v_1, \ldots, v_n) = g(v_1, \ldots, v_n),$$

so that

$$\psi(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) = \psi(a_1, \ldots, a_m, v_1, \ldots, v_n),$$

being the required expression in terms of a fundamental set of integrals.

COROLLARY. When the preceding results are combined, the following existence-theorem is obvious:—

Let $a_1, \ldots, a_{m+n}$ be a set of values of $x_1, \ldots, x_{m+n}$ such that, in their vicinity, all the coefficients U in the complete Jacobian system

$$\frac{\partial u}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u}{\partial x_s} = 0,$$

for $t = 1, \ldots, m$, are regular functions of the variables; and let $h(x_{m+1}, \ldots, x_{m+n})$ denote any regular function of its arguments in the assigned region of variation, which (except for the requirement of being regular) is arbitrary. Then an integral of the Jacobian system exists, which is a regular function of the variables in the vicinity of $a_1, \ldots, a_{m+n}$, and which acquires the value $h(x_{m+1}, \ldots, x_{m+n})$, when $x_1 = a_1, \ldots, x_m = a_m$.

42. It is a part of Cauchy's existence-theorem that an integral satisfying the conditions:

(i) that it is a regular function of the variables within the domain of a set of values where all coefficients in the above linear equation are regular,

(ii) that it acquires the value of an assigned regular function for an initial value of one of the variables,

is a unique integral so determined. Hence the fundamental system of integrals of the equation

$$p_1 + X_2 p_2 + \ldots + X_{n+1} p_{n+1} = 0,$$

required to acquire values $x_2, \ldots, x_{n+1}$ respectively when $x_1 = a_1$, and to be regular functions of the variables, is unique as a set of integrals.

The inductive proof of the establishment of integrals of a Jacobian system shews that, if a set of integrals satisfying the

assigned conditions be unique for a Jacobian system of $m-1$ equations, a set of integrals satisfying the assigned conditions is unique for a Jacobian system of m equations. The proposition just quoted indicates that a fundamental system is unique when there is a single equation: hence a fundamental set of integrals is unique for a Jacobian system.

Similarly, the integral at the end of § 41, defined as an integral of the Jacobian system

$$\frac{\partial u}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u}{\partial x_s} = 0,$$

for $t = 1, \ldots, m$, which is a regular function of the variables and acquires the value of an assigned regular function of $x_{m+1}, \ldots, x_{m+n}$ for initial values of $x_1, \ldots, x_m$, is easily seen to be a unique integral determined by those conditions.

The property of uniqueness of the integrals is thus established in connection with the various existence-theorems belonging to the Jacobian systems. But it must be remembered that the selected initial values of the variables are such that all the coefficients U_{st} are regular in their vicinity: and only on this hypothesis have the theorems been established. Separate investigation is necessary for the consideration of integrals (if any) of the system in the vicinity of a set of selected initial values of the variables, which constitute a singularity or other non-regular place of any of the coefficients.

Two Methods of Integration of complete linear systems.

43. Now that the existence of integrals of a Jacobian system has been established and that the character of the conditions which limit an integral has been indicated, it is desirable to have some means of actually constructing the integral, more especially if there should be an integral which is expressible in finite terms. Two methods seem more direct for this purpose than others: one of these is due to Mayer, the other is based upon the actual stages in the establishment of the existence of the integrals.

Mayer's method has already* been expounded: consequently the discussion need not be repeated, but the results will be restated for convenience. It is as follows:—

* Vol. I of this work, §§ 41, 42.

To obtain a set of n independent integrals of the complete Jacobian system

$$\frac{\partial u}{\partial x_t} + \sum_{s=m+1}^{m+n} U_{st} \frac{\partial u}{\partial x_s} = 0,$$

for $t = 1, \ldots, m$, we transform the variables $x_1, \ldots, x_m$ by the substitutions

$$x_t = \alpha_t + (y_1 - \theta) f_t (y_1, \ldots, y_m),$$

and construct the equation

$$\frac{\partial u}{\partial y_1} + \sum_{s=m+1}^{m+n} Y_s \frac{\partial u}{\partial x_s} = 0.$$

The equations subsidiary to this single equation, viz.

$$dy_1 = \frac{dx_{m+1}}{Y_{m+1}} = \ldots = \frac{dx_{m+n}}{Y_{m+n}},$$

are to be integrated, keeping $y_2, \ldots, y_m$ as invariable quantities: let the n integrals be

$$\phi_p (x_{m+1}, \ldots, x_{m+n}, y_1, \ldots, y_m) = \text{constant},$$

for $p = 1, \ldots, n$. Then the set of n independent integrals of the Jacobian system are given by the following process: in the equations

$$\phi_p (x_{m+1}, \ldots, x_{m+n}, y_1, y_2, \ldots, y_m) = \phi_p (c_1, \ldots, c_n, \theta, y_2, \ldots, y_m),$$

the variables $y_1, \ldots, y_m$ are to be replaced by their values in terms of the variables $x_1, \ldots, x_m$, and if, in any of the equations, $\phi_p (c_1, \ldots, c_n, \theta, y_2, \ldots, y_m)$ should be a pure constant, the changed equation is

$$\phi_p (x_{m+1}, \ldots, x_{m+n}, x_1, \ldots, x_m) = \phi_p (c_1, \ldots, c_n, \alpha_1, \ldots, \alpha_m).$$

These n equations are resolved so as to give $c_1, \ldots, c_n$ (or n independent functional combinations of them): let the result of the resolution be

$$u_\mu (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) = C_\mu, \qquad (\mu = 1, \ldots, n),$$

where $C_1, \ldots, C_n$ are n independent functions of $c_1, \ldots, c_n$: the n integrals of the original system are

$$u = u_\mu (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}), \qquad (\mu = 1, \ldots, n).$$

Note 1. The simplest substitutions for the transformation of the variables appear to be

$$x_1 = y_1,$$
$$x_t = \alpha_t + (y_1 - \alpha_1) y_t,$$

for $t = 2, \ldots, m$; the quantities $Y_{m+1}, \ldots, Y_{m+n}$ in the subsidiary equations are given by

$$Y_{m+s} = U_{s1} + \sum_{t=2}^{m} U_{st} y_t.$$

Note 2. If only a single integral of the Jacobian system is wanted and not a full set, it can certainly be obtained from any one integral of the subsidiary system.

Note 3. If any integrals of a Jacobian system are known, they can be used to modify the system, so as to reduce the amount of integration necessary to complete the set. Thus let

$$u = y_1, \quad u = y_2, \ldots, \quad u = y_p,$$

be known integrals independent of one another, where $p < n$; and use these p quantities to change the variables from $x_1, \ldots, x_{m+n}$ to (say) $x_1, \ldots, x_{m+n-p}, y_1, \ldots, y_p$. Then as y_1 is an integral of the system, the term $\frac{\partial u}{\partial y_1}$ must be absent from each equation of the modified system: its coefficient must vanish in order that the equation may be satisfied. Similarly for $\frac{\partial u}{\partial y_2}, \ldots, \frac{\partial u}{\partial y_p}$. Thus there will be a modified system of m equations: the variables $y_1, \ldots, y_p$ are of the nature of parameters: it involves $m + n - p$ variables $x_1, \ldots, x_{m+n-p}$; and it still is complete. It therefore possesses $n - p$ integrals; and these can be obtained, as in Mayer's method, by the integration of the $n - p$ subsidiary ordinary equations.

44. In outline, and as regards the theoretic amount of inverse operations (such as integration) that are required, Mayer's method for the integration of complete linear systems is the simplest and the briefest: but occasionally, for particular systems, the detailed operations can be complicated. An alternative method of proceeding is provided by an adaptation of Jacobi's method of integrating partial differential equations; the details of the adaptation are almost dictated by the course of the proof of the existence-theorem. In details, it frequently is simpler than Mayer's method, though the number of inverse operations is greater: but the mere number of such operations, without regard paid to their intrinsic difficulty, is not the only trustworthy criterion of practical simplicity.

The method may be described as that of successive reduction. Let the system be taken in its canonical form, the first equation being

$$B_1(u) = \frac{\partial u}{\partial x_1} + \sum_{s=m+1}^{m+n} U_{s1} \frac{\partial u}{\partial x_s} = 0.$$

This equation in $m + n$ variables has $m + n - 1$ functionally independent integrals; of these, $m - 1$ are evidently given by $x_2, \ldots, x_m$, and the remaining n are provided by integrals of the subsidiary equations

$$dx_1 = \frac{dx_{m+1}}{U_{m+1,1}} = \ldots = \frac{dx_{m+n}}{U_{m+n,1}},$$

the quantities $x_2, \ldots, x_m$ being regarded as parametric. If the integrals of these subsidiary equations are

$$u_\mu(x_1, \ldots, x_{m+n}) = \text{constant}, \qquad (\mu = 1, \ldots, n),$$

then the n remaining integrals of $B_1(u) = 0$ are given by

$$u = u_\mu(x_1, \ldots, x_{m+n}) = u_\mu.$$

Every integral of $B_1(u) = 0$ is a functional combination of $x_2, \ldots, x_m, u_1, \ldots, u_n$; the appropriate functional combinations must be such as to satisfy the remaining equations of the system.

We accordingly make $x_2, \ldots, x_m, u_1, \ldots, u_n$ the independent variables for the equation $B_2(u) = 0$. If any integral of this equation be taken in the form

$$u = f(x_2, \ldots, x_m, u_1, \ldots, u_n),$$

which is also an integral of $B_1(u) = 0$, we have

$$B_2(u) = \frac{\partial f}{\partial x_2} + \sum_{r=1}^{n} \frac{\partial f}{\partial u_r} B_2(u_r) = 0.$$

Because the system is complete, we have

$$B_1(B_2 u_r) - B_2(B_1 u_r) = 0;$$

but $B_1(u_r)$ vanishes identically, so that $B_2(B_1 u_r) = 0$, and therefore

$$B_1(B_2 u_r) = 0.$$

Hence $B_2(u_r)$ satisfies the equation $B_1(u) = 0$; it may be zero, or it may be a pure constant: if it is neither of these but is variable, it is an integral of $B_1(u) = 0$, and therefore can be expressed in terms of $x_2, \ldots, x_m, u_1, \ldots, u_n$. Thus all the coefficients in the transformed expression of $B_2(u) = 0$ are functions of the $m + n - 1$

new variables alone. The equation in this form has $m+n-2$ functionally independent integrals; of these, $m-2$ are given by $x_3, \ldots, x_m$; and the remaining n are provided by integrals of the subsidiary equations

$$dx_2 = \frac{du_1}{B_2(u_1)} = \ldots = \frac{du_n}{B_2(u_n)},$$

the quantities $x_3, \ldots, x_m$ being regarded as parametric. All the denominators, if not zero or pure constants, are functions of $x_2, \ldots, x_m, u_1, \ldots, u_n$; let the integrals of this set be

$$v_\rho(x_2, \ldots, x_m, u_1, \ldots, u_n) = \text{constant}, \qquad (\rho = 1, \ldots, n);$$

then the n remaining integrals of $B_2(u) = 0$ are

$$u = v_\rho(x_2, \ldots, x_m, u_1, \ldots, u_n) = v_\rho.$$

Each of these, as a functional combination of $x_2, \ldots, x_m, u_1, \ldots, u_n$, is an integral of $B_1(u) = 0$; and every integral, common to $B_1(u)=0$ and $B_2(u) = 0$, is a functional combination of $x_2, \ldots, x_m, v_1, \ldots, v_n$. The appropriate functional combinations must be chosen so as to satisfy the remaining equations of the Jacobian system.

We now proceed as before: and for the third equation, we make $x_3, \ldots, x_m, v_1, \ldots, v_n$ the independent variables. If any integral of the equation $B_3(u) = 0$ be

$$u = \phi(x_3, \ldots, x_m, v_1, \ldots, v_n),$$

we have

$$B_3(u) = \frac{\partial \phi}{\partial x_3} + \sum_{r=1}^{n} \frac{\partial \phi}{\partial v_r} B_3(v_r) = 0.$$

But, as the system is complete, we have

$$B_1(B_3 v_r) = B_3(B_1 v_r) = 0,$$
$$B_2(B_3 v_r) = B_3(B_2 v_r) = 0,$$

because $B_1(v_r)$ and $B_2(v_r)$ vanish identically; therefore $B_3(v_r)$ is a simultaneous integral of $B_1(u) = 0$ and $B_2(u) = 0$. Consequently $B_3(v_r)$ is either zero, or a pure constant, or a function of $x_3, \ldots, x_m, v_1, \ldots, v_n$; and the coefficients in the modified form of $B_3(u)=0$ involve only the variables which occur in the derivatives of ϕ. The position is now the same as in the preceding stage, except that the number of variables has been decreased by unity.

We pass thus from stage to stage: the integrals at the last stage are n functionally independent integrals of the system.

Note. At first sight, it would appear as though the number of quadratures of ordinary equations, required to make the process effective, is mn, being n for each stage. But the number can be reduced, often very substantially, except at the last stage when n such quadratures are then certainly required. For example, let

$$u'(x_1, \ldots, x_{m+n}) = \text{constant}$$

be any integral of the subsidiary system of $B_1(u) = 0$: then $u = u'(x_1, \ldots, x_{m+n}) = u'$ is an integral of $B_1(u) = 0$. Now for any value of p, we have

$$\begin{aligned} B_1(B_p u') &= B_p(B_1 u') \\ &= 0, \end{aligned}$$

because $B_1(u')$ vanishes identically: hence $B_p(u')$ satisfies $B_1(u) = 0$. If $B_p(u')$ is not zero and is not a pure constant, it is an integral of $B_1(u) = 0$; if it is functionally independent of u', we may write

$$u'' = B_p(u');$$

and we thus obtain a new integral of $B_1(u) = 0$ without any further quadrature, in the case of each operator B_p that leads to a result of this type.

Again, each new integral so obtained may be similarly treated, until possibly an adequate number of integrals has been obtained at the stage. The reduction in the number of quadratures may thus be made by means of the operators in the remaining equations of the system at any stage: it clearly cannot be made at the last stage when no further operator remains for consideration.

Further, if $B_p(u')$ is zero, then u' is an integral common to $B_1(u) = 0$ and $B_p(u) = 0$; when retained as a new independent variable under transformation of the variables, the integration of $B_p(u) = 0$ will be thereby simplified.

Again, if $B_p(u')$ be a pure constant, $= a$ say, and if $B_p(v')$, derived from a functionally distinct integral of $B_1(u) = 0$, be also a pure constant, $= b$ say, then

$$B_p(bu' - av') = 0,$$

that is, $bu' - av'$ is an integral common to $B_1(u) = 0$ and $B_p(u) = 0$; it can be used to simplify the integration of $B_p(u) = 0$ at the appropriate stage.

Thus the number of quadratures necessary for the method may be considerably reduced: but even in the most favourable circumstances, their number is greater than the number in Mayer's method.

Ex. 1. As an example, which will be integrated by both methods, consider the system

$$X_1(z)=x_1p_1-x_2p_2+x_3p_3-x_4p_4=0,$$
$$X_2(z)=x_3p_1-x_1p_3=0,$$
$$X_3(z)=x_4p_2-x_2p_4=0,$$

where $p_\mu=\dfrac{\partial z}{\partial x_\mu}$, for $\mu=1, 2, 3, 4$. We have

$$X_1(X_2z)-X_2(X_1z)=0,$$
$$X_1(X_3z)-X_3(X_1z)=0,$$
$$X_2(X_3z)-X_3(X_2z)=0,$$

so that the system is a complete linear system, being a system in involution. When expressed in a canonical form, it is

$$\xi_1(z)=p_1-\frac{x_1}{x_4}\frac{x_2^2+x_4^2}{x_1^2+x_3^2}p_4=0,$$

$$\xi_2(z)=p_2-\frac{x_2}{x_4}p_4=0,$$

$$\xi_3(z)=p_3-\frac{x_3}{x_4}\frac{x_2^2+x_4^2}{x_1^2+x_3^2}p_4=0.$$

Adopting Mayer's method of integration, we make the transformations

$$x_1=y_1,\quad x_2=a_2+y_1y_2,\quad x_3=a_3+y_1y_3;$$

and then the single equation to be considered is

$$\frac{\partial z}{\partial y_1}-Y\frac{\partial z}{\partial x_4}=0,$$

where

$$Y=\frac{y_1+y_3(a_3+y_1y_3)}{x_4}\frac{(a_2+y_1y_2)^2+x_4^2}{y_1^2+(a_3+y_1y_3)^2}+y_2\frac{a_2+y_1y_2}{x_4}.$$

The subsidiary equation is

$$dy_1+\frac{dx_4}{Y}=0;$$

and an integral is found to be

$$\{y_1^2+(a_3+y_1y_3)^2\}\{(a_2+y_1y_2)^2+x_4^2\}=\text{constant}.$$

Accordingly, by the theorem quoted in § 43, we construct the equation

$$\{y_1^2+(a_3+y_1y_3)^2\}\{(a_2+y_1y_2)^2+x_4^2\}=a_2^2a_3^2,$$

the right-hand side being obtained by putting y_1 equal to zero in the left; and then, replacing the variables x_1, x_2, x_3, we have

$$(x_1^2+x_3^2)(x_2^2+x_4^2)=a_2^2a_3^2,$$

that is, by the theorem, a common integral is

$$z=(x_1^2+x_3^2)(x_2^2+x_4^2).$$

A more general common integral is

$$z=F\{(x_1^2+x_3^2)(x_2^2+x_4^2)\},$$

where F is any arbitrary function.

Proceeding by the other method of integration, we obtain an integral, other than x_2 and x_3, of

$$p_1-\frac{x_1}{x_4}\frac{x_2^2+x_4^2}{x_1^2+x_3^2}p_4=0:$$

the subsidiary system is

$$dx_1+\frac{dx_4}{\dfrac{x_1}{x_4}\dfrac{x_2^2+x_4^2}{x_1^2+x_3^2}}=0,$$

an integral of which is

$$(x_2^2+x_4^2)(x_1^2+x_3^2)=\text{constant}.$$

We take x_2, x_3, v as the independent variables, where

$$v=(x_2^2+x_4^2)(x_1^2+x_3^2).$$

But $\xi_2(v)=0$, so that the second equation becomes

$$\frac{\partial z}{\partial x_2}=0:$$

and any integral common to the first two equations is a function of x_3 and v.

We take x_3 and v as the independent variables for the third equation. But $\xi_3(v)=0$, so that the third equation becomes

$$\frac{\partial z}{\partial x_3}=0.$$

The integral is thus a function of v; a common integral of the system is, as before,

$$z=v=(x_1^2+x_3^2)(x_2^2+x_4^2).$$

If an integral is required to attain an assigned value $g(x_4)$, when $x_1=a$, $x_2=b$, $x_3=c$, it is easily seen to be

$$z=g\left\{\left(\frac{v}{a^2+c^2}-b^2\right)^{\frac{1}{2}}\right\}.$$

Ex. 2. Prove that the system

$$0=p_1+2x_1p_2+3x_2p_3+p_5+x_4p_6,$$

$$0=x_1p_1+2x_2p_2+3x_3p_3+x_5p_5+x_6p_6,$$

$$0=(3x_1^2-2x_2)p_1+(3x_1x_2-x_3)p_2+3x_1x_3p_3+(x_4x_5-x_6)p_4+x_5^2p_5+x_5x_6p_6,$$

is complete: and find a system of three common integrals.

Complete linear Systems that are not homogeneous.

45. The complete linear systems that have been considered are homogeneous in the derivatives of z: and the dependent variable does not explicitly occur. But it is possible to have complete linear systems which are not homogeneous in the derivatives and in which the dependent variable does occur explicitly. This class of equations is a very special example of a system of simultaneous equations and can be treated by the general method devised for general systems: the equations can, however, be more simply treated by being included under the class already considered. We take a new dependent variable u such that

$$u = u(z, x_1, \ldots, x_n),$$

and we transform the equations by means of relations

$$\frac{\partial u}{\partial z} p_m + \frac{\partial u}{\partial x_m} = 0:$$

the transformed equations are homogeneous and u does not occur explicitly. These are amenable to the method already explained: the conditions of coexistence are at once obtainable; and integrals will be given by equations

$$u = \text{constant},$$

provided u involves z.

Note. The same warning must be applied about linear non-homogeneous systems as was applied to a single non-homogeneous equation (§ 35). The method does not necessarily give all the integrals of such a system, for it may fail to give those which belong to the residuary class called *special*.

46. The conditions of the coexistence and the completeness of the system can be easily obtained from the transformed system. Thus let a given linear system be expressed in the form

$$E_1(z) = p_1 + \sum_{s=m+1}^{m+n} a_{s1} p_s \ = Z_1,$$

$$E_2(z) = p_2 + \sum_{s=m+1}^{m+n} a_{s2} p_s \ = Z_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E_m(z) = p_m + \sum_{s=m+1}^{m+n} a_{sm} p_s = Z_m;$$

and consider the system

$$B_1(u) = \frac{\partial u}{\partial x_1} + \sum_{s=m+1}^{m+n} a_{s1} \frac{\partial u}{\partial x_s} + Z_1 \frac{\partial u}{\partial z} = 0,$$

$$B_2(u) = \frac{\partial u}{\partial x_2} + \sum_{s=m+1}^{m+n} a_{s2} \frac{\partial u}{\partial x_s} + Z_2 \frac{\partial u}{\partial z} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B_m(u) = \frac{\partial u}{\partial x_m} + \sum_{s=m+1}^{m+n} a_{sm} \frac{\partial u}{\partial x_s} + Z_m \frac{\partial u}{\partial z} = 0,$$

the quantities $a_{\lambda\mu}$ and Z being functions of $z, x_1, \ldots, x_n$. The conditions of completeness of the latter system are

$$B_i(a_{sj}) = B_j(a_{si}),$$

$$B_i(Z_j) = B_j(Z_i),$$

for all pairs of values $i, j = 1, \ldots, m$, and for $s = m+1, \ldots, m+n$. But for any value of r, we have

$$B_r = E_r + Z_r \frac{\partial}{\partial z},$$

as a relation between the operators; thus the above conditions become

$$E_i(a_{sj}) + Z_i \frac{\partial a_{sj}}{\partial z} = E_j(a_{si}) + Z_j \frac{\partial a_{si}}{\partial z},$$

$$E_i(Z_j) + Z_i \frac{\partial Z_j}{\partial z} = E_j(Z_i) + Z_j \frac{\partial Z_i}{\partial z},$$

for all pairs of values $i, j = 1, \ldots, m$, and for $s = m+1, \ldots, m+n$. These are the conditions, necessary and sufficient to secure the coexistence and completeness of the system.

We know that the system of equations $B_1(u) = 0, \ldots, B_m(u) = 0$, being a complete linear system of m equations in $m+n+1$ variables, possesses $n+1$ functionally distinct integrals. Let a set of these be taken in the form $u_1, \ldots, u_{n+1}$, some of which will certainly involve z; then any integral of the transformed system can be expressed in a form

$$u = f(u_1, \ldots, u_{n+1}),$$

and its most general integral will be obtained by taking f as a completely arbitrary function. It is also obvious that an integral of the original system will be provided by an equation

$$u = 0,$$

if u is an integral of the transformed system which involves z; hence a very general integral of the original system will be provided by the equation

$$f(u_1, \ldots, u_{n+1}) = 0.$$

But for reasons similar to those adduced for a single equation in § 35, we are not in a position to declare (and it is not, in fact, true) that every integral of the original system of equations is included in the equation $f=0$ for an appropriate form of f.

As the quantities $u_1, \ldots, u_{n+1}$, necessary for the construction of the function f, are simultaneous integrals of the system $B_1(u)=0, \ldots, B_m(u)=0$, it is clear that either of the two methods (in §§ 43, 44) effective for the construction of $u_1, \ldots, u_{n+1}$ can be adopted.

Ex. Let it be required to find whether the equations

$$x_1p_1+x_2p_2-x_3p_3+z \;=0,$$
$$x_2p_1-x_1p_2+zp_3 \;+x_3=0,$$

have any common integral.

Expressing these equations in the form

$$p_1+\frac{x_2z-x_1x_3}{x_1^2+x_2^2}p_3=-\frac{x_1z+x_2x_3}{x_1^2+x_2^2},$$

$$p_2-\frac{x_1z+x_2x_3}{x_1^2+x_2^2}p_3=-\frac{x_2z-x_1x_3}{x_1^2+x_2^2},$$

and applying the conditions of the text, we find them satisfied: hence the equations coexist, and they form a complete system.

To obtain the general common integral, we construct the equations

$$\frac{\partial u}{\partial x_1}+\frac{x_2z-x_1x_3}{x_1^2+x_2^2}\frac{\partial u}{\partial x_3}-\frac{x_1z+x_2x_3}{x_1^2+x_2^2}\frac{\partial u}{\partial z}=0,$$

$$\frac{\partial u}{\partial x_2}-\frac{x_1z+x_2x_3}{x_1^2+x_2^2}\frac{\partial u}{\partial x_3}-\frac{x_2z-x_1x_3}{x_1^2+x_2^2}\frac{\partial u}{\partial z}=0,$$

which are a Jacobian system. It possesses two functionally distinct integrals: these are found, by the processes previously explained, to be

$$u=u_1=x_1x_3+x_2z, \qquad u=u_2=x_2x_3-x_1z.$$

A general integral, common to the two original equations, is given by

$$x_1x_3+x_2z=\phi(x_2x_3-x_1z),$$

where ϕ is an arbitrary functional form.

CHAPTER IV.

NON-LINEAR EQUATIONS: JACOBI'S SECOND METHOD, WITH MAYER'S DEVELOPMENTS.

FOR the material of the present chapter, reference may be made to Jacobi's posthumous memoir, "Nova methodus......integrandi," *Crelle*, t. LX (1862), pp. 1—181, *Ges. Werke*, t. v, pp. 1—189; to Mayer's memoir, "Ueber unbeschränkt......Differentialgleichungen," *Math. Ann.*, t. v (1872), pp. 448—470; and to Imschenetsky's memoir "Sur l'intégration......premier ordre," *Grunert's Archiv*, t. L (1869), pp. 278—474. Mention should also be made of Mansion's treatise "Théorie des équations aux derivées partielles du premier ordre" (1875), Book II; and of the exposition given in chapters VI and VII of Goursat's treatise, already (p. 55) quoted.

47. We now proceed to deal with single equations, and with systems of consistent equations, of the first order and of general degree in the derivatives: clearly no generality is lost by assuming that the equations are irreducible. It will be sufficiently obvious from the discussion in the last chapter that the construction of an integral of the equation or of the system of equations is a process of several stages, differing in this respect from the usual construction of an integral of an ordinary equation; and the difficulty, in general, is the discovery of the effective inverse operations that lead from stage to stage.

Now, whatever equation or equations may be assigned for the determination of the value or for the limitation of the form of a dependent variable, one permanent relation subsists between a number of independent variables $x_1, \ldots, x_n$, a dependent variable z, and the derivatives $p_1, \ldots, p_n$ of the latter: the relation is

$$dz = p_1 dx_1 + \ldots + p_n dx_n.$$

The quantities $p_1, \ldots, p_n$ are themselves dependent variables and consequently are functions of $x_1, \ldots, x_n$: but it frequently happens

that they arise as functions of $x_1, \ldots, x_n, z$, the last variable not being explicitly known in terms of the independent variables. Also, there must be only a single functional relation between $z, x_1, \ldots, x_n$, so that the integral equivalent of the preceding differential relation is effectively a single equation among the variables: consequently, the differential relation must be an exact equation. The conditions necessary and sufficient to secure this result are known*: in the present case, they are

$$\frac{\partial p_\mu}{\partial x_m} - \frac{\partial p_m}{\partial x_\mu} + p_m \frac{\partial p_\mu}{\partial z} - p_\mu \frac{\partial p_m}{\partial z} = 0,$$

for the $\frac{1}{2}n(n-1)$ pairs of values of m and μ from the set $1, \ldots, n$. Let

$$\frac{d}{dx_\rho} = \frac{\partial}{\partial x_\rho} + p_\rho \frac{\partial}{\partial z},$$

so that $\dfrac{d}{dx_\rho}$ is the complete derivative with regard to x_ρ, account being taken of the explicit occurrence of x_ρ as well as of its implicit occurrence through z: the necessary and sufficient conditions become

$$\frac{dp_\mu}{dx_m} = \frac{dp_m}{dx_\mu};$$

and these conditions apply, whatever be the quantity z and however its derivatives $p_1, \ldots, p_n$ may be determined. When they are satisfied, the relation

$$dz = p_1 dx_1 + \ldots + p_n dx_n$$

is exact: when an integral equivalent is obtained by the recognised processes of quadrature, that equivalent is an integral relation between $z, x_1, \ldots, x_n$.

Now there are n of these derivatives of z: when regarded for the purpose of quadrature, they will most generally be determined by n equations

$$F_1 = 0, \; F_2 = 0, \; \ldots, \; F_n = 0,$$

where $F_1, \ldots, F_n$ will be assumed to be n regular functions of $x_1, \ldots, x_n, z, p_1, \ldots, p_n$ which, so far as they involve $p_1, \ldots, p_n$, are functionally distinct; consequently the Jacobian

$$J\left(\frac{F_1, \ldots, F_n}{p_1, \ldots, p_n}\right), = J \text{ say},$$

* Part I of this work, § 11.

does not vanish identically. In that case, the n equations can be resolved so as to give expressions for $p_1, \ldots, p_n$ as regular functions of $x_1, \ldots, x_n, z$: when these expressions are substituted in the equations, the latter become identities. Taking two of these equations, say $F_r = 0$ and $F_s = 0$, thus turned into identities, we have, on differentiating with regard to x_m,

$$\frac{dF_r}{dx_m} + \sum_{\mu=1}^{n} \frac{\partial F_r}{\partial p_\mu} \frac{dp_\mu}{dx_m} = 0,$$

$$\frac{dF_s}{dx_m} + \sum_{\mu=1}^{n} \frac{\partial F_s}{\partial p_\mu} \frac{dp_\mu}{dx_m} = 0,$$

and therefore, on the elimination of $\dfrac{dp_m}{dx_m}$,

$$\frac{dF_r}{dx_m}\frac{\partial F_s}{\partial p_m} - \frac{dF_s}{dx_m}\frac{\partial F_r}{\partial p_m} + \sum_{\mu=1}^{n} \frac{\partial (F_r, F_s)}{\partial (p_\mu, p_m)} \frac{dp_\mu}{dx_m} = 0.$$

This holds for each value of m; taking it then in succession for each value of m, and adding all the left-hand sides together, we have

$$\sum_{m=1}^{n} \left(\frac{dF_r}{dx_m}\frac{\partial F_s}{\partial p_m} - \frac{dF_s}{dx_m}\frac{\partial F_r}{\partial p_m}\right) + \sum_{m=1}^{n} \sum_{\mu=1}^{n} \frac{\partial (F_r, F_s)}{\partial (p_\mu, p_m)} \frac{dp_\mu}{dx_m} = 0.$$

The last double summation can be modified: the terms for which $\mu = m$ do not occur: taking a pair of values for μ and m from the set $1, \ldots, n$, and combining them, the summation may be written

$$\sum_{\mu, m}^{n}{}' \frac{\partial (F_r, F_s)}{\partial (p_\mu, p_m)} \left(\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}\right).$$

Moreover, it is convenient to use a symbol to denote the first summation: we write

$$[F_r, F_s] = \sum_{m=1}^{n} \left(\frac{dF_r}{dx_m}\frac{\partial F_s}{\partial p_m} - \frac{dF_s}{dx_m}\frac{\partial F_r}{\partial p_m}\right);$$

and the equation now becomes

$$[F_r, F_s] + \sum_{\mu, m}^{n}{}' \frac{\partial (F_r, F_s)}{\partial (p_\mu, p_m)} \left(\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}\right) = 0.$$

This holds for all combinations of r and s from the set $1, \ldots, n$.

SIGNIFICANCE OF THE JACOBIAN RELATIONS.

48. Two inferences can be drawn from this aggregate of relations.

In the first place, the quantities $p_1, \ldots, p_n$, as determined by the equations $F_1 = 0, \ldots, F_n = 0$, have thus far merely been regarded as variable magnitudes: but, in addition, they are to be derivatives of z. The conditions, necessary and sufficient to secure this last property, are

$$\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu} = 0\,;$$

hence we have

$$[F_r,\ F_s] = 0,$$

for all values of r and s. These equations, $\frac{1}{2}n(n-1)$ in number, are thus a necessary consequence of the hypothesis that the quantities p are the derivatives of z.

In the second place, if these $\frac{1}{2}n(n-1)$ equations are satisfied, then the quantities $p_1, \ldots, p_n$, determined by the equations $F_1 = 0, \ldots, F_n = 0$, are the n first derivatives of z with regard to $x_1, \ldots, x_n$. Assuming the equations to be satisfied, the foregoing aggregate of relations becomes

$$\sum_{\mu,\, m}^{n}{}' \frac{\partial(F_r,\ F_s)}{\partial(p_\mu,\ p_m)}\left(\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}\right) = 0,$$

for all combinations of r and s. We thus have $\frac{1}{2}n(n-1)$ equations, homogeneous and linear in the $\frac{1}{2}n(n-1)$ quantities

$$\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}.$$

Taking the equations in the form

$$\sum_{\mu=1}^{n} \sum_{m=1}^{n} \frac{\partial F_r}{\partial p_\mu} \frac{\partial F_s}{\partial p_m}\left(\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}\right) = 0,$$

which is only a rearrangement, and writing

$$\sum_{m=1}^{n} \frac{\partial F_s}{\partial p_m}\left(\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}\right) = u_{\mu s},$$

we have

$$\sum_{\mu=1}^{n} \frac{\partial F_r}{\partial p_\mu} u_{\mu s} = 0.$$

This equation holds for all values of r and of s: taking one value of s, and the n values of r in turn, we have n equations which are

linear and homogeneous in $u_{1s}, u_{2s}, \ldots, u_{ns}$. The determinant of the coefficients does not vanish, for it is the Jacobian of the functions $F_1, \ldots, F_n$ with regard to $p_1, \ldots, p_n$; hence

$$u_{\mu s} = 0,$$

for all values of μ and s, that is,

$$\sum_{m=1}^{n} \frac{\partial F_s}{\partial p_m}\left(\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu}\right) = 0.$$

Taking the n values of s in turn, we have n equations which are linear and homogeneous in the quantities

$$\frac{dp_\mu}{dx_1} - \frac{dp_1}{dx_\mu}, \ldots, \frac{dp_\mu}{dx_n} - \frac{dp_n}{dx_\mu};$$

the determinant of their coefficients does not vanish, for again it is the Jacobian of $F_1, \ldots, F_n$ with regard to $p_1, \ldots, p_n$: hence

$$\frac{dp_\mu}{dx_m} - \frac{dp_m}{dx_\mu} = 0,$$

for all values of m and μ. These conditions have been proved necessary and sufficient to secure that the quantities p are derivatives of z; and they are a necessary consequence of the equations

$$[F_r . F_s] = 0,$$

which therefore are sufficient to secure that the quantities p are derivatives of z and that, when their values given by $F_1 = 0, \ldots, F_n = 0$ are substituted in the equation

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

this equation is exact.

If, in particular, z does not occur explicitly in any of the equations $F = 0$, then

$$\frac{dF_\mu}{dx_m} = \frac{\partial F_\mu}{\partial x_m}:$$

the equations become

$$\sum_{m=1}^{n}\left(\frac{\partial F_r}{\partial x_m}\frac{\partial F_s}{\partial p_m} - \frac{\partial F_r}{\partial p_m}\frac{\partial F_s}{\partial x_m}\right) = 0,$$

and these are frequently represented by the form

$$(F_r, F_s) = 0.$$

Note. All these conditions will equally be required if the equations determining $p_1, \ldots, p_n$ occur in the form

$$F_1 = a_1, \ldots, F_n = a_n,$$

where $a_1, \ldots, a_n$ are constants.

49. Again, suppose that $n+1$ equations, involving $p_1, \ldots, p_n$, $z, x_1, \ldots, x_n$, are given. Let them be

$$G_1 = 0, \ldots, G_{n+1} = 0;$$

they can be regarded as determining $n+1$ quantities $z, p_1, \ldots, p_n$ in terms of $x_1, \ldots, x_n$. We proceed to shew that the conditions

$$[G_r, G_s] = 0,$$

for all combinations of r and s from the set $1, \ldots, n+1$, must be satisfied, if quantities $z, p_1, \ldots, p_n$ are so related that

$$\frac{\partial z}{\partial x_m} = p_m, \quad \frac{\partial p_m}{\partial x_\mu} = \frac{\partial p_\mu}{\partial x_m};$$

also, that the conditions specified suffice to secure these relations.

When the values of $z, p_1, \ldots, p_n$ are substituted in the equations $G = 0$, each of them becomes an identity; and therefore we have, from any equation $G_r = 0$ after the substitution,

$$\frac{\partial G_r}{\partial x_m} + \frac{\partial G_r}{\partial z}\frac{\partial z}{\partial x_m} + \sum_{\mu=1}^{n} \frac{\partial G_r}{\partial p_\mu}\frac{\partial p_\mu}{\partial x_m} = 0,$$

so that

$$\frac{\partial G_r}{\partial x_m} + p_m \frac{\partial G_r}{\partial z} + \frac{\partial G_r}{\partial z}\left(\frac{\partial z}{\partial x_m} - p_m\right) + \sum_{\mu=1}^{n} \frac{\partial G_r}{\partial p_\mu}\frac{\partial p_\mu}{\partial x_m} = 0,$$

or writing

$$\frac{\partial}{\partial x_m} + p_m \frac{\partial}{\partial z} = \frac{d}{dx_m},$$

for all values of m, we have

$$\frac{dG_r}{dx_m} + \frac{\partial G_r}{\partial z}\left(\frac{\partial z}{\partial x_m} - p_m\right) + \sum_{\mu=1}^{n} \frac{\partial G_r}{\partial p_\mu}\frac{\partial p_\mu}{\partial x_m} = 0.$$

Similarly

$$\frac{dG_s}{dx_m} + \frac{\partial G_s}{\partial z}\left(\frac{\partial z}{\partial x_m} - p_m\right) + \sum_{\mu=1}^{n} \frac{\partial G_s}{\partial p_\mu}\frac{\partial p_\mu}{\partial x_m} = 0.$$

Multiplying the former by $\dfrac{\partial G_s}{\partial p_m}$, the latter by $\dfrac{\partial G_r}{\partial p_m}$, and subtracting we have

$$\frac{dG_r}{dx_m}\frac{\partial G_s}{\partial p_m} - \frac{dG_s}{dx_m}\frac{\partial G_r}{\partial p_m} + \left(\frac{\partial G_r}{\partial z}\frac{\partial G_s}{\partial p_m} - \frac{\partial G_s}{\partial z}\frac{\partial G_r}{\partial p_m}\right)\left(\frac{\partial z}{\partial x_m} - p_m\right)$$
$$+ \sum_{\mu=1}^{n} \frac{\partial(G_r, G_s)}{\partial(p_\mu, p_m)}\frac{\partial p_\mu}{\partial x_m} = 0.$$

Summing the left-hand sides of this equation, taken for all the n values of m in succession, we have

$$\sum_{m=1}^{n}\left(\frac{dG_r}{dx_m}\frac{\partial G_s}{\partial p_m}-\frac{dG_s}{dx_m}\frac{\partial G_r}{\partial p_m}\right)$$

$$+\sum_{m=1}^{n}\frac{\partial(G_r, G_s)}{\partial(z, p_m)}\left(\frac{\partial z}{\partial x_m}-p_m\right)+\sum_{m=1}^{n}\sum_{\mu=1}^{n}\frac{\partial(G_r, G_s)}{\partial(p_\mu, p_m)}\frac{\partial p_\mu}{\partial x_m}=0,$$

or, again using $[G_r, G_s]$ to denote the first summation, we have

$$[G_r, G_s]+\sum_{m=1}^{n}\frac{\partial(G_r, G_s)}{\partial(z, p_m)}\left(\frac{\partial z}{\partial x_m}-p_m\right)+\sum_{m=1}^{n}\sum_{\mu=1}^{n}\frac{\partial(G_r, G_s)}{\partial(p_\mu, p_m)}\frac{\partial p_\mu}{\partial x_m}=0.$$

The last summation can also be written

$$\sum_{m=1}^{n}\sum_{\mu=1}^{n}\frac{\partial G_r}{\partial p_\mu}\frac{\partial G_s}{\partial p_m}\left(\frac{\partial p_\mu}{\partial x_m}-\frac{\partial p_m}{\partial x_\mu}\right);$$

and therefore we have

$$[G_r, G_s]+\sum_{m=1}^{n}\frac{\partial(G_r, G_s)}{\partial(z, p_m)}\left(\frac{\partial z}{\partial x_m}-p_m\right)$$

$$+\sum_{m=1}^{n}\sum_{\mu=1}^{n}\frac{\partial G_r}{\partial p_\mu}\frac{\partial G_s}{\partial p_m}\left(\frac{\partial p_\mu}{\partial x_m}-\frac{\partial p_m}{\partial x_\mu}\right)=0.$$

If then the quantities $z, p_1, \ldots, p_n$, determined as functions of $x_1, \ldots, x_n$ by the $n+1$ equations $G=0$, be such that their values satisfy the relations

$$\frac{\partial z}{\partial x_m}=p_m,\quad \frac{\partial p_\mu}{\partial x_m}=\frac{\partial p_m}{\partial x_\mu},$$

for all values of m and μ, then we must have

$$[G_r, G_s]=0;$$

and this holds for all combinations of r and s.

Conversely, if the relation holds for all the values of r and s, then the values of $z, p_1, \ldots, p_n$, as given by the equations $G=0$, are such that the quantities p are equal to the derivatives of z and satisfy the necessary relations of the foregoing type. When $[G_r, G_s]=0$, the equation becomes

$$\sum_{m=1}^{n}\frac{\partial(G_r, G_s)}{\partial(z, p_m)}\left(\frac{\partial z}{\partial x_m}-p_m\right)+\sum_{m=1}^{n}\sum_{\mu=1}^{n}\frac{\partial G_r}{\partial p_\mu}\frac{\partial G_s}{\partial p_m}\left(\frac{\partial p_\mu}{\partial x_m}-\frac{\partial p_m}{\partial x_\mu}\right)=0;$$

or, if we write

$$u_{rm}=\frac{\partial G_r}{\partial z}\left(\frac{\partial z}{\partial x_m}-p_m\right)+\sum_{\mu=1}^{n}\frac{\partial G_r}{\partial p_\mu}\left(\frac{\partial p_\mu}{\partial x_m}-\frac{\partial p_m}{\partial x_\mu}\right),$$

$$v_r=\sum_{m=1}^{n}\frac{\partial G_r}{\partial p_m}\left(\frac{\partial z}{\partial x_m}-p_m\right),$$

we have

$$-v_r\frac{\partial G_s}{\partial z}+\sum_{m=1}^{n}\frac{\partial G_s}{\partial p_m}u_{rm}=0,$$

for all values of r and s. Taking the $n+1$ values of s in succession, and keeping one and the same value of r, we have $n+1$ equations, homogeneous and linear in the $n+1$ magnitudes v_r, $u_{r1}, \ldots, u_{rn}$. The determinant of the coefficients of these magnitudes is

$$J\left(\frac{G_1, \ldots, G_{n+1}}{z, p_1, \ldots, p_n}\right),$$

which does not vanish, because the $n+1$ equations $G=0$ are presumed to determine $z, p_1, \ldots, p_n$ as functions of the other variables; hence all the magnitudes $v_r, u_{r1}, \ldots, u_{rn}$ vanish, that is,

$$\sum_{m=1}^{n}\frac{\partial G_r}{\partial p_m}\left(\frac{\partial z}{\partial x_m}-p_m\right)=0,$$

$$\frac{\partial G_r}{\partial z}\left(\frac{\partial z}{\partial x_m}-p_m\right)+\sum_{\mu=1}^{n}\frac{\partial G_r}{\partial p_\mu}\left(\frac{\partial p_\mu}{\partial x_m}-\frac{\partial p_m}{\partial x_\mu}\right)=0,$$

the former holding for all values of r, the latter for all values of r and of m. Taking the latter for a single value of m and for all the values of r, we again have $n+1$ equations, homogeneous and linear in the $n+1$ magnitudes

$$\frac{\partial z}{\partial x_m}-p_m,\ \frac{\partial p_1}{\partial x_m}-\frac{\partial p_m}{\partial x_1},\ \ldots,\ \frac{\partial p_n}{\partial x_m}-\frac{\partial p_m}{\partial x_n},$$

one of which is identically zero; the determinant of the coefficients again is the Jacobian of $G_1, \ldots, G_{n+1}$ with regard to $z, p_1, \ldots, p_n$, and so does not vanish: hence the $n+1$ magnitudes are zero, that is,

$$\frac{\partial z}{\partial x_m}-p_m=0,\quad \frac{\partial p_\mu}{\partial x_m}-\frac{\partial p_m}{\partial x_\mu}=0.$$

Next, taking all the values of m in turn, the other set of equations

$$\sum_{m=1}^{n}\frac{\partial G_r}{\partial p_m}\left(\frac{\partial z}{\partial x_m}-p_m\right)=0$$

is satisfied without providing any new condition: accordingly, we have the relations

$$p_m=\frac{\partial z}{\partial x_m},\quad \frac{\partial p_\mu}{\partial x_m}=\frac{\partial p_m}{\partial x_\mu},$$

as a necessary consequence of the equations $[G_r, G_s]=0$.

Note. It will be noticed that, if the n equations

$$F_1 = 0, \ldots, F_n = 0,$$

satisfy all the conditions $[F_r, F_s] = 0$ necessary for coexistence, the determination of a value of z which satisfies them all requires resolution of the equations and a quadrature: while, if the $n+1$ equations

$$G_1 = 0, \ldots, G_{n+1} = 0,$$

satisfy all the conditions $[G_r, G_s] = 0$ necessary for coexistence, the determination of a value of z which satisfies them all requires resolution of the equations only. The former case could be changed into the latter by the provision of an additional appropriate equation: this appropriate equation is actually provided as the result of the necessary quadrature, with the added advantage that it gives a relation between z and the variables $x_1, \ldots, x_n$, free from the quantities $p_1, \ldots, p_n$.

50. These two theorems lead to various issues, as regards the solution of a single equation and of a system of compatible equations. We shall deal with the latter first.

Accordingly, we suppose that several equations

$$F_1 = 0, \ldots, F_s = 0$$

are given: after the foregoing explanations, we can suppose that s is less than n. It may also be assumed that these equations are algebraically independent of one another and, at this stage, that they involve all the variables concerned: also, that it is not possible to eliminate $z, p_1, \ldots, p_n$ from among them, so as to lead to a relation among the independent variables alone. In order that the given equations may coexist, it is clear from the preceding analysis that the further equations

$$[F_i, F_j] = 0$$

must be satisfied, for all combinations of i and j from the set $1, \ldots, s$.

One, or more than one, of these further equations may be impossible: the original equations cannot then coexist as determining a function z of $x_1, \ldots, x_n$ which satisfies $F_1 = 0, \ldots, F_s = 0$ simultaneously. The case requires no further consideration.

One, or more than one, of the further equations may be satisfied identically: no condition is thereby imposed upon the system.

Similarly, no condition is imposed upon the system when any one of the further equations is satisfied in virtue of the original equations.

But it may happen that one of the further equations is not satisfied, either identically or in virtue of the original equations, and yet it must be satisfied: it is a new equation, which must be associated with the original system. Each such further equation, not satisfied either identically or in virtue of the original equations or in virtue of the newly associated equations, must be associated with the system: let the additional aggregate thus provided be

$$F_{s+1} = 0, \ldots, F_t = 0,$$

each of which, as representing a relation $[F_i, F_j] = 0$, is an equation of the first order.

In order that these may coexist with the original system and with one another, each of them must be combined with every other and with every member of the original system in the relation $[F_i, F_j] = 0$. Any new equation thus arising is associated with the increased system: and the process is repeated until the system is so amplified that the relation is satisfied either identically or in virtue of the equations in the amplified system. Such a system, on the analogy of the earlier and simpler case in Chapter III, is called *complete*: if it be denoted by

$$F_1 = 0, \ldots, F_m = 0,$$

the relation $[F_i, F_j] = 0$ is satisfied for every combination of i and j from the set $1, \ldots, m$, either identically or in virtue of the members of the complete system.

If the original system should be such that z does not occur explicitly in any equation, the relation $[F_i, F_j] = 0$ becomes $(F_i, F_j) = 0$; and then the complete system is

$$F_1 = 0, \ldots, F_m = 0,$$

being such that the relation $(F_i, F_j) = 0$ is satisfied for every combination of i and j from the set $1, \ldots, m$, either identically or in virtue of the members of the complete system. Moreover, z does not occur explicitly in any member of the complete system: for it is not introduced by any of the relations $(F_i, F_j) = 0$.

51. In § 22, it was established that the equation

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$$

possesses an integral with one or other of the assigned initial conditions for one of the values $x_1 = a_1, \ldots, x_n = a_n$, except for such values (if any) of the variables as satisfy $f = 0$ and also

$$\frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0.$$

If these equations can coexist, we must have

$$\left[f, \frac{\partial f}{\partial p_\mu}\right] = 0, \quad \left[\frac{\partial f}{\partial p_r}, \frac{\partial f}{\partial p_s}\right] = 0,$$

for all values of μ, r, s, in connection with the $n+1$ equations.

The former condition is

$$\sum_{m=1}^{n} \left\{\frac{df}{dx_m} \frac{\partial}{\partial p_m}\left(\frac{\partial f}{\partial p_\mu}\right) - \frac{\partial f}{\partial p_m} \frac{d}{dx}\left(\frac{\partial f}{\partial p_\mu}\right)\right\} = 0\,;$$

but $\frac{\partial f}{\partial p_m} = 0$ for all values of m, and thus the condition is

$$\sum_{m=1}^{n} \frac{df}{dx_m} \frac{\partial^2 f}{\partial p_m \partial p_\mu} = 0,$$

for all values of μ, so that we have n relations, homogeneous and linear in the n quantities $\frac{df}{dx_1}, \ldots, \frac{df}{dx_n}$. Suppose now that, if the equations $\frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0$ can coexist with $f = 0$, they determine $p_1, \ldots, p_n$, so that

$$J\left(\frac{\frac{\partial f}{\partial p_1}, \ldots, \frac{\partial f}{\partial p_n}}{p_1, \ldots, p_n}\right)$$

is not zero; then the preceding n relations can only be satisfied by

$$\frac{df}{dx_m} = 0,$$

that is, by

$$\frac{\partial f}{\partial x_m} + p_m \frac{\partial f}{\partial z} = 0,$$

for $m = 1, \ldots, n$. These are n additional relations: they must be satisfied by the values of $p_1, \ldots, p_n, z$, provided by the $n+1$ equations.

It is easy to see that these relations must be satisfied whether $\frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0$ are independent of one another or not, quà equations in $p_1, \ldots, p_n$. For the value of z, and the values of $p_1, \ldots, p_n$ deduced from it, must make $f = 0$ satisfied identically when they are substituted: hence

$$\frac{\partial f}{\partial x_m} + p_m \frac{\partial f}{\partial z} + \sum_{\mu=1}^{n} \frac{\partial f}{\partial p_\mu} \frac{\partial p_\mu}{\partial x_m} = 0,$$

that is,

$$\frac{\partial f}{\partial x_m} + p_m \frac{\partial f}{\partial z} = 0,$$

for all the values of m, which are the conditions in question.

The other set of conditions is

$$\sum_{m=1}^{n} \left[\left\{ \frac{d}{dx_m} \left(\frac{\partial f}{\partial p_r} \right) \right\} \frac{\partial^2 f}{\partial p_m \partial p_s} - \left\{ \frac{d}{dx_m} \left(\frac{\partial f}{\partial p_s} \right) \right\} \frac{\partial^2 f}{\partial p_m \partial p_r} \right] = 0,$$

for all values of r and s.

When all these conditions are satisfied, and when the $n+1$ equations $f = 0$, $\frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0$, determine $p_1, \ldots, p_n, z$ as functions of $x_1, \ldots, x_n$, the value of z is certainly an integral of the original equation $f = 0$. Clearly, it then is not capable of obeying assigned initial conditions: for it possesses no arbitrary element which is at our disposal.

Such integrals are of the class usually called *singular*: we shall recur to them later. When they exist, they result from the elimination of $p_1, \ldots, p_n$ between

$$f = 0, \quad \frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0.$$

Ex. 1. As an example, we may take

$$f = -z + x_1 p_1 + \ldots + x_n p_n + g(p_1, \ldots, p_n) = 0;$$

the additional equations are

$$\frac{\partial f}{\partial p_m} = x_m + \frac{\partial g}{\partial p_m} = 0,$$

so that evidently g must involve all the quantities $p_1, \ldots, p_m$. The relations

$$\frac{\partial f}{\partial x_m} + p_m \frac{\partial f}{\partial z} = 0$$

are satisfied identically; likewise the other set of relations. Moreover, the form of g is known: hence, eliminating $p_1, \ldots, p_n$ from the equations

$$f=0, \quad x_m+\frac{\partial g}{\partial p_m}=0, \qquad (m=1, \ldots, n),$$

the value of z given by the resulting equation is an integral of the original equation. But it contains no arbitrary element.

Ex. 2. It must not be supposed that elimination of $p_1, \ldots, p_n$ is always possible among the $n+1$ equations. Taking $n=2$, a simple instance is provided by the equation

$$f=(px+qy-z)^2-p^2-q^2+\frac{z^2}{x^2+y^2-1}=0.$$

All the equations

$$f=0, \quad \frac{\partial f}{\partial p}=0, \quad \frac{\partial f}{\partial q}=0,$$

and all the relations

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0, \quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0,$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial p}\right)\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial q}\right)-\frac{d}{dx}\left(\frac{\partial f}{\partial q}\right)\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial p}\right)$$
$$+\frac{d}{dy}\left(\frac{\partial f}{\partial p}\right)\frac{\partial}{\partial q}\left(\frac{\partial f}{\partial q}\right)-\frac{d}{dy}\left(\frac{\partial f}{\partial q}\right)\frac{\partial}{\partial q}\left(\frac{\partial f}{\partial p}\right)=0,$$

are satisfied by the two equations

$$\frac{p}{x}=\frac{q}{y}=\frac{z}{x^2+y^2-1};$$

attempted elimination gives no further equation. An integral for the particular example is clearly given by

$$z^2=a\,(x^2+y^2-1),$$

where a is an arbitrary constant.

To equations having integrals of this kind, we shall recur later.

THE COMBINANTS (F, G), $[F, G]$: SOME PROPERTIES.

52. Before passing to the further consideration of a complete system of given equations, it is convenient to note a property of the operation, represented by (F, G) when F and G do not involve z, and by $[F, G]$ when F and G do involve z.

Let F, G, H be any three independent functions of $2n$ quantities $x_1, \ldots, x_n, p_1, \ldots, p_n$; then it is not difficult to verify that the equation

$$((F, G)\,H)+((G, H)\,F)+((H, F)\,G)=0$$

is satisfied identically. From this identical relation, one inference can be made at once. Let $u = \phi$, and $u = \psi$, where ϕ and ψ are functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$, be two independent integrals of

$$(F, u) = 0,$$

which is a homogeneous linear equation in u; then

$$(F, \phi) = 0, \quad (F, \psi) = 0,$$

both equations being satisfied identically. Hence

$$((F, \phi)\psi) = 0, \quad ((\psi, F)\phi) = -((F, \psi)\phi) = 0,$$

and therefore

$$((\phi, \psi)F) = 0,$$

that is,

$$(F(\phi, \psi)) = 0.$$

Thus, taking

$$u = (\phi, \psi),$$

we have

$$(F, u) = 0.$$

Now (ϕ, ψ) may be zero identically, or it may be a pure constant: in either case, the equation $u = (\phi, \psi)$ gives a trivial (and negligible) integral of the differential equation. But if (ϕ, ψ) be a variable quantity, then $u = (\phi, \psi)$ gives an integral of $(F, u) = 0$; and if it be distinct from ϕ and from ψ, it is a new integral. We therefore have the theorem*:

If $u = \phi$ and $u = \psi$ are integrals of the equation

$$(F, u) = 0,$$

then $u = (\phi, \psi)$ also satisfies the equation: and if (ϕ, ψ) be a variable quantity distinct from ϕ and from ψ, then $u = (\phi, \psi)$ is a new integral of the equation.

Another mode of stating this result is as follows: *Let $f_1 = 0, \ldots, f_m = 0$ be a complete system of equations which do not involve z explicitly, so that the relation*

$$(f_r, f_s) = 0,$$

for all values of r and s from the set $1, \ldots, m$, is satisfied; and let $u = \theta$ be an integral of the homogeneous linear equation $(f_1, u) = 0$. Then if the relation $(f_1, f_r) = 0$ is satisfied identically, the quantity

* The theorem is customarily associated with Poisson's name. It was used by Jacobi without explicit indication of the limitations, though he uses it only generally, not universally: see Jacobi, *Ges. Werke*, t. v, pp. 49, 50.

$u=(f_r, \theta)$ also satisfies the equation $(f_1, u)=0$; and it is a new integral of that equation if, being a variable quantity, it is functionally independent of θ.

The result is derived, on the lines of the earlier explanation, from the relation

$$((f_1, f_r)\,\theta)+((f_r, \theta)f_1)+((\theta, f_1)f_r)=0.$$

For $(f_1, \theta)=0$ identically, and $(f_1, f_r)=0$ identically, so that $((f_1, f_r)\,\theta)=0$ and $((\theta, f_1)f_r)=0$: thus $u=(f_r, \theta)$ satisfies the equation $(f_1, u)=0$.

The relation $(f_1, f_r)=0$ must always be satisfied; but if it is satisfied only in virtue of equations of the system, the inference as to the significance of (f_r, θ) cannot be drawn.

53. Next, let f, g, h be any three independent functions of $2n+1$ magnitudes $x_1, \ldots, x_n, z, p_1, \ldots, p_n$; then it is not difficult to verify that the equation

$$[[f, g]\,h]+[[g, h]f]+[[h, f]\,g]=-\frac{\partial f}{\partial z}[g, h]-\frac{\partial g}{\partial z}[h, f]-\frac{\partial h}{\partial z}[f, g]$$

is satisfied identically.

A corresponding inference can be drawn from this identity: but it is not so completely useful as in the former case. Let $u=\phi$ and $u=\psi$, where ϕ and ψ are functions of $x_1, \ldots, x_n, z, p_1, \ldots, p_n$, be two independent integrals of

$$[f, u]=0,$$

which is a homogeneous linear equation in u: then

$$[f, \phi]=0, \quad [f, \psi]=0$$

are satisfied identically. Hence also

$$[[f, \phi]\,\psi]=0, \quad [[\psi, f]\,\phi]=-[[f, \psi]\,\phi]=0;$$

therefore, taking $\phi=g$ and $\psi=h$ in the above identity, and using these relations, we have

$$[[\phi, \psi]f]=-\frac{\partial f}{\partial z}[\phi, \psi],$$

or, writing

$$v=[\phi, \psi],$$

we have

$$[f, v]=v\frac{\partial f}{\partial z}.$$

Thus, in general, $u = v = [\phi, \psi]$ does not satisfy the equation

$$[f, u] = 0;$$

and thus, in general, a new integral of that equation is not obtained. If, however, f does not explicitly involve z, so that $\frac{\partial f}{\partial z}$ is zero, then

$$[f, v] = 0;$$

hence $u = v$ satisfies the equation. As before, $[\phi, \psi]$ may be zero identically, or it may be a pure constant: then $u = v$ gives a trivial (and negligible) integral of the equation. But if $[\phi, \psi]$ is a variable quantity, then $u = [\phi, \psi]$ is an integral of the equation: and it is a new integral if distinct from ϕ and from ψ, that is, if it is not expressible in terms of ϕ and of ψ alone. Hence we have the theorem*:—

If $u = \phi$ and $u = \psi$ are integrals of the equation

$$[f, u] = 0,$$

then $u = [\phi, \psi]$ is a new integral of that equation, only if f does not explicitly involve z and if $[\phi, \psi]$ is a variable quantity not expressible in terms of ϕ and ψ alone.

Another mode of stating the result is as follows: *Let $f_1 = 0, \ldots, f_m = 0$ be a complete system of equations, some at least of which involve z explicitly, so that the relation*

$$[f_r, f_s] = 0,$$

for all values of r and s from the set $1, \ldots, m$, is satisfied; and let $u = \vartheta$ be an integral of the homogeneous linear equation $[f, u] = 0$. Then if the relation $[f_1, f_r] = 0$ is satisfied identically and if the equation $f_1 = 0$ does not involve z explicitly, the quantity $u = [f_r, \vartheta]$ also satisfies the equation $[f_1, u] = 0$; and it is a new integral of that equation if, being a variable quantity, it is functionally independent of ϑ.

The result is derived from the same identity as before. Taking $f = f_1$, $g = f_r$, $h = \vartheta$, we have $[f_1, \vartheta] = 0$ identically and $[f_1, f_r] = 0$ also identically, so that

$$[[f_1, f_r]\vartheta] = 0, \quad [[f_1, \vartheta]f_r] = 0:$$

* The correct statement of the theorem appears to have been given first by Mayer, *Math. Ann.*, t. IX (1876), p. 370.

also $\dfrac{\partial f_1}{\partial z} = 0$, by the hypothesis adopted: thus

$$[[f_r, \vartheta] f_1] = 0,$$

and so $u = [f_r, \vartheta]$ satisfies the equation $[f_1, u] = 0$.

The relation $[f_1, f_r] = 0$ must always be satisfied: if it is satisfied only in virtue of the equations of the system, the inference as to the significance of $[f_r, \vartheta]$ cannot be drawn.

54. Now the ultimate object of investigations, connected with a single equation or with complete systems of equations, is either the construction of the most general integral that is possessed or the formation of processes effective for such construction. Moreover, speaking generally, such processes will be made simpler by every reduction in the number of inverse operations to be performed and by every increase in the number of direct operations.

It is clear, from the two preceding sets of results, that a direct operation for the construction of a new integral of $(F_1, u) = 0$ will more frequently be effective than a direct operation for the construction of a new integral of $[f_1, u] = 0$: indeed, the latter is effective only when the equation $f_1 = 0$ is more limited than is generally permitted to the system of equations in which it is included.

We know that it is always possible, by means of a transformation

$$u = u(z, x_1, \ldots, x_n) = 0,$$

to remove the dependent variable from explicit occurrence in an equation, or in a system of equations, involving only one dependent variable: the number of independent variables is, however, thereby increased by unity. When the integral u of the transformed system has been obtained in the most general form, which comprehends all its integrals, a general integral of the original system is at once deduced from the equation $u = 0$; but this general integral is not completely comprehensive, for it need not include special integrals if any such exist. But as has been seen in the case of a single equation, that is non-homogeneous and of the first order, the processes adopted for the untransformed equation do not lead to the special integrals, if any.

Thus there would appear to be no real loss of generality and no real diminution in the number of integrals obtainable, if we pass

to a transformed system in which the dependent variable does not explicitly occur. On the other hand, there is an added element of effectiveness, because the quantity $u = (F_r, \theta)$ is often an integral of $(F_1, u) = 0$, whereas the quantity $u = [f_r, \vartheta]$ requires compliance with an additional condition in order that it may be an integral of $[f_1, u] = 0$.

Accordingly, for the immediate present, it will be assumed that the dependent variable does not occur explicitly: hence we have to deal with a system of equations

$$F_1 = 0, \ \ldots, \ F_m = 0,$$

involving the quantities $x_1, \ldots, x_n, p_1, \ldots, p_n$. We may further assume that the equations are linearly independent of one another, so that no one of the quantities F can be expressed as a linear combination of the remainder with coefficients whether variable or constant. And after the discussion in § 50, we shall assume that the system is complete, so that the relation

$$(F_r, F_s) = 0$$

is satisfied, for all values of r and s from the set $1, \ldots, m$, either identically or in virtue of the equations of the system.

Moreover, after the same discussion, it will be assumed that $m < n$. What is required is a value of z satisfying all the equations of the system: in order to proceed by quadratures, other $n - m$ compatible and independent equations are needed.

MAYER'S DEVELOPMENT OF JACOBI'S SECOND METHOD.

55. There are various ways of deducing the further $n - m$ equations that are requisite: one of the simplest of these ways is Mayer's development of what is often called Jacobi's second method.

The m equations in the complete system

$$F_1 = 0, \ \ldots, \ F_m = 0$$

are linearly independent of one another, in the sense that no one of the quantities F can be expressed as a linear combination of the remainder: thus there can be no effective functional relation among the quantities F. Consequently, the m equations can be resolved so as to express m of the involved variables in terms of the rest.

In the first instance, let it be supposed that the equations can be resolved so as to express m of the variables $p_1, \ldots, p_n$, say to express $p_1, \ldots, p_m$, in terms of all the other quantities involved; and let the result of the resolution be denoted by

$$p_i - \phi_i(p_{m+1}, \ldots, p_n, x_1, \ldots, x_n) = 0,$$

or by

$$p_i - \phi_i = 0,$$

for $i = 1, \ldots, m$. We prove, as follows, that the resolved system of equations is complete, the original system being complete: that is to say, the relation

$$(p_r - \phi_r, \; p_s - \phi_s) = 0$$

is satisfied, for all values of r and of s from the set $1, \ldots, m$.

When the values $\phi_1, \ldots, \phi_m$ for $p_1, \ldots, p_m$ respectively are substituted in all the equations of the original system, each of the latter becomes an identity. Therefore

$$\frac{\partial F_r}{\partial x_i} + \sum_{k=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial \phi_k}{\partial x_i} = 0,$$

for all values of $i = 1, \ldots, n$: that is,

$$\frac{\partial F_r}{\partial x_i} = \sum_{k=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial (p_k - \phi_k)}{\partial x_i},$$

for all these values. Similarly, we have

$$\frac{\partial F_r}{\partial p_j} + \sum_{k=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial \phi_k}{\partial p_j} = 0,$$

for the values $j = m + 1, \ldots, n$; that is,

$$\frac{\partial F_r}{\partial p_j} = \sum_{k=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial (p_k - \phi_k)}{\partial p_j},$$

for these values. Also

$$\frac{\partial F_r}{\partial p_{j'}} = \sum_{k=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial (p_k - \phi_k)}{\partial p_{j'}},$$

for the values $j' = 1, \ldots, m$, because for each of these values only a single term on the right-hand side occurs: hence

$$\frac{\partial F_r}{\partial p_i} = \sum_{k=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial (p_k - \phi_k)}{\partial p_i},$$

for the values $i = 1, \ldots, n$. And these results hold for all the values $r = 1, \ldots, m$.

Substituting in (F_r, F_s), we find

$$(F_r, F_s) = \sum_{i=1}^{n} \left(\frac{\partial F_r}{\partial x_i} \frac{\partial F_s}{\partial p_i} - \frac{\partial F_r}{\partial p_i} \frac{\partial F_s}{\partial x_i} \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial F_s}{\partial p_l} \left\{ \frac{\partial (p_k - \phi_k)}{\partial x_i} \frac{\partial (p_l - \phi_l)}{\partial p_i} - \frac{\partial (p_k - \phi_k)}{\partial p_i} \frac{\partial (p_l - \phi_l)}{\partial x_i} \right\}$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial F_s}{\partial p_l} (p_k - \phi_k,\ p_l - \phi_l).$$

Now $(F_r, F_s) = 0$, for all values of r and of s from the set $1, \ldots, m$: hence

$$\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial F_r}{\partial p_k} \frac{\partial F_s}{\partial p_l} (p_k - \phi_k,\ p_l - \phi_l) = 0,$$

for all these combinations of values. Taking the relation for one value of s and for all the values of r, we have m equations, homogeneous and linear in the m quantities

$$\sum_{l=1}^{m} \frac{\partial F_s}{\partial p_l} (p_k - \phi_k,\ p_l - \phi_l),$$

for $k = 1, \ldots, m$. The determinant of the coefficients of these quantities in the m equations is

$$J\left(\frac{F_1, \ldots, F_m}{p_1, \ldots, p_m}\right),$$

which, by hypothesis, does not vanish: consequently,

$$\sum_{l=1}^{m} \frac{\partial F_s}{\partial p_l} (p_k - \phi_k,\ p_l - \phi_l) = 0,$$

for all the values of s and k. Taking this relation for all the m values of s, we again have m equations, homogeneous and linear in the m quantities

$$(p_k - \phi_k,\ p_l - \phi_l),$$

for $l = 1, \ldots, m$; and the determinant of the coefficients of these quantities in the m equations is again

$$J\left(\frac{F_1, \ldots, F_m}{p_1, \ldots, p_m}\right),$$

which does not vanish: consequently,

$$(p_k - \phi_k,\ p_l - \phi_l) = 0,$$

for k and $l = 1, \ldots, m$. Hence the system of equations

$$p_i - \phi_i = 0, \qquad (i = 1, \ldots, m),$$

is complete.

It is easy to see that the completeness of the system of equations

$$p_i - \phi_i = 0,$$

for $i = 1, \ldots, m$, is of a special kind. The relation

$$(p_k - \phi_k,\ p_l - \phi_l) = 0,$$

k and l having any values from the set $1, \ldots, m$, is

$$0 = -\frac{\partial \phi_l}{\partial x_k} + \frac{\partial \phi_k}{\partial x_l} + \sum_{j=m+1}^{n} \left(\frac{\partial \phi_k}{\partial x_j} \frac{\partial \phi_l}{\partial p_j} - \frac{\partial \phi_k}{\partial p_j} \frac{\partial \phi_l}{\partial x_j} \right).$$

This relation is to be satisfied, and it clearly is not satisfied in virtue of the equations

$$p_i - \phi_i = 0,$$

for $i = 1, \ldots, m$, because it does not involve any of the quantities $p_1, \ldots, p_m$; hence the relations for the modified system are satisfied identically.

When the relations, necessary and sufficient to secure the completeness of a system, are satisfied identically, the system is said to be *in involution.* The resolved system of complete equations is a system in involution, because each of the relations

$$(p_k - \phi_k,\ p_l - \phi_l) = 0$$

is satisfied identically*.

Note. Even when a complete system of equations $F_1 = 0, \ldots, F_m = 0$, is such that z occurs explicitly, a corresponding result is obtainable. Suppose that the m equations can be resolved with regard to z and to $m-1$ of the quantities p, say $p_1, \ldots, p_{m-1}$, in the form

$$z - \psi = 0, \quad p_1 - \psi_1 = 0, \quad \ldots, \quad p_{m-1} - \psi_{m-1} = 0,$$

where $\psi, \psi_1, \ldots, \psi_{m-1}$ do not involve $z, p_1, \ldots, p_{m-1}$; then the relations

$$[p_i - \psi_i,\ p_j - \psi_j] = 0,$$

$$[z - \psi - x_1(p_1 - \psi_1) - \ldots - x_{m-1}(p_{m-1} - \psi_{m-1}),\ p_i - \psi_i] = 0,$$

* Sometimes, for convenience, the unresolved complete system is said to be *in involution.*

are satisfied in virtue of the relations

$$[F_r, F_s] = 0,$$

for all values of i and j. When the quantities

$$[p_i - \psi_i,\ p_j - \psi_j]$$

are expressed in full, they contain none of the variables $z, p_1, \ldots, p_{m-1}$; they cannot vanish, therefore, in virtue of the resolved equations: hence they must vanish identically. Similarly, the quantity

$$[z - \psi - x_1(p_1 - \psi_1) - \ldots - x_{m-1}(p_{m-1} - \psi_{m-1}),\ p_i - \psi_i]$$

vanishes identically.

The resolved system can be regarded as a system in involution.

56. In order to obtain common integrals of the system, a satisfactory method will be devised if, by its means, other $n-m$ equations are associated with the m equations in the system: and the remaining stage will be a quadrature with reference to the variables $x_1, \ldots, x_n$, if the Jacobian of

$$p_1 - \phi_1, \ \ldots,\ p_m - \phi_m,\quad u_{m+1},\ \ldots,\ u_n,$$

(where $u_{m+1} = \text{constant}, \ldots, u_n = \text{constant}$, are the additional $n-m$ equations) with regard to $p_1, \ldots, p_n$ does not vanish identically, that is, if

$$J\left(\frac{u_{m+1}, \ \ldots,\ u_n}{p_{m+1},\ \ldots,\ p_n}\right)$$

does not vanish identically. Accordingly, this method requires the determination of $n-m$ equations $u = \text{constant}$.

Each such equation, as it is to coexist with the equations of the given system, must satisfy the conditions which are necessary and sufficient to secure the coexistence: that is, it must satisfy the relations

$$(p_1 - \phi_1, u) = 0,\ \ldots,\ (p_m - \phi_m, u) = 0,$$

which, in effect, are m equations for the determination of u. Now these equations constitute a complete system of the type considered in the last chapter. We have

$$((p_r - \phi_r, u), p_s - \phi_s) - ((p_s - \phi_s, u), p_r - \phi_r) + ((p_s - \phi_s, p_r - \phi_r), u) = 0$$

identically; also $(p_s - \phi_s, p_r - \phi_r) = 0$ identically for all values of r and s, so that $((p_s - \phi_s, p_r - \phi_r), u) = 0$: hence

$$(p_r - \phi_r, (p_s - \phi_s, u)) = (p_s - \phi_s, (p_r - \phi_r, u)).$$

Let

$$(p_i - \phi_i, u) = A_i(u),$$

where A_i is a linear operator: the foregoing relation becomes

$$A_r(A_s u) = A_s(A_r u),$$

for all values of r and s. This aggregate of relations is necessary and sufficient to secure that the system of equations $A_1(u) = 0, \ldots, A_m(u) = 0$, is complete.

Written in full, the equations are

$$\frac{\partial u}{\partial x_1} + \sum_{i=m+1}^{n} \left(\frac{\partial u}{\partial p_i}\frac{\partial \phi_1}{\partial x_i} - \frac{\partial u}{\partial x_i}\frac{\partial \phi_1}{\partial p_i}\right) + \sum_{j=1}^{m} \frac{\partial u}{\partial p_j}\frac{\partial \phi_1}{\partial x_j} = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial u}{\partial x_m} + \sum_{i=m+1}^{n} \left(\frac{\partial u}{\partial p_i}\frac{\partial \phi_m}{\partial x_i} - \frac{\partial u}{\partial x_i}\frac{\partial \phi_m}{\partial p_i}\right) + \sum_{j=1}^{m} \frac{\partial u}{\partial p_j}\frac{\partial \phi_m}{\partial x_j} = 0,$$

a system of m equations in the $2n$ variables $x_1, \ldots, x_n, p_1, \ldots, p_n$. When any integral of the system has been obtained involving any of the variables $p_1, \ldots, p_m$, the relations $p_1 = \phi_1, \ldots, p_m = \phi_m$, can be used (without affecting its value or its significance) so as to remove these variables. In the transformed expression for u, we have

$$\frac{\partial u}{\partial p_1} = 0, \ \ldots, \ \frac{\partial u}{\partial p_m} = 0,$$

that is, we may take the m equations in the form

$$\frac{\partial u}{\partial x_1} + \sum_{i=m+1}^{n} \left(\frac{\partial u}{\partial p_i}\frac{\partial \phi_1}{\partial x_i} - \frac{\partial u}{\partial x_i}\frac{\partial \phi_1}{\partial p_i}\right) = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial u}{\partial x_m} + \sum_{i=m+1}^{n} \left(\frac{\partial u}{\partial p_i}\frac{\partial \phi_m}{\partial x_i} - \frac{\partial u}{\partial x_i}\frac{\partial \phi_m}{\partial p_i}\right) = 0.$$

The original system of m equations was complete: the transformed system, with the condition that the integrals u do not involve $p_1, \ldots, p_m$, is also complete. The number of variables involved is $2n - m$, being $x_1, \ldots, x_n, p_{m+1}, \ldots, p_n$: so that, as the system for the quantity u is now a complete Jacobian system, it possesses $2n - 2m$ integrals*, which are functionally independent of one another.

* It is easy to see that this result comes also from the earlier form of the equations for u: that form involves $2n$ variables, and so there are $2n - m$ integrals. But $u = p_1 - \phi_1, \ldots, u = p_m - \phi_m$ are seen to be m integrals: and therefore there are other $2n - 2m$ integrals.

Two limitations are, however, imposed upon these integrals, so that not all of them can be retained for our purpose. In the first place, the aggregate of $n-m$ equations required must be such that

$$J\left(\frac{u_{m+1}, \ldots, u_n}{p_{m+1}, \ldots, p_n}\right)$$

does not vanish identically. In the second place, let $u_{m+1}=a_{m+1}$ be an integral of the system: it must involve one or more of the quantities $p_{m+1}, \ldots, p_n$, and it must be resoluble with regard to one of them, because otherwise all the derivatives $\frac{\partial u}{\partial p_{m+1}}, \ldots, \frac{\partial u}{\partial p_n}$ would vanish: let the resolved form be

$$p_{m+1}-\phi_{m+1}=0,$$

where ϕ_{m+1} involves a_{m+1}. Any other of the $2n-2m-1$ remaining integrals, say v, undoubtedly satisfies

$$(p_1-\phi_1, v)=0, \ldots, (p_m-\phi_m, v)=0;$$

but, for our purpose of proceeding to the determination of the n quantities p, it must also satisfy the relation

$$(p_{m+1}-\phi_{m+1}, v)=0;$$

and this relation will not, in general, be satisfied for any one of the $2n-2m-1$ integrals, selected at random. Accordingly, it is not necessary to obtain all the $2n-2m$ integrals of the system though, if they are known, they can be used in the construction of v as an appropriate functional combination of the $2n-2m-1$ integrals other than u_{m+1}: it is sufficient at this stage to obtain a single integral of the transformed system. Denoting this single integral by u_{m+1}, we resolve the equation $u_{m+1}=a_{m+1}$ with regard (say) to p_{m+1}, in the form $p_{m+1}=\phi_{m+1}$; and then any other equation $v=$ constant, that can coexist with the original system and with $p_{m+1}=\phi_{m+1}$, must satisfy the necessary and sufficient conditions

$$(p_r-\phi_r, v)=0,$$

for $r=1, \ldots, m+1$. As before, v may be assumed not to contain $p_1, \ldots, p_m$: and for reasons similar to those adduced before, it may be assumed not to contain p_{m+1}, so that

$$\frac{\partial v}{\partial p_1}=0, \ldots, \frac{\partial v}{\partial p_{m+1}}=0;$$

and then the system of equations for v is

$$\frac{\partial v}{\partial x_r} + \sum_{i=m+2}^{n} \left(\frac{\partial v}{\partial p_i} \frac{\partial \phi_r}{\partial x_i} - \frac{\partial v}{\partial x_i} \frac{\partial \phi_r}{\partial p_i} \right) = 0,$$

for $r = 1, \ldots, m+1$. The system of $m+1$ equations is complete: it involves the $2n-m-1$ variables $x_1, \ldots, x_n, p_{m+2}, \ldots, p_n$; and so it possesses $2n-2m-2$ functionally independent integrals.

57. At each stage, we have a complete Jacobian system for the determination of a quantity u, such that an equation $u = a$ can be associated with the system of equations for the variable z. The theory of these Jacobian systems, as explained in the preceding chapter, shews that they do possess a number of integrals; and therefore quantities u of the appropriate type do exist, so that we require only their explicit expressions in order to formulate the successive equations $u = a$.

We thus may pass from stage to stage: at each step, an integral of a number of simultaneous equations, forming a complete Jacobian system, is required: and as, at any stage, the number of equations has become greater while the number of variables has become less than at the preceding stage, the construction of the integrals in succession is successively simpler.

At each stage, what is required is a single integral belonging to the complete Jacobian system then framed: this integral must involve one of the variables p still surviving in the system*. For this purpose, we may use either Mayer's method or the amplified Jacobian method devised for complete linear systems; but it is not necessary to work either method to the complete issue, because all that is wanted is a single integral of the simultaneous system, not the aggregate of functionally independent integrals of the system.

Each new equation of the type $u =$ constant, associated with the system in its amplified condition before the derivation of the particular u, introduces an arbitrary constant. Thus, at the end of the series of operations which result in giving n equations, the number of arbitrary constants introduced is $n-m$; when the

* In case, at any stage, an appropriate integral of this type may not conveniently be obtainable, while an integral involving the variables x and some of the old variables may be forthcoming, a transformation similar to that adopted for a corresponding difficulty, hereafter (§§ 58, 59) discussed, will be effective.

n equations are fully resolved for $p_1, \ldots, p_n$, the expressions for these quantities involve $x_1, \ldots, x_n, a_1, \ldots, a_{n-m}$. When these values are introduced into the equation

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

the right-hand side is an exact differential; when the quadrature of this exact differential is effected, we have

$$z = \phi(x_1, \ldots, x_n, a_1, \ldots, a_{n-m}) + b,$$

where b is an arbitrary constant. This equation gives the required value of z as an integral common to the system of equations: its expression contains $n - m + 1$ constants.

Ex. Obtain a common integral (if it exist) of the simultaneous equations

$$\left.\begin{aligned} F_1 &= p_1 p_2 - x_3 x_4 = 0 \\ F_2 &= p_3 p_4 - x_1 x_2 = 0 \end{aligned}\right\}.$$

We have

$$(F_1, F_2) = p_1 x_1 + p_2 x_2 - p_3 x_3 - p_4 x_4;$$

the right-hand side must vanish, and it clearly does not vanish in virtue of $F_1 = 0$, $F_2 = 0$; hence we have a new equation to be associated with the first two, and we write

$$F_3 = p_1 x_1 + p_2 x_2 - p_3 x_3 - p_4 x_4 = 0.$$

We now have

$$\begin{aligned} (F_1, F_2) &= F_3, \\ (F_1, F_3) &= -2F_1, \\ (F_2, F_3) &= 2F_2; \end{aligned}$$

all the quantities of the type (F_r, F_s) vanish in virtue of the three equations; hence these equations are a complete system.

Resolving the three equations $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, so as to express p_1, p_2, p_3 in terms of the other variables that occur, we find two systems, viz.

$$\text{(i)} \qquad p_1 = \frac{x_2 x_3}{p_4}, \quad p_2 = \frac{x_4 p_4}{x_2}, \quad p_3 = \frac{x_1 x_2}{p_4};$$

$$\text{(ii)} \qquad p_1 = \frac{x_4 p_4}{x_1}, \quad p_2 = \frac{x_1 x_3}{p_4}, \quad p_3 = \frac{x_1 x_2}{p_4}.$$

The second of these two sets is derivable from the first by interchanging the variables x_1 and x_2; hence its integral must be similarly derivable from the integral of the first.

To obtain this integral, we need an equation $u = a$, where a is a constant and u must involve p_4; and u is determined by the equations

$$\left(p_1 - \frac{x_2 x_3}{p_4},\ u\right) = 0, \quad \left(p_2 - \frac{x_4 p_4}{x_2},\ u\right) = 0, \quad \left(p_3 - \frac{x_1 x_2}{p_4},\ u\right) = 0,$$

together with the justifiable assumption that u is explicitly independent of p_1, p_2, p_3. These equations are

$$0=\frac{\partial u}{\partial x_1}+\frac{x_2x_3}{p_4^2}\frac{\partial u}{\partial x_4},$$

$$0=\frac{\partial u}{\partial x_2}-\frac{x_4}{x_2}\frac{\partial u}{\partial x_4}+\frac{p_4}{x_2}\frac{\partial u}{\partial p_4},$$

$$0=\frac{\partial u}{\partial x_3}+\frac{x_1x_2}{p_4^2}\frac{\partial u}{\partial x_4};$$

and the fact that they are complete can easily be verified.

The Mayer solution of these equations is as follows. We transform the variables by the relations

$$x_1=y_1,$$

$$x_2=x_2+(y_1-a_1)\,y_2,$$

$$x_3=x_3+(y_1-a_1)\,y_3;$$

and we form the single equation

$$\frac{\partial u}{\partial y_1}+Y_1\frac{\partial u}{\partial x_4}+Y_2\frac{\partial u}{\partial p_4}=0,$$

where

$$Y_1=\frac{x_2x_3}{p_4^2}-\frac{x_4}{x_2}y_2+\frac{x_1x_2}{p_4^2}y_3,$$

$$Y_2=\frac{p_4}{x_2}y_2.$$

An integral of the subsidiary system

$$dy_1=\frac{dx_4}{Y_1}=\frac{dp_4}{Y_2},$$

where y_2 and y_3 are arbitrary parameters, is required involving p_4: one such integral is clearly derivable from

$$dy_1=\frac{dp_4}{Y_2}=\frac{dp_4}{p_4}\,\frac{a_2+(y_1-a_1)\,y_2}{y_2},$$

in the form

$$\frac{p_4}{a_2+(y_1-a_1)\,y_2}=\text{constant}.$$

Then an integral of the original system is given by

$$\frac{p_4}{a_2+(y_1-a_1)\,y_2}=\frac{c}{a_2},$$

that is, by

$$\frac{p_4}{x_2}=\frac{c}{a_2}.$$

Hence $u=\frac{p_4}{x_2}$ is an integral of the three equations; as it involves p_4, it is of the required type.

To deduce the value of z, we take

$$\frac{p_4}{x_2}=a,$$

and then the values of p_1, p_2, p_3, p_4 are

$$p_1=\frac{x_3}{a},\quad p_2=ax_4,\quad p_3=\frac{x_1}{a},\quad p_4=ax_2:$$

inserting these values in

$$dz=p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4,$$

we find that

$$z=\frac{x_1x_3}{a}+ax_2x_4+b$$

is an integral of the first set of equations derived from the resolution of $F_1=0$, $F_2=0$, $F_3=0$, and therefore is an integral of the original equations.

Effecting upon this integral the interchange of variable whereby the first and the second resolved sets are interchanged, we find that

$$z=\frac{x_2x_3}{a}+ax_1x_4+b$$

is also an integral of the original equations $F_1=0$, $F_2=0$

We thus have two distinct integrals, each involving two arbitrary constants: and they are the only integrals that are thus obtainable. Their relation (if any) to one another, and the derivation of other integrals (if any) from them, belong to a range of subsequent investigation.

The amplified Jacobian method of solution is simple in the present case and leads very directly to the integral

$$u=\frac{p_4}{x_2}:$$

the rest of the analysis is the same as before.

58. The preceding investigation has rested on the two assumptions: (i) that the equations $F_1=0, \ldots, F_m=0$ of the complete system can be resolved with regard to m of the variables $p_1, \ldots, p_n$: (ii) that the equations of the complete amplified system $F_1=0, \ldots, F_m=0, u_{m+1}=a_{m+1}, \ldots, u_n=a_n$, can be resolved with regard to $p_1, \ldots, p_n$, so that the Jacobian

$$J\left(\frac{F_1, \ldots, F_m, u_{m+1}, \ldots, u_n}{p_1, \ldots\ldots\ldots\ldots, p_n}\right)$$

does not vanish identically. The latter assumption is, however, unnecessary: and, as has been proved by Mayer*, it is sufficient that the n functionally independent equations $F_1=0, \ldots, F_m=0$, $u_{m+1}=a_{m+1}, \ldots, u_n=a_n$ should be resoluble with regard to n of the quantities which they involve.

* *Math. Ann.*, t. VIII (1875), p. 313.

Still retaining the first assumption, let the m equations $F_1=0$, ..., $F_m=0$ be resoluble with respect to p_1, ..., p_m; and let the resolved set be

$$p_1-\phi_1=0,\ \ldots,\ p_m-\phi_m=0.$$

Let X be a function of all the variables such that

$$(F_r,\ X)=0,$$

for $r=1$, ..., m; and let ξ denote the value of X which results from substituting ϕ_1, ..., ϕ_m as the values of p_1, ..., p_m in X; then

$$(p_r-\phi_r,\ \xi)=0.$$

For

$$\frac{\partial\xi}{\partial x_i}=\frac{\partial X}{\partial x_i}+\sum_{k=1}^{m}\frac{\partial X}{\partial p_k}\frac{\partial\phi_k}{\partial x_i},$$

so that

$$\frac{\partial X}{\partial x_i}=\frac{\partial\xi}{\partial x_i}+\sum_{k=1}^{m}\frac{\partial X}{\partial p_k}\frac{\partial(p_k-\phi_k)}{\partial x_i},$$

for $i=1$, ..., m. Similarly

$$\frac{\partial X}{\partial p_j}=\frac{\partial\xi}{\partial p_j}+\sum_{k=1}^{m}\frac{\partial X}{\partial p_k}\frac{\partial(p_k-\phi_k)}{\partial p_j},$$

for $j=m+1$, ..., n; and this last relation is identically true for $j=1$, ..., m, because neither ξ nor any one of the quantities ϕ_1, ..., ϕ_m, involves p_1, ..., p_m; that is, the relation is true for $j=1$, ..., n. Also, when the values of p_1, ..., p_m are substituted in the equations $F_1=0$, ..., $F_m=0$, these become identities: hence

$$\frac{\partial F_r}{\partial x_i}=\sum_{h=1}^{m}\frac{\partial F_r}{\partial p_h}\frac{\partial(p_h-\phi_h)}{\partial x_i},$$

for values of $i=1$, ..., n; and (as above)

$$\frac{\partial F_r}{\partial p_j}=\sum_{h=1}^{m}\frac{\partial F_r}{\partial p_h}\frac{\partial(p_h-\phi_h)}{\partial p_j},$$

first for $j=m+1$, ..., n, from the identical equation, and obviously identically for $j=1$, ..., m, that is, for values of $j=1$, ..., n. Hence

$$(F_r,\ X)=\sum_{h=1}^{m}\frac{\partial F_r}{\partial p_h}(p_h-\phi_h,\ \xi)+\sum_{h=1}^{m}\sum_{k=1}^{m}\frac{\partial F_r}{\partial p_h}\frac{\partial X}{\partial p_k}(p_h-\phi_h,\ p_k-\phi_k).$$

Now we have

$$(p_h-\phi_h,\ p_k-\phi_k)=0,$$

and by hypothesis,

$$(F_r, X) = 0,$$

for all values of r; hence

$$\sum_{h=1}^{m} \frac{\partial F_r}{\partial p_h} (p_h - \phi_h, \xi) = 0,$$

holding for $r = 1, \ldots, m$. But

$$J\left(\frac{F_1, \ldots, F_m}{p_1, \ldots, p_m}\right)$$

is not zero, because of the assumed resolubility of the equations $F_1 = 0, \ldots, F_m = 0$ with respect to $p_1, \ldots, p_m$: hence the preceding m relations can only be satisfied by

$$(p_h - \phi_h, \xi) = 0;$$

which was to be proved.

Next, suppose that (by some method or other) we possess $n - m$ equations

$$u_{m+1} = a_{m+1}, \ldots, u_n = a_n,$$

which coexist with $F_1 = 0, \ldots, F_m = 0$, and with one another; the n equations in the aggregate being functionally independent of one another. The original system of m equations is certainly resoluble with regard to $p_1, \ldots, p_m$; the amplified system of n equations is resoluble with regard to n of the variables, which can certainly be chosen so as to include $p_1, \ldots, p_m$ and may include others of the quantities p though perhaps not all of them. Suppose, then, that the variables chosen for resolution include $p_1, \ldots, p_\mu$, where $\mu \geqslant m$, but not more than μ of the quantities p; the resolved equations will be equivalent to μ equations of the amplified system, say to

$$F_1 = 0, \ldots, F_m = 0, \quad u_{m+1} = a_{m+1}, \ldots, u_\mu = a_\mu.$$

In the remaining equations of the system, let the values of $p_1, \ldots, p_\mu$ be substituted, and suppose that they become

$$v_{\mu+1} = a_{\mu+1}, \ldots, v_n = a_n,$$

these equations not being resoluble for any of the quantities $p_{\mu+1}, \ldots, p_n$, and consequently not involving any of these quantities. Then, exactly as in the preceding case, we have

$$(p_h - \phi_h, v_k) = 0,$$

for $h = 1, \ldots, \mu$, and $k = \mu + 1, \ldots, n$: taking account of the fact that v_k involves none of the variables $p_1, \ldots, p_n$, we may write this set of equations in the form

$$\frac{\partial v_k}{\partial x_h} - \sum_{i=\mu+1}^{n} \frac{\partial v_k}{\partial x_i} \frac{\partial \phi_h}{\partial p_i} = 0,$$

for all the values of h and k.

The quantities $v_{\mu+1}, \ldots, v_n$ involve the variables $x_1, \ldots, x_n$: we prove, as follows, that they are functionally independent combinations of $x_{\mu+1}, \ldots, x_n$. Otherwise, there would be some relation

$$f(x_1, \ldots, x_\mu, v_{\mu+1}, \ldots, v_n) = 0,$$

which would have to be satisfied identically when the values of $v_{\mu+1}, \ldots, v_n$ in terms of the variables $x_1, \ldots, x_n$ are substituted; and as it would involve one or more of the quantities v, it could be resolved with regard (say) to v_n in the form

$$v_n = g(x_1, \ldots, x_\mu, v_{\mu+1}, \ldots, v_{n-1}).$$

This relation would also be an identity when the values of $v_{\mu+1}, \ldots, v_n$ in terms of $x_1, \ldots, x_n$ are substituted. Now

$$\frac{\partial v_n}{\partial x_h} - \sum_{i=\mu+1}^{n} \frac{\partial v_n}{\partial x_i} \frac{\partial \phi_h}{\partial p_i} = 0;$$

substituting $g(x_1, \ldots, x_\mu, v_{\mu+1}, \ldots, v_{n-1})$ for v_n, this gives

$$\left(\frac{\partial g}{\partial x_h} + \sum_{\lambda=\mu+1}^{n-1} \frac{\partial g}{\partial v_\lambda} \frac{\partial v_\lambda}{\partial x_h}\right) - \sum_{\lambda=\mu+1}^{n-1} \frac{\partial g}{\partial v_\lambda} \sum_{i=\mu+1}^{n} \frac{\partial v_\lambda}{\partial x_i} \frac{\partial \phi_h}{\partial p_i} = 0;$$

but

$$\frac{\partial v_\lambda}{\partial x_h} - \sum_{i=\mu+1}^{n} \frac{\partial v_\lambda}{\partial x_i} \frac{\partial \phi_h}{\partial p_i} = 0,$$

for $\lambda = \mu + 1, \ldots, n - 1$, and therefore

$$\frac{\partial g}{\partial x_h} = 0,$$

for all the values of h. Thus the above expression for v_n would give

$$v_n = g(v_{\mu+1}, \ldots, v_{n-1}),$$

and the quantities $v_{\mu+1}, \ldots, v_n$ would not be functionally independent of one another, contrary to the construction of the quantities u. Hence $v_{\mu+1} = a_{\mu+1}, \ldots, v_n = a_n$, are functionally independent combinations of $x_{\mu+1}, \ldots, x_n$.

The equations $v_{\mu+1}=a_{\mu+1}, \ldots, v_n=a_n$ can therefore be resolved with regard to $x_{\mu+1}, \ldots, x_n$; and consequently the system

$$F_1=0, \ldots, F_m=0, \qquad u_{m+1}=a_{m+1}, \ldots, u_n=a_n$$

can be resolved with regard to $p_1, \ldots, p_\mu, x_{\mu+1}, \ldots, x_n$. Let a resolved set be

$$p_h=\psi_h(x_1, \ldots, x_\mu, p_{\mu+1}, \ldots, p_n),$$
$$x_k=\theta_k(x_1, \ldots, x_\mu, p_{\mu+1}, \ldots, p_n),$$

for $h=1, \ldots, \mu$, and $k=\mu+1, \ldots, n$, the functions θ and ψ involving the arbitrary constants.

59. Now take a new dependent variable Z, defined by the contact-transformation*

$$Z=z-p_{\mu+1}x_{\mu+1}-\ldots-p_nx_n,$$

being a transformation of a type first used by Lagrange†; then

$$dZ=p_1dx_1+\ldots+p_\mu dx_\mu-x_{\mu+1}\,dp_{\mu+1}-\ldots-x_ndp_n.$$

We write

$$x_1, \ldots, x_\mu, -p_{\mu+1}, \ldots, -p_n=y_1, \ldots, y_\mu, y_{\mu+1}, \ldots, y_n,$$

respectively, and regard $y_1, \ldots, y_n$ as new independent variables: then, denoting by $q_1, \ldots, q_n$ the derivatives of the new dependent variable with regard to the new independent variables, we have

$$p_h=q_h, \quad x_k=q_k,$$

for $h=1, \ldots, \mu$, and $k=\mu+1, \ldots, n$. Let

$$F_r(x_1, \ldots, x_n, p_1, \ldots, p_n)=G_r(y_1, \ldots, y_n, q_1, \ldots, q_n),$$
$$u_s(x_1, \ldots, x_n, p_1, \ldots, p_n)=w_s(y_1, \ldots, y_n, q_1, \ldots, q_n),$$

on effecting these changes: then as the equations

$$F_1=0, \ldots, F_m=0, \qquad u_{m+1}=a_{m+1}, \ldots, u_n=a_n,$$

are resoluble with regard to $p_1, \ldots, p_\mu, x_{\mu+1}, \ldots, x_n$, the equations

$$G_1=0, \ldots, G_m=0, \qquad w_{m+1}=a_{m+1}, \ldots, w_n=a_n,$$

are resoluble with regard to $q_1, \ldots, q_n$. Moreover, the equations

$$F_1=0, \ldots, F_m=0, \qquad u_{m+1}=a_{m+1}, \ldots, u_n=a_n,$$

* Called *tangential transformation* in Part I of this work: the phrase *contact-transformation* is now customary, and it will be herein adopted whenever reference to it is required.

† *Œuvres complètes*, t. IV, p. 84.

satisfy the relations

$$(F_r, F_s) = 0, \quad (F_r, u_i) = 0, \quad (u_i, u_j) = 0,$$

for $r, s = 1, \ldots, m$, and $i, j = m+1, \ldots, n$. Now

$$\frac{\partial F_r}{\partial x_h} = \frac{\partial G_r}{\partial y_h}, \quad \frac{\partial F_r}{\partial p_h} = \frac{\partial G_r}{\partial q_h},$$

$$\frac{\partial F_r}{\partial x_k} = \frac{\partial G_r}{\partial q_k}, \quad \frac{\partial F_r}{\partial p_k} = -\frac{\partial G_r}{\partial y_k},$$

for $h = 1, \ldots, \mu$, and $k = \mu+1, \ldots, n$; and similarly for relations between derivatives of u and w. Thus

$$\begin{aligned}(F_r, F_s) &= \sum_{h=1}^{\mu} \left(\frac{\partial G_r}{\partial y_h}\frac{\partial G_s}{\partial q_h} - \frac{\partial G_r}{\partial q_h}\frac{\partial G_s}{\partial y_h}\right) \\ &\quad + \sum_{k=\mu+1}^{n} \left\{\frac{\partial G_r}{\partial q_k}\left(-\frac{\partial G_s}{\partial y_k}\right) - \left(-\frac{\partial G_r}{\partial y_k}\right)\frac{\partial G_s}{\partial q_k}\right\} \\ &= \sum_{i=1}^{n} \left(\frac{\partial G_r}{\partial y_i}\frac{\partial G_s}{\partial q_i} - \frac{\partial G_r}{\partial q_i}\frac{\partial G_s}{\partial y_i}\right) \\ &= (G_r, G_s);\end{aligned}$$

and similarly

$$(F_r, u_i) = (G_r, w_i), \quad (u_i, u_j) = (w_i, w_j).$$

Consequently

$$(G_r, G_s) = 0, \quad (G_r, w_i) = 0, \quad (w_i, w_j) = 0;$$

and the equations

$$G_1 = 0, \ldots, G_m = 0, \quad w_{m+1} = a_{m+1}, \ldots, w_n = a_n$$

are resoluble with regard to $q_1, \ldots, q_n$, expressing these quantities in terms of $y_1, \ldots, y_n, a_{m+1}, \ldots, a_n$. Moreover, the earlier results shew that the values of $q_1, \ldots, q_n$ thus given make

$$dZ = q_1 dy_1 + \ldots + q_n dy_n$$

an exact equation: when the quadrature is effected, the result will be of the form

$$Z = \psi(y_1, \ldots, y_n, a_{m+1}, \ldots, a_n) + b,$$

where b is an arbitrary constant. To obtain the value of z, we note that

$$x_k = \frac{\partial Z}{\partial y_k} = \frac{\partial \psi}{\partial y_k},$$

for $k=\mu+1, \ldots, n$; and

$$\begin{aligned} Z &= z - x_{\mu+1}p_{\mu+1} - \ldots - x_n p_n \\ &= z + x_{\mu+1}y_{\mu+1} + \ldots + x_n y_n ; \end{aligned}$$

hence

$$\begin{aligned} z + x_{\mu+1}y_{\mu+1} &+ \ldots + x_n y_n \\ &= \psi(y_1, \ldots, y_n, a_{m+1}, \ldots, a_n) + b \\ &= \psi(x_1, \ldots, x_\mu, y_{\mu+1}, \ldots, y_n, a_{m+1}, \ldots, a_n) + b, \\ x_k &= \frac{\partial \psi}{\partial y_k}, \end{aligned}$$

for $k=\mu+1, \ldots, n$. Eliminating $y_{\mu+1}, \ldots, y_n$ among these $n-\mu+1$ relations, and resolving the eliminant with regard to z, we have a relation

$$z = I(x_1, \ldots, x_n, a_{m+1}, \ldots, a_n) + b,$$

for z occurs in the elimination only in the combination $z-b$; this relation is the integral of the original system of equations, and it involves $n-m+1$ constants.

One such integral will arise for each resolved set of equations arising out of the resolution of the equations

$$F_1 = 0, \ldots, F_m = 0, \quad u_{m+1} = a_{m+1}, \ldots, \quad u_n = a_n ;$$

the aggregate of these integrals includes all the integrals that are thus obtainable. But other integrals may be deduced by other processes, which will form the subject of subsequent explanations.

Ex. Consider an equation

$$f = 3px + qy + q^3x^2 = 0.$$

An equation $u=a$ is required, which may coexist with $f=0$: it is given by

$$(f, u) = 0,$$

an equation that is homogeneous and linear in u: and an integral is required which involves either p or q or both. The subsidiary equations are

$$\frac{dx}{3x} = \frac{dp}{-3p-2q^3x^2} = \frac{dy}{y+3q^2x^2} = \frac{dq}{-q};$$

one integral of these equations is

$$q^3x = \text{constant};$$

and another integral is

$$\frac{y}{xq^2} - x = \text{constant}.$$

Taking the former integral, we have to resolve the equations

$$f=0, \quad q^3x = c^3,$$

where c is a constant. Resolution with respect to p and q is simple, giving

$$q=cx^{-\frac{1}{3}},\quad p=-\tfrac{1}{3}c^3-\tfrac{1}{3}cyx^{-\frac{4}{3}};$$

substituting in

$$dz=p\,dx+q\,dy,$$

and effecting the quadrature, we find

$$z=A-\tfrac{1}{3}c^3x+cx^{-\frac{1}{3}}y,$$

which is an integral involving two arbitrary constants.

Taking the other integral of the subsidiary equations, we have to resolve the equations

$$f=0,\quad \frac{y}{xq^2}-x=2a,$$

where a is a constant. Resolution with regard to p and q is possible: it is simpler with regard to p and y, and with respect to these variables gives the relations

$$F=y-q^2(x^2+2ax)=0,\quad G=p+\tfrac{2}{3}q^3(x+a)=0,$$

which satisfy the relation $(F, G)=0$ identically. After the investigations above, we take q and x as the new independent variables and Z as the new dependent variable, where

$$Z=z-qy.$$

Thus

$$\begin{aligned}dZ&=p\,dx-y\,dq\\&=-\tfrac{2}{3}q^3(x+a)\,dx-q^2(x^2+2ax)\,dq,\end{aligned}$$

so that

$$Z=B-\tfrac{1}{3}q^3(x^2+2ax).$$

Now

$$y=-\frac{\partial Z}{\partial q}=q^2(x^2+2ax),$$

so that we have to eliminate q between the equations

$$y=q^2(x^2+2ax),\quad z-qy=B-\tfrac{1}{3}q^3(x^2+2ax).$$

The result is

$$z-B=\tfrac{2}{3}y^{\frac{3}{2}}(x^2+2ax)^{-\frac{1}{2}},$$

another integral involving two arbitrary constants.

Later, the relation between different integrals will be considered.

60. Reasons were adduced in § 54 for discussing equations in a form which does not explicitly contain the dependent variable; but it should be added that the preceding method can be applied also when the dependent variable does occur explicitly. In that case, the investigation follows the same lines as before, but the analysis is rather more complicated on account of the occurrence of z: it will be sufficient to give merely an outline.

Let $f_1 = 0, \ldots, f_m = 0$ be a complete system of equations in the n independent variables, involving the dependent variable and its first derivatives: then the relation

$$[f_r, f_s] = 0$$

is satisfied for all values of r and s, either explicitly or in virtue of the equations of the system. Any other equation, coexisting with the equations of the system in a form $u =$ constant, must be such that

$$[f_i, u] = 0,$$

for all values $i = 1, \ldots, m$.

Suppose that the system of equations $f_1 = 0, \ldots, f_m = 0$ is resolved with regard to z and $m - 1$ of the variables p, say $p_1, \ldots, p_{m-1}$, in the form

$$z - \psi = 0, \quad p_1 - \psi_1 = 0, \ldots, p_{m-1} - \psi_{m-1} = 0;$$

the resolved system is in involution, for (§ 55, Note) the relations

$$[p_i - \psi_i, p_j - \psi_j] = 0,$$

$$[z - \psi - x_1 (p_1 - \psi_1) - \ldots - x_{m-1} (p_{m-1} - \psi_{m-1}), p_i - \psi_i] = 0,$$

for all values of i and j from the set $1, \ldots, m-1$, are satisfied identically. Let these values of $z, p_1, \ldots, p_{m-1}$, in terms of $x_1, \ldots, x_n, p_m, \ldots, p_n$, be substituted in u and let the resulting value be denoted by w; then the equations

$$[z - \psi - x_1 (p_1 - \psi_1) - \ldots - x_{m-1} (p_{m-1} - \psi_{m-1}), w] = 0,$$

$$[p_i - \psi_i, w] = 0,$$

are satisfied in virtue of $[f_i, u] = 0$, and conversely.

Moreover, the system of equations determining w is a complete system. For if

$$g_1 = 0, \ldots, g_m = 0$$

is a complete system in involution, then the identical relation

$$[[g_r, g_s] w] + [[g_s, w] g_r] + [[w, g_r] g_s]$$

$$= - \frac{\partial w}{\partial z} [g_r, g_s] - \frac{\partial g_r}{\partial z} [g_s, w] - \frac{\partial g_s}{\partial z} [w, g_r]$$

becomes

$$[g_r, [g_s, w]] - [g_s, [g_r, w]] = \frac{\partial g_r}{\partial z} [g_s, w] - \frac{\partial g_s}{\partial z} [g_r, w],$$

because $[g_r, g_s]$ vanishes identically: hence, denoting $[g_i, w]$ by $B_i(w)$, we have

$$B_r(B_s w) - B_s(B_r w) = \frac{\partial g_r}{\partial z} B_s(w) - \frac{\partial g_s}{\partial z} B_r(w)$$

$$= 0,$$

in virtue of $B_r(w)=0$, $B_s(w)=0$, which is the test of a complete system.

As the equations

$$[z-\psi-x_1(p_1-\psi_1)-\ldots-x_{m-1}(p_{m-1}-\psi_{m-1}),\ w]=0,$$
$$[p_i-\psi_i,\ w]=0,$$

are a complete system, they possess a simultaneous set of integrals: let one such integral involving some one of the variables $p_m, \ldots, p_n$ be obtainable in the form

$$w=w(x_1, \ldots, x_n, p_m, \ldots, p_n);$$

then the equation

$$w(x_1, \ldots, x_n, p_m, \ldots, p_n)=a,$$

where a is an arbitrary constant, coexists with

$$z-\psi=0, \quad p_1-\psi_1=0, \ldots, p_{m-1}-\psi_{m-1}=0.$$

Let it be resolved so as to give (say) p_m in terms of the other variables it contains, and denote the result by

$$p_m=\chi_m;$$

and let this value be inserted in the other equations so that they take the form

$$z-\chi=0, \quad p_1-\chi_1=0, \ldots, p_{m-1}-\chi_{m-1}=0.$$

Then for the next stage, we proceed from the $m+1$ equations

$$z-\chi=0, \quad p_1-\chi_1=0, \ldots, p_m-\chi_m=0,$$

as in this stage from the m equations.

When $n+1$ equations have been obtained, the first of them has a form

$$z-\theta=0,$$

where θ involves $n-m+1$ constants; $z=\theta$ is an integral of the original system.

JACOBI'S SECOND METHOD, WHEN z DOES NOT OCCUR.

61. The preceding investigation has been carried out after an initial assumption that the m equations in the given complete system are resoluble with regard to m of the variables $p_1, \ldots, p_n$: the selection of $p_1, \ldots, p_m$ was merely typical. This assumption is not any real limitation: for if the m equations

$$F_1=0, \ \ldots, \ F_m=0$$

are not theoretically resoluble with regard to any m of the variables $p_1, \ldots, p_n$, so that all the determinants

$$\left\| \begin{array}{c} \frac{\partial F_1}{\partial p_1}, \ldots, \frac{\partial F_1}{\partial p_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial F_m}{\partial p_1}, \ldots, \frac{\partial F_m}{\partial p_n} \end{array} \right\|$$

vanish, then $p_1, \ldots, p_n$ can be eliminated among the m equations: as the m equations are functionally distinct, the eliminant cannot vanish identically and so would take a form

$$\Theta(x_1, \ldots, x_n) = 0,$$

a relation among the independent variables alone. Such a result is excluded: and so the m equations are resoluble with regard to some selection of m variables from the set $p_1, \ldots, p_n$.

The forms of the resolved equations may, however, be complicated: and then it might be desirable to proceed from the unresolved equations. Such a process was given by Jacobi, and it is sometimes called his *second method*; naturally, it is less simple than the method that has just been expounded, for it deals with equations of a less simple form than those to which Mayer's method is applied. Indeed, the preceding process is really a form of Jacobi's method: but it has been simplified and shortened by the improvements and the developments due to Lie and to Mayer.

Thus far in the range of these discussions, we have been considering m equations: and though there is no intrinsic element in the analysis which makes m greater than unity, all the superficial appearance suggests that m is not unity. For variety, we shall now deal with the integration of a single equation: and it will be found that, in general, the process leads to the issue through the integration of systems. For this purpose, we shall use Jacobi's method: a sufficient indication of its detailed working, whether for single equations or for detailed systems, will thus be provided.

As already hinted, Jacobi's method of integration (without the modifications and amplifications introduced by the investigations of Lie and Mayer) appears to be most useful when, from whatever cause, the equation or equations are not resolved with regard to one or more of the derivatives. We begin with a single irreducible

equation, unresolved with regard to any of the variables p and not explicitly containing the dependent variable: it may be taken in the form

$$f = f(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0.$$

By the process adopted, other $n-1$ equations are required which, speaking generally*, would suffice for the expression of $p_1, \ldots, p_n$ in terms of $x_1, \ldots, x_n$.

If $u =$ constant be such an equation, then the relation

$$(f, u) = 0$$

must be satisfied; any integral of this equation, distinct from f (which manifestly is an integral) and involving some of the variables $p_1, \ldots, p_n$, will suffice for the purpose. The system of ordinary equations, subsidiary to the construction of this integral, is

$$\frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \ldots = \frac{dx_n}{-\frac{\partial f}{\partial p_n}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \ldots = \frac{dp_n}{\frac{\partial f}{\partial x_n}};$$

let $f_1 =$ constant be one integral of the system, where f_1 involves one at least of the quantities $p_1, \ldots, p_n$: then we may take

$$u = f_1.$$

The relation

$$(f, f_1) = 0$$

is satisfied identically: and the two equations

$$f_1 = a, \quad f = 0,$$

where a is an arbitrary constant, satisfy the conditions of co-existence.

62. We now proceed to obtain another equation, involving some of the variables p and coexisting with the two equations; if it be $v =$ constant, then the relations

$$(f, v) = 0, \quad (f_1, v) = 0,$$

must be satisfied. These effectively are two equations for the determination of v; any common integral of the appropriate form and functionally distinct from f and f_1 (both of which manifestly are integrals) will suffice. Now the equation $(f, v) = 0$ is the

* That is to say, omitting from consideration the alternative already discussed in §§ 58, 59.

same as that for the determination of u, so that the subsidiary ordinary equations are the same as before: let

$$\phi = \phi(x_1, \ldots, x_n, p_1, \ldots, p_n) = \text{constant},$$

be an integral, which involves some of the variables $p_1, \ldots, p_n$ and is functionally distinct from f and f_1; then the equation

$$(f, \phi) = 0$$

is satisfied identically.

If ϕ is such that $(f_1, \phi) = 0$, then we may take

$$v = \phi$$

as a common integral of the two equations.

If ϕ is such that (f_1, ϕ) does not vanish, then (f_1, ϕ) is either a constant, say c, or is a variable quantity, say ϕ_1. In the latter case, ϕ_1 is an integral of the equation $(f, u) = 0$, by Poisson's theorem (§ 52); and it is a new integral, if it is functionally distinct from f, f_1, ϕ.

Similarly, if ϕ_1 is a new integral of $(f, u) = 0$, we may have $(f_1, \phi_1) = 0$, in which case we may take

$$v = \phi_1$$

as a common integral of the two equations; or if (f_1, ϕ_1) is not zero, it is either a constant, say c', or is a variable quantity, say ϕ_2. As before, Poisson's theorem shews that ϕ_2 is an integral of the equation $(f, u) = 0$: it is a new integral, if it is functionally distinct from f, f_1, ϕ, ϕ_1.

Proceeding in this sequence, we have a number of functions $\phi, \phi_1, \phi_2, \ldots$; and provided (f_1, ϕ_μ) is a variable quantity, it is a new integral of the equation $(f, u) = 0$ if it is functionally distinct from $f, f_1, \phi, \phi_1, \ldots, \phi_\mu$. Now the number of functionally distinct integrals of $(f, u) = 0$ is not greater than $2n - 1$; hence, if the series of functions either should not cease, by the occurrence of a zero-value for (f, ϕ_r), or should not give a constant non-zero value for (f, ϕ_r), then we must sooner or later obtain a function ϕ_r which is expressible in terms of those already found. Let ϕ_i be the first function in the sequence which either is zero, or is a pure constant different from zero, or is expressible in terms of the preceding functions.

Then no new distinct integrals will arise from continuing the construction of the functions (f_1, ϕ). For in the first alternative and in the second alternative, we have $(f_1, \phi_i) = 0$: and if, in the third alternative

$$\phi_i = \theta(f, f_1, \phi, \phi_1, \ldots, \phi_{i-1}),$$

then

$$(f_1, \phi_i) = (f_1, f)\frac{\partial\theta}{\partial f} + (f_1, f_1)\frac{\partial\theta}{\partial f_1} + (f_1, \phi)\frac{\partial\theta}{\partial\phi} + \ldots + (f_1, \phi_{i-1})\frac{\partial\theta}{\partial\phi_{i-1}}$$

$$= \phi_1\frac{\partial\theta}{\partial\phi} + \ldots + \phi_i\frac{\partial\theta}{\partial\phi_{i-1}},$$

which is expressible in terms of the functions anterior to ϕ_i; and so for each succeeding function.

Accordingly, consider a functional combination of $\phi, \phi_1, \ldots, \phi_{i-1}$ represented by

$$v = g(\phi, \phi_1, \ldots, \phi_{i-1});$$

then

$$(f, v) = 0,$$

whatever be the form of the function g. Also, as above,

$$(f_1, v) = \phi_1\frac{\partial g}{\partial\phi} + \phi_2\frac{\partial g}{\partial\phi_1} + \ldots + \phi_i\frac{\partial g}{\partial\phi_{i-1}};$$

hence, if g can be determined so that the right-hand side vanishes, we shall have $(f_1, v) = 0$. In order to determine g from the relation

$$\phi_1\frac{\partial g}{\partial\phi} + \phi_2\frac{\partial g}{\partial\phi_1} + \ldots + \phi_i\frac{\partial g}{\partial\phi_{i-1}} = 0,$$

we consider the system of $i-1$ ordinary equations

$$\frac{d\phi}{\phi_1} = \frac{d\phi_1}{\phi_2} = \ldots = \frac{d\phi_{i-1}}{\phi_i};$$

their integral equivalent consists of $i-1$ distinct integral equations of the form

$$h_r(\phi, \phi_1, \phi_2, \ldots, \phi_{i-1}) = \text{constant}, \qquad (r = 1, \ldots, i-1),$$

whether ϕ_i be zero, or a constant, or be the foregoing quantity θ; and each of these functions h_r is such that

$$\phi_1\frac{\partial h_r}{\partial\phi} + \phi_2\frac{\partial h_r}{\partial\phi_1} + \ldots + \phi_i\frac{\partial h_r}{\partial\phi_{i-1}} = 0.$$

Hence, taking

$$v = h_r(\phi, \phi_1, \phi_2, \ldots, \phi_{i-1}),$$

we have

$$(f,\ v)=0,\quad (f_1,\ v)=0\,;$$

and thus we have $i-1$ distinct integrals common to the two equations.

If ϕ_i is zero, the simplest of these integrals is

$$v=\phi_{i-1}\,;$$

even so, it is only one of $i-1$ distinct integrals common to the two equations.

Also i is greater than zero, because we have assumed that $(f_1,\ \phi)$, which is ϕ_1, does not vanish. Hence, if i is greater than unity, a common integral has been obtained; in that case, indeed, we have obtained $i-1$ common integrals of $(f,\ u)=0$, $(f_1,\ u)=0$, distinct from f and f_1. Consequently, this stage is completed except only when $(f_1,\ \phi)$, though not zero, either is a constant or is not functionally independent of f, f_1, ϕ: that is, in the case when $i=1$.

In the case when $i=1$ in connection with a quantity ϕ, we return to the equations subsidiary to $(f,\ u)=0$: and we determine another integral of them in the form

$$\psi=\psi(x_1,\ \ldots,\ x_n,\ p_1,\ \ldots,\ p_n)=\text{constant},$$

where ψ is functionally distinct from f, f_1, ϕ. We proceed with ψ in the same way as with ϕ, by forming the functions

$$(f_1,\ \psi)=\psi_1,\quad (f_1,\ \psi_1)=\psi_2,\ \ldots,$$

in succession; and, as before, we obtain an integral or a number of integrals common to the two equations

$$(f,\ u)=0,\quad (f_1,\ u)=0,$$

save only in the case where ψ_1, though not zero, is either a constant or is not functionally independent of f, f_1, ψ.

Even if the integral required is not provided because of the double lapse of the process into this exceptional stage, an integral as required can be obtained by a combination of the two integrals ϕ and ψ. Take any function $g(\phi,\ \psi,\ f_1)$: owing to the origin of ϕ and ψ, we have

$$(f,\ g)=0\,;$$

and

$$(f_1, g) = \frac{\partial g}{\partial \phi}(f_1, \phi) + \frac{\partial g}{\partial \psi}(f_1, \psi) + \frac{\partial g}{\partial f_1}(f_1, f_1)$$
$$= \frac{\partial g}{\partial \phi}\phi_1 + \frac{\partial g}{\partial \psi}\psi_1.$$

We form the equations

$$\frac{d\phi}{\phi_1} = \frac{d\psi}{\psi_1} = \frac{df_1}{0}:$$

in these equations, ϕ_1 either is a constant or is a functional combination of f, f_1, ϕ, say $\Phi(f, f_1, \phi)$; and likewise for ψ_1, which either is a constant or is a functional combination of f, f_1, ψ, say $\Psi(f, f_1, \psi)$. For our purposes, f is zero: one integral of the two ordinary equations is $f_1 = a_1$, where a_1 is an arbitrary constant; another integral is given by integrating

$$\frac{d\phi}{\Phi(0, a_1, \phi)} = \frac{d\psi}{\Psi(0, a_1, \psi)}.$$

Let an integral equivalent of this be

$$u(a_1, \phi, \psi) = \text{constant},$$

or say

$$u(f_1, \phi, \psi) = \text{constant}:$$

then if we take

$$g(\phi, \psi, f_1) = u(f_1, \phi, \psi),$$

we have

$$(f_1, u) = 0.$$

In other words, $u = u(f_1, \phi, \psi)$ is an integral common to the two equations $(f, u) = 0$, $(f_1, u) = 0$.

The simplest instance occurs when $\phi_1 = c$, $\psi_1 = c'$, where c and c' are constants: then

$$u = c'\phi - c\psi.$$

In every case, an integral common to the two equations $(f, u) = 0$, $(f_1, u) = 0$ has been obtained. It has required the assignment of certainly one integral of the equations subsidiary to $(f, u) = 0$: even when the functions (f_1, ϕ_r) have to be formed, each of them gives an integral of that subsidiary system, and so does each combination of the type $h_1(\phi, \phi_1, \ldots)$, $h_2(\phi, \phi_1, \ldots)$, ...; and only one of these combinations is assigned. The most unfavourable association is that in which the ϕ-series ends with ϕ_1 and a ψ-series ends with ψ_1; and then the two integrals ϕ and ψ

of the subsidiary system of $(f, u) = 0$ must be assigned for the construction of an integral common to $(f, u) = 0$, $(f_1, u) = 0$.

Now the subsidiary system consists of $2n-1$ ordinary equations; its integral equivalent must consist of $2n-1$ independent equations. One of these is $f = 0$, and another consists of $f_1 = a_1$; hence there are other $2n-3$ independent equations, which may be denoted by

$$\phi = \text{constant}, \quad \psi = \text{constant}, \quad \chi = \text{constant}, \quad \vartheta = \text{constant}, \quad \ldots.$$

If $(f_1, \phi) = 0$, then $u = \phi$ is the quantity desired. If (f_1, ϕ) is neither zero, nor a constant, nor a functional combination of f, f_1, ϕ, then there is a ϕ-series: and a single combination of the members of the series, (which must also, in the circumstances, be a combination of some of the quantities $\phi, \psi, \chi, \vartheta, \ldots$), will give a quantity u as required. The most unfavourable set of results possible is that in which the ϕ-series terminates with (f_1, ϕ), the ψ-series terminates with (f_1, ψ), and so on, no one of these quantities vanishing: then each of the quantities

$$\int \frac{d\phi}{(f_1, \phi)} - \int \frac{d\psi}{(f_1, \psi)},$$
$$\int \frac{d\phi}{(f_1, \phi)} - \int \frac{d\chi}{(f_1, \chi)},$$
$$\int \frac{d\phi}{(f_1, \phi)} - \int \frac{d\vartheta}{(f_1, \vartheta)},$$
$$\vdots$$

is an integral common to $(f, u) = 0$, $(f_1, u) = 0$. As there are $2n-3$ quantities $\phi, \psi, \chi, \vartheta, \ldots$, it follows that, even with the most unfavourable set of results, the two equations $(f, u) = 0$ and $(f_1, u) = 0$ possess $2n-4$ integrals in common, independent of f, of f_1, and of one another, and obtainable in this manner.

Let $u = f_2$ be one of these integrals: then the equation

$$f_2 = a_2,$$

where a_2 is an arbitrary constant, associates itself with

$$f = 0, \quad f_1 = a_1.$$

We thus have succeeded in associating two new equations $f_1 = a_1$, and $f_2 = a_2$, with f and with one another.

63. The next stage is the determination of a new equation $u = \text{constant}$, consistent with

$$f = 0, \quad f_1 = a_1, \quad f_2 = a_2;$$

the necessary and sufficient conditions for coexistence are

$$(f, u)=0, \quad (f_1, u)=0, \quad (f_2, u)=0.$$

Let $u=\lambda$ be an integral, common to $(f, u)=0$, $(f_1, u)=0$, and functionally distinct from f, f_1, f_2, where

$$\lambda=\lambda(x_1, \ldots, x_n, p_1, \ldots, p_n):$$

it may be taken as one of the $2n-5$ common integrals, other than f, f_1, f_2. We proceed as before, and form a series of functions

$$(f_2, \lambda)=\lambda_1, \quad (f_2, \lambda_1)=\lambda_2, \ldots.$$

Each of these quantities is a common integral of $(f, u)=0$, $(f_1, u)=0$. For

$$(f(f_2, \theta))+(f_2(\theta, f))+(\theta(f, f_2))=0,$$
$$(f_1(f_2, \theta))+(f_2(\theta, f_1))+(\theta(f_1, f_2))=0;$$

and $(f, f_2)=0$, $(f_1, f_2)=0$, both identically, so that

$$(\theta(f, f_2))=0, \quad (\theta(f_1, f_2))=0;$$

and therefore

$$(f(f_2, \theta))=(f_2(f, \theta)),$$
$$(f_1(f_2, \theta))=(f_2(f_1, \theta)).$$

Let $\theta=\lambda$; these results give

$$(f, \lambda_1)=(f_2(f, \lambda))=0,$$
$$(f_1, \lambda_1)=(f_2(f_1, \lambda))=0,$$

because $(f, \lambda)=0$, $(f_1, \lambda)=0$, both identically satisfied; thus λ_1 is an integral common to $(f, u)=0$, $(f_1, u)=0$. Let $\theta=\lambda_1$; then the two relations give

$$(f, \lambda_2)=(f_2(f, \lambda_1))=0,$$
$$(f_1, \lambda_2)=(f_2(f_1, \lambda_1))=0,$$

as before: that is, λ_2 is an integral common to $(f, u)=0$, $(f_1, u)=0$. And so for all the functions λ in succession.

The number of independent integrals is limited: and thus the λ-series will terminate either in a zero, or in a pure constant, or in a function expressible in terms of the anterior functions. Proceeding as before, we obtain some λ-function, or some combination of λ-functions, say Λ, such that

$$(f_2, \Lambda)=0,$$

save only in the case when (f_2, λ) is either a constant (not zero) or is not distinct from f, f_1, f_2, λ.

In the latter circumstance, we take another integral μ, common to $(f, u)=0$, $(f_1, u)=0$, and distinct from f, f_1, f_2, λ. Proceeding in the same way, we obtain some μ-function or some combination of μ-functions, say M, such that

$$(f_2, M)=0,$$

save only in the case when (f_2, μ) either is a constant (not zero) or is not distinct from f, f_1, f_2, μ.

And should the latter happen, then if

$$N=\int\frac{d\lambda}{(f_2, \lambda)}-\int\frac{d\mu}{(f_2, \mu)},$$

we have

$$(f_2, N)=0.$$

Thus in every case we obtain an integral common to the three equations

$$(f, u)=0, \quad (f_1, u)=0, \quad (f_2, u)=0:$$

and in the least favourable combination of circumstances, there are $2n-6$ such integrals, independent of f, f_1, f_2, and of one another.

Let f_3 be one of those integrals; then the equation

$$f_3=a_3,$$

where a_3 is an arbitrary constant, associates itself with

$$f=0, \quad f_1=a_1, \quad f_2=a_2.$$

64. We proceed in this way from stage to stage, obtaining equations $f_4=a_4$, ... in succession which are associated with all the equations that precede them. The last stage of all is the construction of an equation $f_{n-1}=a_{n-1}$. Our earlier results shew that, when the equations

$$f=0, \quad f_1=a_1, \quad \ldots, \quad f_{n-1}=a_{n-1}$$

are resolved for $p_1, \ldots, p_n$ in terms of $x_1, \ldots, x_n$, the values thus obtained are such as to make

$$p_1dx_1+\ldots+p_ndx_n$$

an exact differential; after quadrature, an integral of the original equation $f=0$ is given by

$$z-a_n=\int(p_1dx_1+\ldots+p_ndx_n),$$

involving n arbitrary constants.

If it is not possible or not convenient to resolve the equations $f=0, \ldots, f_{n-1}=a_{n-1}$ with regard to $p_1, \ldots, p_n$, we choose another set of the variables involved and, resolving with regard to these, adopt the process explained in §§ 58, 59.

JACOBI'S SECOND METHOD WHEN z DOES OCCUR.

65. In the preceding account of Jacobi's method of solving an equation $f=0$, the dependent variable z has been supposed not to occur explicitly. If it should occur explicitly, we have already seen that there is a mode of proceeding by a change of dependent variable, associated with a unit increase in the number of independent variables. This mode of proceeding may be cumbrous: and in any case, it is desirable (if possible) to have a direct method for constructing an integral.

Accordingly, let

$$f=f(x_1, \ldots, x_n, z, p_1, \ldots, p_n)=0$$

be an irreducible equation which involves z explicitly: if

$$u=u(x_1, \ldots, x_n, z, p_1, \ldots, p_n)=\text{constant}$$

be an equation which can coexist with $f=0$, it is necessary and sufficient that the relation

$$[f, u]=0$$

should be satisfied. This equation is homogeneous and linear in the derivatives of u; written in full, it is

$$\sum_{i=1}^{n}\left\{\left(\frac{\partial f}{\partial x_i}+p_i\frac{\partial f}{\partial z}\right)\frac{\partial u}{\partial p_i}-\frac{\partial u}{\partial x_i}\frac{\partial f}{\partial p_i}-\frac{\partial u}{\partial z}p_i\frac{\partial f}{\partial p_i}\right\}=0.$$

To obtain a value of u, we construct the system of subsidiary equations

$$\frac{dx_1}{-\frac{\partial f}{\partial p_1}}=\cdots=\frac{dx_n}{-\frac{\partial f}{\partial p_n}}=\frac{dp_1}{\frac{\partial f}{\partial x_1}+p_1\frac{\partial f}{\partial z}}=\cdots=\frac{dp_n}{\frac{\partial f}{\partial x_n}+p_n\frac{\partial f}{\partial z}}$$

$$=\frac{dz}{-\left(p_1\frac{\partial f}{\partial p_1}+\ldots+p_n\frac{\partial f}{\partial p_n}\right)},$$

which (for reasons that will appear hereafter) are called the *equations of the characteristics*; and we take an integral of these

equations, choosing by preference one that is not free from $z, p_1, \ldots, p_n$, if any such exist. Let such an integral be

$$g(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = \text{constant};$$

then the equation

$$u = g(x_1, \ldots, x_n, z, p_1, \ldots, p_n).$$

gives a value of u as required; and the relation

$$[f, g] = 0$$

is satisfied identically, so far as concerns $g =$ constant, but not necessarily identically, so far as concerns $f = 0$: indeed, it may be satisfied only in virtue of $f = 0$.

Ex. The characteristics of the equation

$$pxz + qyz - xy = 0$$

are given by

$$\frac{dx}{-xz} = \frac{dy}{-yz} = \frac{dp}{pz - y + p(px + qy)} = \frac{dq}{qz - x + q(qx + py)} = \frac{dz}{-pxz - qyz}.$$

An integral, as required, is given by

$$z^2 - xy = \text{constant};$$

the relation

$$[pxz + qyz - xy,\ z^2 - xy] = 0$$

is satisfied only in virtue of $f = 0$. Another integral, as required, is given by

$$\frac{pz - \frac{1}{2}y}{qz - \frac{1}{2}x} = \text{constant};$$

the relation

$$\left[pxz + qyz - xy,\ \frac{pz - \frac{1}{2}y}{qz - \frac{1}{2}x}\right] = 0$$

is satisfied identically.

66. Accordingly, at this stage it is convenient, for the sake of very substantial simplification of the analysis, to resolve the two equations

$$f = 0, \quad g = a,$$

for z and one of the variables p, chosen so as to give the simplest resolution: let the selected variable be p_1, and let the result of the resolution be denoted by

$$z - \psi = 0, \quad p_1 - \psi_1 = 0,$$

where ψ and ψ_1 are functions of $x_1, \ldots, x_n, p_2, \ldots, p_n$. Then, after the explanations in § 55, Note, and § 60, we take these two equations in the form

$$z - \psi - x_1(p_1 - \psi_1) = 0, \quad p_1 - \psi_1 = 0;$$

any equation $w = c$ that can coexist with them must satisfy the equations

$$[z - \psi - x_1(p_1 - \psi_1), w] = 0,$$
$$[p_1 - \psi_1, w] = 0.$$

These two equations to determine w are, by § 60, a complete system.

As any integral of these two equations is to furnish an equation w = constant, which shall coexist with

$$z - \psi = 0, \quad p_1 - \psi_1 = 0,$$

it can be transformed so that, if z and p_1 do occur, they are replaced by ψ and ψ_1 respectively: that is, without loss of generality, w may be assumed not to involve either z or p_1 explicitly. Let

$$w = \phi = \phi(x_1, \ldots, x_n, p_2, \ldots, p_n)$$

be an integral of the equation

$$[z - \psi - x_1(p_1 - \psi_1), w] = 0;$$

then, as $[z - \psi - x_1(p_1 - \psi_1), \phi]$ does not contain z or p_1, so that it cannot vanish in virtue of $z - \psi = 0$ or $p_1 - \psi_1 = 0$, and as it must vanish, it vanishes identically. Construct the function $[p_1 - \psi_1, \phi]$, $= \phi_1$ say. If ϕ_1 vanishes identically, this last condition is satisfied: also $[p_1 - \psi_1, \phi] = 0$; and therefore $w = \phi$ is a common integral of the two equations. In that case, the equation

$$\phi = a_1,$$

where a_1 is an arbitrary constant, can be associated with

$$z - \psi = 0, \quad p_1 - \psi_1 = 0.$$

Suppose, on the other hand, that ϕ_1 does not vanish identically; then, as

$$[[\zeta, \pi]\phi] + [[\pi, \phi]\zeta] + [[\phi, \zeta]\pi]$$
$$= -\frac{\partial\phi}{\partial z}[\zeta, \pi] - \frac{\partial\zeta}{\partial z}[\pi, \phi] - \frac{\partial\pi}{\partial z}[\phi, \zeta]$$

identically, we have, on writing

$$\zeta = z - \psi - x_1(p_1 - \psi_1), \quad \pi = p_1 - \psi_1,$$

the relation

$$[[p_1 - \psi_1, \phi]\zeta] = -[p_1 - \psi_1, \phi],$$

that is,

$$[\phi_1, \zeta] = -\phi_1.$$

Thus ϕ_1 is not an integral of

$$[z-\psi-x_1(p_1-\psi_1),\ w]=0;$$

and as $[\phi_1, \zeta]$ involves derivatives of ϕ_1, it is clear that ϕ_1 cannot be a pure constant.

In that case, let

$$w=\chi=\chi(x_1, \ldots, x_n, p_2, \ldots, p_n)$$

be another integral of the equation

$$[z-\psi-x_1(p_1-\psi_1),\ w]=0,$$

functionally distinct from $w=\phi$; and construct the function $[p_1-\psi_1, \chi], =\chi_1$ say. If χ_1 vanishes identically, it follows that $w=\chi$ is an integral of the two equations determining w; and then the equation

$$\chi=c_1,$$

where c_1 is an arbitrary constant, can be associated with

$$z-\psi=0, \quad p_1-\psi_1=0.$$

But if χ_1 does not vanish identically, then we have

$$[\chi_1, \zeta]=-\chi_1,$$

as before; and χ_1 cannot be a constant. Also

$$[\phi_1, \zeta]=-\phi_1,$$

so that

$$\left[\frac{\chi_1}{\phi_1}, \zeta\right]=-\frac{1}{\phi_1}\chi_1-\frac{\chi_1}{\phi_1^2}(-\phi_1)=0;$$

and therefore

$$w_1=\frac{\chi_1}{\phi_1}$$

is an integral of the equation

$$[z-\psi-x_1(p_1-\psi_1),\ w]=0.$$

Now both ϕ_1 and χ_1 are variable: but $\dfrac{\chi_1}{\phi_1}$ may be a constant, say c. Then

$$\begin{aligned}[p_1-\psi_1, \chi]&=\chi_1\\&=c\phi_1\\&=[p_1-\psi_1, c\phi],\end{aligned}$$

and therefore

$$[p_1-\psi_1, \chi-c\phi]=0;$$

hence as

$$[z-\psi-x_1(p_1-\psi_1),\ \chi-c\phi]=0,$$

so that, under the particular hypothesis, $w = \chi - c\phi$ is an integral common to the two equations. But, in general, w_1 will be a variable quantity.

Assuming w_1 now not to be a constant, construct the function $[p_1 - \psi_1, w_1], = \chi_2$ say. If χ_2 vanishes identically, it follows that $w = w_1$ is an integral common to the two equations for the determination of w; and then the equation

$$w_1 = c_1,$$

where c_1 is an arbitrary constant, can be associated with

$$z - \psi = 0, \quad p_1 - \psi_1 = 0.$$

If χ_2 does not vanish identically, then

$$w_2 = \frac{\chi_2}{\phi_1}$$

is an integral of the equation

$$[z - \psi - x_1(p_1 - \psi_1), w] = 0.$$

If w_2 be constant and equal to α, then

$$w = w_1 - \alpha\phi$$

is an integral common to the two equations. But, in general, w_2 will be a variable quantity.

Assuming that w_2 is variable, we construct the function $[p_1 - \psi_1, w_2], = \chi_3$ say. As before, if χ_3 vanishes identically, we have an integral $w = w_2$ common to the two equations. If $\chi_3 = \beta\phi_1$, where β is a constant, then $w = w_2 - \beta\phi$ is an integral common to the two equations. If $\dfrac{\chi_3}{\phi_1}$ is not zero nor a constant, then

$$w_3 = \frac{\chi_3}{\phi_1}$$

is an integral of the equation

$$[z - \psi - x_1(p_1 - \psi_1), w] = 0.$$

Proceeding in this way, either we shall at some stage obtain an integral common to the two equations, or we shall obtain an integral of the equation

$$[z - \psi - x_1(p_1 - \psi_1), w] = 0$$

which is expressible in terms of the preceding integrals; for the number of functionally distinct integrals of that equation is limited.

When the last alternative occurs, all succeeding integrals are also so expressible; for if

$$w_m = f(w_{m-1}, \ldots, \chi, \phi),$$

then, as

$$[p_1 - \psi_1, w_\mu] = w_{\mu+1}\phi_1,$$

we have

$$w_{m+1} = \frac{1}{\phi_1}[p_1 - \psi_1, w_m]$$
$$= \frac{\partial f}{\partial w_{m-1}} w_m + \frac{\partial f}{\partial w_{m-2}} w_{m-1} + \ldots + \frac{\partial f}{\partial \chi} w_1 + \frac{\partial f}{\partial \phi},$$

shewing that w_{m+1} is expressible in terms of the earlier integrals: and so for all succeeding integrals. Now take some functional combination of ϕ, χ, $w_1, \ldots, w_{m-1}$, say

$$g = g(\phi, \chi, w_1, \ldots, w_{m-1});$$

then

$$[p_1 - \psi_1, g] = \phi_1 \left\{ \frac{\partial g}{\partial \phi} + \frac{\partial g}{\partial \chi} w_1 + \frac{\partial g}{\partial w_1} w_2 + \ldots + \frac{\partial g}{\partial w_{m-1}} f \right\}:$$

if g can be chosen so that the right-hand side vanishes, then $[p_1 - \psi_1, g] = 0$, and we shall have an integral common to our two equations. Let any integral of the system of ordinary equations

$$\frac{d\phi}{1} = \frac{d\chi}{w_1} = \frac{dw_1}{w_2} = \ldots = \frac{dw_{m-1}}{f(w_{m-1}, \ldots, \chi, \phi)}$$

be

$$g_1(\phi, \chi, \ldots, w_{m-1}) = \text{constant};$$

then taking

$$g = g_1(\phi, \chi, \ldots, w_{m-1}),$$

we have

$$[p_1 - \psi_1, g] = 0.$$

Moreover, there are m functionally distinct integrals of the system of ordinary equations: hence there are m distinct integrals common to the two equations

$$[z - \psi - x_1(p_1 - \psi_1), w] = 0, \quad [p_1 - \psi_1, w] = 0;$$

and these are constructed out of $m + 1$ distinct integrals of the first equation*.

* The simplest case occurs when w_1 is not functionally distinct from the integrals that precede it, viz. from ϕ and χ, so that we then have

$$w_1 = f(\chi, \phi);$$

if the integral of the equation

$$d\phi = \frac{d\chi}{f(\chi, \phi)}$$

be $g(\phi, \chi) = \text{constant}$, we take $g(\phi, \chi)$ as a common integral of the two equations.

Let one of these integrals be $u(x_1, \ldots, x_n, p_2, \ldots, p_n)$: then the equation

$$u(x_1, \ldots, x_n, p_2, \ldots, p_n) = a_2$$

coexists with the equations

$$z - \psi = 0, \quad p_1 - \psi_1 = 0.$$

In every case, therefore, an equation has been constructed which coexists with the equations already obtained.

67. To proceed to the next stage, we resolve the equation

$$u(x_1, \ldots, x_n, p_2, \ldots, p_n) = a_2$$

with regard to one of the variables p which it contains: let the resolved form be

$$p_2 = \chi_2,$$

where χ_2 involves $a_2, x_1, \ldots, x_n, p_3, \ldots, p_m$. Let this value of p_2 be inserted in ψ and ψ_1, and let the resulting expressions be χ and χ_1; then we have the simultaneous equations

$$z - \chi = 0, \quad p_1 - \chi_1 = 0, \quad p_2 - \chi_2 = 0.$$

Now χ, χ_1, χ_2 do not involve z: hence, writing

$$\zeta = z - \chi - x_1(p_1 - \chi_1) - x_2(p_2 - \chi_2), \quad \pi_1 = p_1 - \chi_1, \quad \pi_2 = p_2 - \chi_2,$$

and denoting by θ any quantity which does not involve z, we have

$$\begin{aligned}
&[[\pi_1, \theta]\zeta] + [[\theta, \zeta]\pi_1] + [[\zeta, \pi_1]\theta] = -[\pi_1, \theta],\\
&[[\pi_2, \theta]\zeta] + [[\theta, \zeta]\pi_2] + [[\zeta, \pi_2]\theta] = -[\pi_2, \theta],\\
&[[\pi_1, \theta]\pi_2] + [[\theta, \pi_2]\pi_1] + [[\pi_2, \pi_1]\theta] = 0;
\end{aligned}$$

also we have

$$[\zeta, \pi_1] = 0, \quad [\zeta, \pi_2] = 0, \quad [\pi_2, \pi_1] = 0,$$

identically.

Let σ and ρ be integrals common to the two equations

$$[\zeta, v] = 0, \quad [\pi_1, v] = 0,$$

obtained as in the preceding sections, and limited so that they do not involve z and that they are functionally distinct from π_2 and from one another; and let

$$[\pi_2, \sigma] = \sigma_1, \quad [\pi_2, \rho] = \rho_1.$$

If either σ_1 or ρ_1 vanishes, then we have a common integral of the three equations

$$[\zeta, v] = 0, \quad [\pi_1, v] = 0, \quad [\pi_2, v] = 0.$$

If neither of them vanishes, we make θ equal to σ and then to ρ in succession in the above identities. The first of the identities gives no condition; the second gives

$$[\zeta, \sigma_1] = \sigma_1, \quad [\zeta, \rho_1] = \rho_1;$$

and the third gives

$$[\pi_1, \sigma_1] = 0, \quad [\pi_1, \rho_1] = 0.$$

Hence $\dfrac{\rho_1}{\sigma_1}$ is an integral common to the two equations

$$[\zeta, v] = 0, \quad [\pi_1, v] = 0,$$

unless it is a constant; and if $\dfrac{\rho_1}{\sigma_1}$ is a constant, say equal to a, then

$$[\zeta, \rho - a\sigma] = 0, \quad [\pi_1, \rho - a\sigma] = 0,$$
$$[\pi_2, \rho - a\sigma] = \rho_1 - a\sigma_1 = 0,$$

so that $\rho - a\sigma$ would be a common integral of the three equations determining v.

Writing $\tau = \dfrac{\rho_1}{\sigma_1}$, and

$$[\pi_2, \tau] = \tau_1,$$

then if τ_1 vanishes, a common integral of the three equations is $v = \tau$; while if τ_1 does not vanish, we have

$$[\zeta, \tau_1] = \tau_1, \quad [\pi_1, \tau_1] = 0,$$

and therefore

$$\left[\zeta, \frac{\tau_1}{\sigma_1}\right] = 0, \quad \left[\pi_1, \frac{\tau_1}{\sigma_1}\right] = 0,$$

shewing that $\dfrac{\tau_1}{\sigma_1}$ is an integral common to the two equations $[\zeta, v] = 0$, $[\pi_1, v] = 0$.

We proceed as in the former stage: sooner or later, an integral of the two equations $[\zeta, v] = 0$ and $[\pi_1, v] = 0$ is obtained which is expressible in terms of the earlier integrals, or an integral is obtained which also satisfies $[\pi_2, v] = 0$. In the former alternative, we construct (as in the earlier stage) a combination of all these independent integrals of $[\zeta, v] = 0$ and $[\pi_1, v] = 0$ which shall also satisfy $[\pi_2, v] = 0$. Let it be

$$v = v(x_1, \ldots, x_n, p_3, \ldots, p_m);$$

then the equation

$$v(x_1, \ldots, x_n, p_3, \ldots, p_m) = a_3$$

coexists with the equations

$$\zeta = 0, \quad \pi_1 = 0, \quad \pi_2 = 0.$$

Let it be resolved for one of the variables p, say p_3, in the form

$$p_3 - \theta_3 = 0,$$

where θ_3 involves $a_3, x_1, \ldots, x_n, p_4, \ldots, p_m$; when this value is substituted in χ, χ_1, χ_2, let them become $\theta, \theta_1, \theta_2$; then our equations are

$$z - \theta = 0, \quad p_1 - \theta_1 = 0, \quad p_2 = \theta_2, \quad p_3 = \theta_3.$$

So we proceed from stage to stage. In each stage the construction of the new equation requires, in the least favourable combination of circumstances, the assignment of two integrals of the subsidiary system associated with the initial equation

$$[f, u] = 0.$$

This subsidiary system contains $2n$ differential equations: its integral equivalent must therefore contain $2n$ integral equations, that is, it possesses $2n$ integrals. Hence there are sufficient integrals for the achievement of n stages; at the end of the last, we shall have

$$z = \text{function of } x_1, \ldots, x_n, a_1, \ldots, a_n,$$

(where $a_1, \ldots, a_n$ are arbitrary constants) as the integral of the original equation. Or at the completion of the $(n-1)$th stage, we can resolve the n equations then coexisting, and express $p_1, \ldots, p_n$ in terms of $z, x_1, \ldots, x_n, a_1, \ldots, a_{n-1}$; substitution in the relation

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

and quadrature, lead to the integral required.

Ex. Let Z denote $z - p_1 x_1 - \ldots - p_n x_n$; and suppose that a set of equations

$$F_\mu = F_\mu (p_1, \ldots, p_n, Z) = 0, \qquad (\mu = 1, \ldots, m),$$

where $m < n$, is propounded for solution.

We have

$$\frac{dF_\mu}{dx_i} = \frac{\partial F_\mu}{\partial x_i} + p_i \frac{\partial F_\mu}{\partial z} = 0,$$

for all values of μ and of i: consequently

$$[F_r, F_s] = 0,$$

for all values of r and s, so that the system is in involution.

To obtain other equations consistent with the system, we need simultaneous integrals of

$$[F_1,\ u]=0,\ \ldots,\ [F_m,\ u]=0.$$

The equations subsidiary to the solution of $[F_1,\ u]=0$ are

$$\ldots=\frac{dp_i}{\dfrac{\partial F_1}{\partial x_i}+p_i\dfrac{\partial F_1}{\partial z}}=\ldots=\frac{dz}{-p_1\dfrac{\partial F_1}{\partial p_1}-\ldots-p_n\dfrac{\partial F_1}{\partial p_n}}=\ldots;$$

but $\dfrac{\partial F_1}{\partial x_i}+p_i\dfrac{\partial F_1}{\partial z}=0$, and so an integral of these equations is given by

$$p_1=\text{constant}.$$

Also

$$[p_1,\ F_r]=0,$$

for $r=2,\ \ldots,\ m$; so that $u=p_1$ is an integral common to all the equations $[F_\mu,\ u]=0$. We therefore associate the equation

$$p_1=a_1$$

with the given set; the new system is

$$F_1=0,\ \ldots,\ F_\mu=0,\ \ p_1=a_1,$$

and it is easily seen to be in involution.

Similarly, we may associate the equations

$$p_2=a_2,\ \ldots,\ p_{n-m+1}=a_{n-m+1},$$

where $a_2,\ \ldots,\ a_{n-m+1}$ are arbitrary constants, with the amplified system and with one another: and the whole system thus extended, viz.

$$F_1=0,\ \ldots,\ F_m=0,\ \ p_1=a_1,\ \ldots,\ p_{n-m+1}=a_{n-m+1},$$

is in involution. If therefore the quantities $p_1,\ \ldots,\ p_n$ can be eliminated from the system, the eliminant will give an integral of the original set.

Now the $n+1$ equations thus obtained are independent of one another, and they involve the $n+1$ quantities $p_1,\ \ldots,\ p_n,\ Z$; when resolved with regard to these quantities, they give

$$Z=c,\ \ p_i=a_i,$$

that is,

$$z-a_1x_1-a_2x_2-\ldots-a_nx_n=c,$$

where the constants $a_1,\ \ldots,\ a_{n-m+1}$ are arbitrary, and the remaining constants $a_{n-m+2},\ \ldots,\ a_n,\ c$ satisfy the m relations

$$F_\mu(a_1,\ \ldots,\ a_n,\ c)=0,$$

for the values $\mu=1,\ \ldots,\ m$. The equation

$$z=a_1x_1+a_2x_2+\ldots+a_nx_n+c,$$

with the limitations upon the constants, provides an integral of the propounded system.

CHARPIT'S METHOD: INTEGRALS WHEN THE CONDITIONS IN CAUCHY'S THEOREM ARE NOT SATISFIED.

68. Naturally, the simplest case of the preceding method arises when the number of independent variables is two. With the usual notation in this case, the equation may be written

$$f(x, y, z, p, q) = 0;$$

and the condition $[f, u] = 0$, which must be satisfied by u if $u = \text{constant}$ is to coexist with $f = 0$, is

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial u}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial u}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial q}\frac{\partial u}{\partial y}$$
$$- \left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\frac{\partial u}{\partial z} = 0.$$

To obtain an integral of this homogeneous linear equation which shall involve p or q or both, the system of ordinary equations

$$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}} = \frac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}}$$

is formed: if

$$u(x, y, z, p, q) = \text{constant}$$

be any integral, distinct from $f = 0$, involving p or q or both, then the equations

$$f(x, y, z, p, q) = 0, \quad u(x, y, z, p, q) = a,$$

where a is an arbitrary constant, are resolved with respect* to p and q. These values make the equation

$$dz - p\,dx - q\,dy = 0$$

exact. For from the equations $f = 0$, $u = a$, we find

$$\frac{\partial(f, u)}{\partial(x, p)} + p\frac{\partial(f, u)}{\partial(z, p)} + \frac{\partial(f, u)}{\partial(y, q)} + q\frac{\partial(f, u)}{\partial(z, q)} = \left(\frac{dq}{dx} - \frac{dp}{dy}\right)\frac{\partial(f, u)}{\partial(p, q)};$$

and because $u(x, y, z, p, q) = \text{constant}$ is an integral of the system of ordinary equations, the left-hand side of this equation vanishes, so that

$$\left(\frac{dq}{dx} - \frac{dp}{dy}\right)\frac{\partial(f, u)}{\partial(p, q)} = 0;$$

* Or with respect to other variables, with a modification in the rest of the process, similar to that in §§ 58, 59.

and therefore, as the Jacobian of f and u with regard to p and q does not vanish *, we have

$$\frac{dq}{dx} - \frac{dp}{dy} = 0,$$

the necessary and sufficient condition. Effecting the necessary quadrature of the equation

$$dz - pdx - qdy = 0,$$

we have an equation giving z in terms of x, y, and two arbitrary constants.

This mode of obtaining the integral of the original equation by means of a single integral of the subsidiary system was first devised by Charpit †.

The method of Jacobi, whether in its original form as developed by himself or in the amplified form as developed by Lie and Mayer, and (for the case of two independent variables) the method of Charpit, aim at the construction of an integral containing a number of arbitrary constants; and the results do not indicate any particular suggestion of Cauchy's existence-theorem. The association will be made later, partly by a modified use of the equations of the characteristics; and it will be necessary to indicate the kinds of integrals which can be deduced from those provided by the methods of Jacobi and of Charpit.

69. All the examples, that follow, have been chosen, so as to give some initial indications of one investigation hitherto practically omitted by mathematicians. When an equation

$$f(x, y, z, p, q) = 0$$

is resolved with regard to p, or is given in a resolved form, so that it may be written

$$p = g(x, y, z, q),$$

Cauchy's existence-theorem can be applied only if the function $g(x, y, z, q)$ is a regular function of its arguments within the

* It would vanish if u involved neither p nor q.

† In a memoir, presented 30 June, 1784, to the Académie des Sciences, Paris; he died soon afterwards, and the memoir was never printed: see Lacroix, *Traité du calcul différentiel et du calcul intégral*, 2e éd., 1814, t. II, p. 548. Lacroix indicates (*ib.*, p. 567) that Charpit tried to extend his method to partial differential equations of the first order and degree higher than the first, involving more than two dependent variables.

domains of the initial values adopted: it ceases to apply if initial values are selected in the domains of which the function $g(x, y, z, q)$ is regular.

In all these examples, it is possible to choose initial values which make p infinite or indeterminate: the known method of constructing an integral has been used so as to give indications of the kind of integral (if any) which exists in association with such initial conditions.

What is required for the full discussion of an equation

$$f(x, y, z, p, q) = 0,$$

(and, *à fortiori*, of an equation in more than two independent variables), is a classification of all the non-regular forms arising out of the resolution of the equation with regard to p or, what is the same thing, a classification of all the non-regular forms of $g(x, y, z, q)$ in an equation

$$p = g(x, y, z, q).$$

Each of these would need to be considered in turn, as was done* for the non-regular forms of an equation

$$\frac{dw}{dz} = f(w, z);$$

the following set of examples give a few of the simplest types.

Meanwhile, some indications of results can be given: the methods of Charpit and of Jacobi are entirely independent even of the results given by Cauchy's theorem.

Ex. 1. Consider the equation

$$p(ax + by + cz) = 1.$$

It is clear that Cauchy's general theorem will not apply to this equation if, when $x = 0$, we require z to acquire the value of a function of y regular in the vicinity of $y = 0$ and vanishing there: the initial value of p is infinite and the proof no longer is valid.

But an integral can be obtained by Charpit's method. One of the subsidiary equations is

$$\frac{dp}{ap + cp^2} = \frac{dq}{bp + cpq},$$

so that

$$\frac{dp}{a + cp} = \frac{dq}{b + cq},$$

* In Chapters III and IV in Part II of this work.

an integral of which is

$$\frac{b+cq}{a+cp}=\alpha,$$

where α is an arbitrary constant. Accordingly, we combine this equation with the original equation, and we resolve them for p and q: substituting these found values in $dz-pdx-qdy=0$, we have

$$dz=\frac{dx}{ax+by+cz}+\left\{-\frac{b}{c}+\frac{a}{c}\alpha+\left(\frac{\alpha}{ax+by+cz}\right)\right\}dy,$$

and therefore

$$a\,dx+b\,dy+c\,dz=\left(\frac{c}{ax+by+cz}+a\right)(dx+\alpha dy).$$

Writing

$$u=ax+by+cz,$$

a simple quadrature leads to the equation

$$u-\frac{c}{a}\log(c+au)=\beta+a(x+\alpha y).$$

The value of z thus provided is an integral which contains the two arbitrary constants α and β.

In order to see whether any integral z exists, which vanishes when $x=0$ and $y=0$, these being values which make p infinite initially, we note that the foregoing equation is satisfied by $z=0$, $x=0$, $y=0$, provided

$$\beta=-\frac{c}{a}\log c.$$

Assuming this value of β, we have

$$e^u\left(1+\frac{a}{c}u\right)^{-\frac{c}{a}}=e^{a(x+\alpha y)};$$

and therefore, in the vicinity of the initial values assigned, we have

$$\left(1+\frac{a}{c}u\right)e^{-\frac{a}{c}u}=e^{-\frac{a^2}{c}(x+\alpha y)},$$

that is,

$$u^2+\ldots=2c(x+\alpha y)+\ldots,$$

so that, unless $c=0$ (and this will be excluded), we have

$$ax+by+cz=u=(x+\alpha y)^{\frac{1}{2}}R\{(x+\alpha y)^{\frac{1}{2}}\},$$

where R is a regular function of its argument and does not vanish when $x=0$ and $y=0$.

Ex. 2. In the same way it may be proved that an integral of the equation

$$p(ax+by+cz)^m=1,$$

where m is a positive integer, is given by

$$\int\frac{u^m}{c+au^m}du=\beta+x+\alpha y,$$

where α and β are arbitrary constants, and

$$u = ax + by + cz;$$

and that an integral, which vanishes when $x=0$ and $y=0$, is given by

$$ax + by + cz = (x + ay)^{\frac{1}{1+m}} R\{(x+ay)^{\frac{1}{1+m}}\},$$

where R is a regular function of its argument and does not vanish with x and y.

Ex. 3. It is easy to see that the integral of the equation

$$(p + a'q)(ax + by + cz)^m = 1,$$

where a' is a constant and m is a positive integer, is of the same type as in the preceding example: obtain the integral.

Ex. 4. Consider the equation

$$p(ax + by + cz + kq) = 1,$$

where a, b, c, k are constants.

Proceeding from subsidiary equations as in Ex. 1, we find that they have an integral

$$\frac{b + cq}{a + cp} = \alpha,$$

where α is an arbitrary constant.

There are two ways of continuing. We may either resolve the original equation and the new equation for p and y, and introduce a new dependent variable ζ, where

$$\zeta = z - qy,$$

and then we have

$$d\zeta = p\,dx - y\,dq:$$

we substitute for p and y; and, effecting the necessary quadrature, we eliminate q by the relation

$$\frac{\partial \zeta}{\partial q} = -y.$$

Or we may resolve the two equations for p and q, substitute in $dz = p\,dx + q\,dy$, and effect the quadrature. The result is

$$-\frac{1}{c}\left(1 + \frac{c^2}{a^2 k\alpha}\right)\log\{\Delta^{\frac{1}{2}} - cu + k(b - a\alpha)\} + \frac{c}{a^2 k\alpha}\log\{\Delta^{\frac{1}{2}} - cu + k(b + a\alpha)\}$$

$$= \frac{-2\dfrac{c}{a}}{\Delta^{\frac{1}{2}} - cu + k(b - a\alpha)} + \frac{x + \alpha y}{ak} + \beta,$$

where α and β are arbitrary constants, $u = ax + by + cz$, and

$$\Delta = (cu + bk - ak\alpha)^2 - 4ck\{(b - a\alpha)u - c\alpha\}.$$

It is possible (but the analysis is somewhat laborious) to deduce, from this result when $k=0$, the integral of the equation in Ex. 1.

We can make one more inference. If it were possible that the equation could possess an integral such that when $x=0$, the dependent variable acquires the value of a function of y such that z and q vanish when $y=0$, then p would become infinite for the initial values $x=0$ and $y=0$: Cauchy's theorem no longer applies. Now we are to have

$$\frac{b+cq}{a+cp}=a\,;$$

therefore for such integral (if any) we have $a=0$ because initially p is infinite, and then $b+cq=0$. But q is to vanish initially, so that $b=0$; and thus $q=0$ always; or z is merely a function of x, vanishing with x and given by

$$x=\frac{c}{a^2}e^{az}-\frac{c}{a^2}(1+az).$$

Excluding this trivial case, it follows that the given equation has no integral of the kind indicated, provided c is different from zero.

Ex. 5. Integrate the equation

$$p\,(ax+by+cz+kyq)=1\,;$$

and discuss the question whether it possesses an integral which, when $x=0$, acquires the value of a regular function of y that vanishes when $y=0$.

[An integral is given by eliminating q between the two equations

$$\left(z-qy+\frac{ax}{c}+\frac{1}{a}\right)(b+c'q)^{-\frac{c}{c'}}$$
$$=\beta+\frac{x}{ac}-\frac{1}{\alpha a^2}\log\left\{1-\alpha a\,(b+c'q)^{-\frac{c}{c'}}\right\},$$

$$\beta+\frac{x}{ac}+\frac{y}{c}(b+c'q)^{\frac{k}{c'}}-\frac{1}{a}\,\frac{1}{(b+c'q)^{\frac{c}{c'}}-\alpha a}=\frac{1}{\alpha a^2}\log\left\{1-\alpha a\,(b+c'q)^{-\frac{c}{c'}}\right\},$$

where α and β are arbitrary constants, and $c'=c+k$.]

Ex. 6. Obtain an integral of the equation

$$p=\frac{a}{yq-\frac{1}{2}z}$$

in the form

$$\tfrac{1}{4}z^2=(x-\beta)(\alpha y-a),$$

where α and β are arbitrary constants; and discuss the integrals of the equation (if any) which are such that yq and z vanish when $x=0$.

Ex. 7. As another example, consider the equation

$$p=\frac{cq}{yq-\frac{1}{2}z},$$

with a view to inquiring whether it possesses an integral which, when $x=0$, can be a function of y that vanishes, when $y=0$, in an order higher than the first, so that then q may vanish when $x=0$, $y=0$.

Forming the subsidiary equations in Charpit's method, we find one integral of them in a form

$$pq = a^2,$$

where a is an arbitrary constant. Resolving this equation and the original equation with regard to p and q, substituting in

$$dz = p\,dx + q\,dy,$$

and effecting the quadrature, we find

$$3ac\,(2cx - yz) - \beta = (a^2y^2 - 2cz)^{\frac{3}{2}} - a^3y^3,$$

where β is an arbitrary constant. This equation gives an integral involving two arbitrary constants.

If the equation is to provide an integral of the kind indicated, it is clear that $\beta = 0$. To discuss the consequent value of z when $x = 0$, we proceed from the equation

$$(a^2y^2 - 2cz)^{\frac{3}{2}} = a^3y^3 - 3acyz.$$

This equation certainly gives a value of z which vanishes when $y = 0$; two roots are zero, and the third is

$$z = \frac{3a^2}{8c}\,y^2,$$

which is of the required type.

Accordingly, the equation possesses an infinitude of integrals (because of the parameter a) which, when $x = 0$, give z and q as functions of y that vanish when $y = 0$; these integrals are provided by the equation

$$(a^2y^2 - 2cz)^3 = (a^3y^3 - 3acyz + 6ac^2x)^2,$$

where a is an arbitrary constant, that is, by the equation

$$8c^3z^3 - 3a^2c^2y^2z^2 - x\,(36a^2c^3yz - 12a^4c^2y^3) + 36a^2c^4x^2 = 0.$$

It is easy to see that, though, when $x = 0$, the integral becomes the simple regular function for the vicinity of $y = 0$, the integral itself is not a regular function of x and y in the specified domains.

Ex. 8. Prove that an integral of the equation

$$pz = aq + x,$$

where a is a constant, can be obtained by eliminating p and q between the equation itself and the equations

$$z\,(p^2 - 1)^{\frac{1}{2}} + ay + \beta = \alpha a\left\{p + \tfrac{1}{2}\log\left(\frac{p-1}{p+1}\right)\right\}, \quad q = \alpha\,(p^2 - 1)^{\frac{1}{2}},$$

where α and β are arbitrary constants. Discuss the integrals in the vicinity of $x = 0$.

Ex. 9. Consider the equation

$$p\,(ax + by + cz) + a'x + b'y + c'z = 0.$$

Changing the dependent variable so that

$$z' = z - a''x - b''y,$$

where a'' and b'' are constants, we can choose a'' and b'' so that the new equation has the form

$$p'(ax+\beta y+\gamma z')+\gamma' z'=0.$$

Accordingly, we consider the equation in the form

$$p(ax+by+cz)+c'z=0;$$

as it is homogeneous in the constants a, b, c, c', we can imagine it multiplied by such a constant factor as to make $a+c'=1$ unless $a+c'=0$.

Firstly, if $a+c'=1$, prove that an integral is given by the elimination of p between the equations

$$\left.\begin{aligned} &p(ax+by+cz)+c'z=0 \\ &\frac{z-px}{p^{c'}\left(p+\frac{1}{c}\right)^{a}}=Ay+B+\frac{by}{c}\int p^{-c'}\left(p+\frac{1}{c}\right)^{-a-1}dp \end{aligned}\right\},$$

where A and B are arbitrary constants.

Secondly, if $a+c'=0$, prove that an integral is given by the elimination of p between the equations

$$\left.\begin{aligned} &p\,(ax+by+cz)+c'z=0 \\ &\frac{z}{x}-p+\frac{by}{c'}-(Ay+B)\,e^{-\frac{c'}{cp}}=0 \end{aligned}\right\},$$

where A and B are arbitrary constants.

Discuss these integrals in the vicinity of $x=0$.

CHAPTER V.

Classes of Integrals possessed by Equations of the first order: Generalisation of Integrals.

The customary classification of integrals of a partial differential equation of the first order into three kinds was first made by Lagrange: see his *Œuvres Complètes*, t. III, p. 572, t. IV, pp. 65, 74. A full exposition is given in Imschenetsky's memoir, quoted on p. 100: it will be found in chapter I of the memoir. Other expositions are given by Goursat, *Leçons sur l'intégration...premier ordre*, by Mansion, *Théorie des équations...premier ordre*, and by Jordan, *Cours d'Analyse*, t. III.

That the theory is not complete even for the simplest case is pointed out by Goursat, in the book just quoted, § 18. Some further exceptions are indicated in the present chapter.

70. Before proceeding to the exposition of further methods of integration, and partly in order to facilitate the discussion of characteristics in particular, it is convenient to develop the relations, to one another, of the different integrals that have been obtained or have been proved to exist.

We have seen that, in the case of a homogeneous linear equation of the first order, it is possible to construct an integral which, on appropriate determination of its arbitrary elements, comprehends any integral of the equation: also that, in the case of a linear non-homogeneous equation of the first order, it is possible to construct an integral which similarly comprehends any integral that is not of the type called special. Consequently, no further discussion is necessary in those cases.

But in the case of equations that are not linear, it has been seen that there certainly are two kinds of integrals. On the one hand, there is Cauchy's existence-theorem according to which an arbitrary functional element occurs in the expression of the

integral proved to exist. On the other hand, Jacobi's method of integration, either in its original form or in any of its modified forms, has led to integrals which contain arbitrary constants in their expression. It is natural to enquire what is the relation, if any, between integrals of such widely distinct types and, further, whether integrals of other types exist.

Variation of Parameters.

71. Accordingly, beginning with a single equation which (after the preceding explanations) may be taken as not linear, we shall suppose it given in the form

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0;$$

and we may imagine that it has been integrated by the Jacobian method, with a result that z is given as a function of the variables and of n arbitrary constants $a_1, \ldots, a_n$ by means of an equation

$$\phi(z, x_1, \ldots, x_n, a_1, \ldots, a_n) = 0.$$

The values of the derivatives are given by equations

$$\phi_m = \frac{\partial \phi}{\partial z} p_m + \frac{\partial \phi}{\partial x_m} = 0,$$

for $m = 1, \ldots, n$; these values of p_m, together with the value of z deduced from $\phi = 0$, will, when substituted in the differential equation, make it satisfied identically. Moreover, the elimination of the n arbitrary constants between the $n+1$ equations

$$\phi = 0, \quad \phi_1 = 0, \ \ldots, \ \phi_n = 0$$

leads to the differential equation, and to that differential equation alone, provided that not all the Jacobians

$$J\left(\left(\frac{\phi, \phi_1, \ldots, \phi_n}{a_1, \ldots, a_n}\right)\right)$$

vanish; and conversely, when there is only a single differential equation, the Jacobians do not all vanish.

In the process of returning from the $n+1$ equations

$$\phi = 0, \quad \phi_1 = 0, \ \ldots, \ \phi_n = 0$$

to the differential equation, the quantities $a_1, \ldots, a_n$ are to be eliminated: but no regard is paid, during the operation, to their

constant values; and the resulting differential equation will be the same, provided the $n+1$ equations have the same form, when these quantities are made variable. We therefore make $a_1, \ldots, a_n$ functions of $x_1, \ldots, x_n$, subject to this proviso. This change leaves the equation $\phi = 0$ unaltered in form: in order that $\phi_m = 0$ (for $m = 1, \ldots, n$) may remain unaltered in form, it is necessary that the equation

$$\frac{\partial \phi}{\partial a_1}\frac{\partial a_1}{\partial x_m} + \ldots + \frac{\partial \phi}{\partial a_n}\frac{\partial a_n}{\partial x_m} = 0$$

should be satisfied, for each of the n values of m: and if these equations are satisfied, then $\phi = 0$ (with the changed values of $a_1, \ldots, a_n$) will still give an integral of the differential equation.

Multiplying the n equations by $dx_1, \ldots, dx_n$ respectively and adding, we find

$$\frac{\partial \phi}{\partial a_1} da_1 + \ldots + \frac{\partial \phi}{\partial a_n} da_n = 0,$$

where $da_1, \ldots, da_n$ are the complete variations of the quantities $a_1, \ldots, a_n$; and conversely this equation, when satisfied, yields the n conditions. The coefficients of the differential elements are functions of $z, x_1, \ldots, x_n, a_1, \ldots, a_n$ in general: but z is given by $\phi = 0$ in terms of the other quantities; and, as $a_1, \ldots, a_n$ are (unknown) functions of $x_1, \ldots, x_n$, so the latter may be regarded in the most general case as functions of $a_1, \ldots, a_n$: that is, the coefficients may, in the most general case, be regarded as functions of $a_1, \ldots, a_n$. Thus we have a Pfaffian equation: by the general theory of Pfaffian equations*, the integral equivalent consists of one equation or of several equations connecting the quantities $a_1, \ldots, a_n$.

In the argument, one exceptional case has been omitted: it may be that the Pfaffian equation is evanescent, on account of vanishing coefficients: we then have

$$\frac{\partial \phi}{\partial a_1} = 0, \ \ldots, \ \frac{\partial \phi}{\partial a_n} = 0,$$

concurrently with $\phi = 0$.

After noting this exceptional case, we return to the integral equivalent of the Pfaffian equation. Let it consist of μ equations

$$g_1(a_1, \ldots, a_n) = 0, \ \ldots, \ g_\mu(a_1, \ldots, a_n) = 0,$$

* See Part I of this work, *passim*.

and of these solely: then the only relations among the differential elements are

$$dg_1 = 0, \; \dots, \; dg_\mu = 0,$$

and the Pfaffian equation must be satisfied in virtue of these. Thus μ quantities $\lambda_1, \dots, \lambda_\mu$ must exist such that

$$\frac{\partial \phi}{\partial a_1} da_1 + \dots + \frac{\partial \phi}{\partial a_n} da_n = \lambda_1 dg_1 + \dots + \lambda_\mu dg_\mu ;$$

and therefore

$$\frac{\partial \phi}{\partial a_m} = \lambda_1 \frac{\partial g_1}{\partial a_m} + \dots + \lambda_\mu \frac{\partial g_\mu}{\partial a_m},$$

for the n values of m. These n equations, together with

$$\phi = 0, \quad g_1 = 0, \; \dots, \; g_\mu = 0,$$

make up $n + \mu + 1$ equations, involving $a_1, \dots, a_n, \lambda_1, \dots, \lambda_\mu$: eliminating the quantities a and λ, we have a single equation as the result, and it expresses z in terms of $x_1, \dots, x_n$. The value of z determined by this final equation is an integral of the original differential equation: the functional forms $g_1, \dots, g_\mu$ are involved in its expression.

72. It might appear as if there were integrals of a character intermediate between those of the two kinds considered. Thus we might have $a_{m+1}, \dots, a_n$ as constants, so that the differential relation would then be

$$\frac{\partial \phi}{\partial a_1} da_1 + \dots + \frac{\partial \phi}{\partial a_m} da_m = 0.$$

If the integral equivalent of this relation consists of σ equations in the form

$$g_1(a_1, \dots, a_m) = 0, \; \dots, \; g_\sigma(a_1, \dots, a_m) = 0,$$

and of these only, then the same argument as before leads to equations

$$\frac{\partial \phi}{\partial a_i} = \rho_1 \frac{\partial g_1}{\partial a_i} + \dots + \rho_\sigma \frac{\partial g_\sigma}{\partial a_i},$$

for $i = 1, \dots, m$. These m equations, together with

$$\phi = 0, \quad g_1 = 0, \dots, g_\sigma = 0,$$

are $m + \sigma + 1$ equations involving $z, x_1, \dots, x_n, a_1, \dots, a_m, \rho_1, \dots, \rho_\sigma$: eliminating these m quantities $a_1, \dots, a_m$ and the σ quantities ρ, we have a single equation between $z, x_1, \dots, x_n$. The value of z

thus given is an integral of the original equation. The functional forms $g_1, \ldots, g_\sigma$ are involved in its expression; and the arbitrary constants $a_{m+1}, \ldots, a_n$ also occur. The latter can be regarded as given by $n-m$ relations

$$h_1(a_{m+1}, \ldots, a_n) = 0, \ldots, h_{n-m}(a_{m+1}, \ldots, a_n) = 0,$$

involving the $n-m$ constants: they are such that the equations

$$dh_1 = 0, \ldots, dh_{n-m} = 0$$

are satisfied identically. Now it is known from the theory of Pfaffian equations that

$$\sigma + n - m \geqslant \mu,$$

so that the total number of equations among the quantities $a_1, \ldots, a_n$ is greater than before: their range of value is therefore more restricted than in the preceding case. Accordingly, we can regard the present mode of satisfying the differential relation as a specialisation of the preceding mode or as a special instance of the preceding mode involving a greater number of relations some of which are of restricted forms.

In this argument, as in the preceding argument in § 71, one exceptional case is omitted: it may be that the reduced Pfaffian equation is evanescent, on account of vanishing coefficients: we then have

$$\frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_m} = 0,$$

concurrently with $\phi = 0$.

It thus appears that, while the completed process leads in every case to a single equation providing an integral, there are intrinsic differences according to the circumstances of the cases. It is clear that distinctions will arise according to the number of relations postulated among the quantities $a_1, \ldots, a_n$; it is customary to regard a class of integrals as being defined according to the number of relations so postulated. When μ relations of the indicated character occur, the corresponding class of integrals is frequently called the μth *class*: and if

$$0 < \mu < n,$$

the integrals of all the classes may be regarded as falling within the category of what will presently be called general integrals. Thus there will be $n-1$ classes of general integrals.

The extreme cases must also be taken into consideration. It is possible that $\mu = n$: there are then n functional relations connecting the n quantities $a_1, \ldots, a_n$, independent of one another; all these quantities are constants and, when the relations are quite arbitrary, the constants are arbitrary: the integral then provided is what will be called the complete integral. It is possible that $\mu = 0$: if the equations can be satisfied, and an integral is provided, we have what will be called the singular integral.

Of the general integrals, the most comprehensive is that in which only a single functional form occurs, say

$$a_1 = \psi(a_2, \ldots, a_n),$$

and ψ can be taken as the most general and arbitrary function of its arguments. The equations which determine the integral are

$$\phi = 0, \quad a_1 = \psi(a_2, \ldots, a_n),$$

$$\frac{\partial \phi}{\partial a_m} + \frac{\partial \phi}{\partial a_1}\frac{\partial \psi}{\partial a_m} = 0,$$

for $m = 2, \ldots, n$; and the integral itself is given by the elimination of $a_1, \ldots, a_n$ among these $n+1$ equations.

That it is the most extensive class of general integral can easily be seen by the following argument, whereby it is proved to include all the other classes. When μ relations are postulated among the n quantities $a_1, \ldots, a_n$ in the form

$$g_r(a_1, \ldots, a_n) = 0,$$

for $r = 1, \ldots, \mu$, the integral is given by these equations, together with

$$\phi = 0,$$

$$\frac{\partial \phi}{\partial a_m} = \lambda_1 \frac{\partial g_1}{\partial a_m} + \ldots + \lambda_\mu \frac{\partial g_\mu}{\partial a_m},$$

for $m = 1, \ldots, n$. Let

$$\theta(a_1, \ldots, a_n) = \lambda_1 g_1 + \ldots + \lambda_\mu g_\mu,$$

so that the relation $\theta = 0$ is certainly satisfied for the integral in question; moreover, the equations

$$\frac{\partial \phi}{\partial a_m} = \frac{\partial \theta}{\partial a_m}$$

are certainly satisfied for this integral. Now let $\theta = 0$ be resolved for a_1 so as to express it in terms of $a_2, \ldots, a_n$ in a form

$$a_1 = \chi(a_2, \ldots, a_n):$$

we have

$$\frac{\partial \theta}{\partial a_m} + \frac{\partial \theta}{\partial a_1}\frac{\partial \chi}{\partial a_m} = 0.$$

Hence, for the integral in question, the equations

$$\frac{\partial \phi}{\partial a_m} + \frac{\partial \phi}{\partial a_1}\frac{\partial \chi}{\partial a_m} = 0$$

are satisfied: and conversely, when these are satisfied, the original set of equations also is satisfied. Now in the case when there is only a single relation

$$a_1 = \psi(a_2, \ldots, a_n),$$

ψ is the most general function possible: so that the relation

$$a_1 = \chi(a_2, \ldots, a_n)$$

is included as a special case, and consequently the equations

$$\frac{\partial \phi}{\partial a_m} + \frac{\partial \phi}{\partial a_1}\frac{\partial \chi}{\partial a_m} = 0$$

are a special case of the equations

$$\frac{\partial \phi}{\partial a_m} + \frac{\partial \phi}{\partial a_1}\frac{\partial \psi}{\partial a_m} = 0;$$

that is, the general integral in question is a special case of the general integral, which arises when there is only a single relation between the quantities $a_1, \ldots, a_n$. The latter general integral is accordingly the most comprehensive.

In passing, we may note that the general integral includes the exceptional case noted, in which $a_{m+1}, \ldots, a_n$ are arbitrary constants and the equations

$$\frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_m} = 0$$

are satisfied. We can represent it by relations

$$a_\mu = \psi_\mu(a_1, \ldots, a_m),$$

for $\mu = m + 1, \ldots, n$, and by restricting the functions ψ_μ to be constants; for then

$$\frac{\partial \psi_\mu}{\partial a_i} = 0,$$

for $i = 1, \ldots, m$, and the relation

$$\frac{\partial \phi}{\partial a_i} + \sum_{\mu} \frac{\partial \phi}{\partial a_\mu} \frac{\partial \psi_\mu}{\partial a_i} = 0$$

simply becomes

$$\frac{\partial \phi}{\partial a_i} = 0,$$

which (for $i = 1, \ldots, m$) are the equations for the exceptional case.

Classes of Integrals.

73. Three kinds of integrals may thus arise. One of them is given by an equation containing n arbitrary constants; it is called the *complete integral.* Another of them is given by equations that involve a functional form or several functional forms, and in the most general type these forms are arbitrary; these integrals are called *general integrals* and often, when there is only a single functional form so that the widest range of variation is provided, the integral is called *the general integral.* And, lastly, the equations

$$\phi = 0, \quad \frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_n} = 0,$$

may be possible and be consistent with one another; if the result of eliminating $a_1, \ldots, a_n$ among them provides a single equation involving no arbitrary element, and if the equation determines an integral*, the integral thus furnished is called the *singular integral.*

It must however be noticed that an integral, containing the appropriate number of arbitrary constants, is not necessarily the complete integral, any more than one which contains no arbitrary element is necessarily a singular integral. On the one hand, since an arbitrary function can be regarded as containing any number of arbitrary constants, a general integral may be simply specialised so as to contain the appropriate number of arbitrary constants: it will not thereby necessarily become a complete integral, for it may

* The reason for this limitation will appear subsequently: meanwhile, it may be sufficient to point out that, while the equations $\frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_n} = 0$ are consistent with the existence of an integral, it has not been proved (and, indeed, cannot be proved) that their significance is only co-extensive with that existence. Even in the case of ordinary equations of the first order, the corresponding process frequently gives rise to relations that do not provide integrals of the equations in question: and the same holds, to a wider extent, in partial equations.

be only a special case of the general integral. On the other hand, by assigning particular values to the arbitrary constants in a complete integral, the latter becomes free from all arbitrary elements: it will not thereby become a singular integral (even if such an integral is possessed by the equation), for it is only a special case of the complete integral. It is therefore important to devise tests which shall shew to what category any given integral should, if possible, be assigned: and this necessity raises a further question as to how comprehensive is the retained aggregate of integrals.

Special Integrals.

74. Suppose, then, that we have an integral of the differential equation

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$$

given by the equation

$$\theta(x_1, \ldots, x_n, z) = 0;$$

and let the values of z thus determined be denoted by ζ. Also, let a complete integral be given in the form

$$g(x_1, \ldots, x_n, z, a_1, \ldots, a_n) = 0;$$

and let the value of z thus determined be denoted by Z. We have to consider whether it is possible to associate with $g = 0$ equations or relations which will change Z into ζ; if this should be possible, then the character of the added equations or relations will indicate the character of the integral ζ.

In order to obtain the tests that may be both sufficient and necessary, assume that $a_1, \ldots, a_n$ are changed into functions of $x_1, \ldots, x_n$, such that Z is still an integral of the differential equation and such that, if possible, it becomes the integral ζ. As the two integrals are now hypothetically the same functions of $x_1, \ldots, x_n$, the derivatives of these functions with regard to the variables are respectively the same. For the integral ζ, they are given by

$$\frac{\partial \theta}{\partial x_m} + p_m \frac{\partial \theta}{\partial z} = 0,$$

for $m = 1, \ldots, n$, when z is replaced by ζ in these equations; and for the integral Z, they are given by

$$\frac{\partial g}{\partial x_m} + p_m \frac{\partial g}{\partial z} = 0, \quad \frac{\partial g}{\partial a_1}\frac{\partial a_1}{\partial x_m} + \ldots + \frac{\partial g}{\partial a_n}\frac{\partial a_n}{\partial x_m} = 0,$$

for $m=1, \ldots, n$, when z is replaced by Z in these equations. Consequently, we must have

$$\frac{\partial g}{\partial x_m}\frac{\partial \theta}{\partial z}-\frac{\partial g}{\partial z}\frac{\partial \theta}{\partial x_m}=0,$$

$$\frac{\partial g}{\partial a_1}\frac{\partial a_1}{\partial x_m}+\ldots+\frac{\partial g}{\partial a_n}\frac{\partial a_n}{\partial x_m}=0,$$

for $m=1, \ldots, n$, when z is replaced by the supposed common value of ζ and Z.

Now when this common value is substituted, the n equations

$$\frac{\partial g}{\partial x_m}\frac{\partial \theta}{\partial z}-\frac{\partial g}{\partial z}\frac{\partial \theta}{\partial x_m}=0$$

are a set of equations involving the quantities $a_1, \ldots, a_n$. If they determine values for these quantities, we can proceed to the identification of the integral; but they do not necessarily determine such values, and then we cannot proceed.

Suppose that such values are determined. If they are constants, then ζ is a more or less particular form of the complete integral: all the equations

$$\frac{\partial g}{\partial a_1}\frac{\partial a_1}{\partial x_m}+\ldots+\frac{\partial g}{\partial a_n}\frac{\partial a_n}{\partial x_m}=0$$

are satisfied. If values are found, so that some at least have the form of functions of $x_1, \ldots, x_n$, there may be some functional relation or several functional relations among them: let these be denoted by

$$g_1(a_1, \ldots, a_n)=0, \ldots, g_\mu(a_1, \ldots, a_n)=0.$$

Then the other n equations are satisfied by means of the equations

$$\frac{\partial g}{\partial a_m}=\lambda_1\frac{\partial g_1}{\partial a_m}+\ldots+\lambda_\mu\frac{\partial g_\mu}{\partial a_m},$$

for $m=1, \ldots, n$, with appropriately determinate values of $\lambda_1, \ldots, \lambda_\mu$. All the conditions then are satisfied; and ζ then is a more or less particular form of the general integral. If on the other hand the variable values found (say m in number) are such that no functional relation subsists among these m quantities, the n remaining equations can only be satisfied by having

$$\frac{\partial g}{\partial a_i}=0,$$

for each of the m quantities a_i found to be variable; the integral ζ would then be a degenerate form of the general integral of the differential equation. Lastly, if all the quantities a are variable and if there is no functional relation among them, the n remaining equations can only be satisfied by having

$$\frac{\partial g}{\partial a_1} = 0, \ldots, \frac{\partial g}{\partial a_n} = 0;$$

the integral ζ would then be a singular integral of the differential equation.

It thus appears that, subject to the determination of the quantities $a_1, \ldots, a_n$ from the equations

$$\frac{\partial g}{\partial x_m}\frac{\partial \theta}{\partial z} - \frac{\partial g}{\partial z}\frac{\partial \theta}{\partial x_m} = 0,$$

the integral ζ is comprehended within the aggregate of the complete integral, the general integral, and the singular integral. This aggregate is widely comprehensive: it cannot be declared to be completely comprehensive, because occasions arise in which the equations refuse to provide a consistent set of values of $a_1, \ldots, a_n$ needed to secure inclusion. The whole of this theory is formal: it does not take account of the peculiarities of equations: and examples will be indicated to which it fails to apply.

Such integrals, as do occur but are not included in any of the three classes, will be called *special*.

Ex. 1. It is easy to see that the equation

$$p_1 x_1 + \ldots + p_n x_n = z$$

has an integral

$$z = a_1 x_1 + \ldots + a_n x_n,$$

which is a complete integral. To obtain a general integral, the most general possible, we take only a single relation among the quantities $a_1, \ldots, a_n$ in the form

$$a_1 = f(a_2, \ldots, a_n),$$

where f is an arbitrary function of its arguments. The associated equations are

$$x_s + x_1 \frac{\partial f}{\partial a_s} = 0,$$

for $s = 2, \ldots, n$; these give

$$a_r = g_r\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right),$$

where the character of g_r is dominated by the arbitrary form of f. Inserting these values of a, we have

$$\frac{z}{x_1} = F\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right),$$

where F is an arbitrary function.

This is the integral which would be obtained by the process of § 30; accordingly, the most comprehensive integral given by that process is the general integral.

The equations, which would give the singular integral if it existed, are

$$z = a_1 x_1 + \ldots + a_n x_n,$$
$$x_1 = 0, \ \ldots, \ x_n = 0:$$

clearly there is no singular integral of the equation, though $z=0$ is a particular case of the complete integral.

Ex. 2. The equation

$$xp + 2yq = 2\left(z - \frac{x^2}{y}\right)^2$$

has been discussed (§ 34, Ex. 3); in particular, it was shewn that the integral

$$z = \frac{x^2}{y}$$

was not derivable from the general integral there obtained. The equation does not possess a singular integral.

Is the integral $z = \dfrac{x^2}{y}$ comprehended in the complete integral?

Ex. 3. At the end of § 59, it was shewn that the equation

$$3px + qy + q^3x^2 = 0$$

possesses two complete integrals

$$z = a - \tfrac{1}{3} b^3 x + b x^{-\frac{1}{3}} y,$$
$$z = A + \tfrac{2}{3} y^{\frac{3}{2}} (x^2 + 2Bx)^{-\frac{1}{2}}.$$

The general integral deduced from the first of these complete integrals is obtained by associating with it the equations

$$a = \phi(b), \quad \phi'(b) - xb^2 + x^{-\frac{1}{3}} y = 0,$$

where ϕ is arbitrary: the general integral deduced from the second of them is obtained by associating with it the equations

$$A = \psi(B), \quad \psi'(B) - \tfrac{2}{3} y^{\frac{3}{2}} x (x^2 + 2Bx)^{-\frac{3}{2}},$$

where ψ is arbitrary. Clearly there is no singular integral.

To obtain the relations to one another of the two complete integrals, we adopt the method in the text. When we equate the different respective derivatives, we have the relations

$$-\tfrac{1}{3} b^3 - \tfrac{1}{3} b x^{-\frac{4}{3}} y = -\tfrac{2}{3} y^{\frac{3}{2}} (x^2 + 2Bx)^{-\frac{3}{2}} (x + B),$$
$$b x^{-\frac{1}{3}} = y^{\frac{1}{2}} (x^2 + 2Bx)^{-\frac{1}{2}};$$

these relations are consistent with one another, in virtue of the single relation

$$b=y^{\frac{1}{3}}x^{\frac{1}{3}}(x^2+2Bx)^{-\frac{1}{3}}.$$

When we equate the two integrals themselves, we find

$$a=A-\tfrac{2}{3}Bxy^{\frac{3}{3}}(x^2+2Bx)^{-\frac{3}{3}}.$$

The values of a and b are thus variable quantities; and it is easy to see that they are connected by the relation

$$a-A=-\tfrac{2}{3}Bb^3.$$

In virtue of this relation, and of the values of a and b, the other necessary relations

$$\frac{\partial z}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial z}{\partial b}\frac{\partial b}{\partial x}=0,$$

$$\frac{\partial z}{\partial a}\frac{\partial a}{\partial y}+\frac{\partial z}{\partial b}\frac{\partial b}{\partial y}=0,$$

are satisfied.

Hence each of the two complete integrals is a particular case of the general integral deduced from the other: the generalising relation is

$$a-A+\tfrac{2}{3}Bb^3=0.$$

Ex. 4. The equation

$$pq=4xy$$

has

$$z=\frac{x^2}{a}+ay^2+b$$

for a complete integral; it has no singular integral: and its general integral is given by

$$z=\frac{x^2}{a}+ay^2+b,\quad 0=-\frac{x^2}{a^2}+y^2+f'(a),\quad b=f(a).$$

Another integral is given by

$$z=2xy+b.$$

To investigate its relation to the complete integral, we proceed as before. Equating the derivatives, we find

$$\frac{2x}{a}=2y,\quad 2ay=2x,$$

giving

$$a=\frac{x}{y};$$

with this value, the two quantities z are the same.

The other equations

$$\frac{\partial g}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial g}{\partial b}\frac{\partial b}{\partial x}=0,\quad \frac{\partial g}{\partial a}\frac{\partial a}{\partial y}+\frac{\partial g}{\partial b}\frac{\partial b}{\partial y}=0,$$

are satisfied by

$$\frac{\partial g}{\partial a}=0,\quad b=\text{arbitrary constant}.$$

The new integral is a special case of the general integral; for we have

$$z=\frac{x^2}{a}+ay^2+b, \quad b=f(a), \quad -\frac{x^2}{a^2}+y^2+f'(a)=0,$$

as the equations of the general integral; and they lead to the new integral, when $f(a)$ is regarded as a pure constant.

Ex. 5. Classify the integral $z=3x_1^{\frac{1}{3}}x_2^{\frac{1}{3}}x_3^{\frac{1}{3}}+b$ of the equation

$$p_1p_2p_3=1.$$

Ex. 6. Consider the equation

$$\{1+(z-x-y)^{\frac{1}{2}}\}p+q=2,$$

which has already (§ 34, Ex. 4) been discussed from the point of view of the general integral. The equation is clearly satisfied by

$$z=x+y:$$

the question is, does this integral fall within the three classes of integrals considered?

Proceeding to integrate by Charpit's method, we find

$$\frac{q-1}{p-1}=a,$$

as one integral of the subsidiary equations. When this relation is combined with the original equation, we have values of p and q: these are substituted in

$$dz=p\,dx+q\,dy,$$

and the quadrature is effected: the result is

$$z+(a-1)y+2(a+1)(z-x-y)^{\frac{1}{2}}=b,$$

where a and b are arbitrary. Writing

$$u=z+(a-1)y+2(a+1)(z-x-y)^{\frac{1}{2}},$$

the singular integral (if any) is given by

$$\frac{\partial(u-b)}{\partial a}=0, \quad \frac{\partial(u-b)}{\partial b}=0;$$

the latter equation shews that the singular integral does not exist: consequently

$$z=x+y$$

is not a singular integral.

The general integral is given by the elimination of a between

$$u=\phi(a),$$

$$\frac{\partial u}{\partial a}=\phi'(a).$$

If ϕ can be determined, so that the result of the elimination is to give

$$z=x+y,$$

the second of the equations for the elimination must become

$$y=\phi'(a),$$

and the first of them must become

$$x+ay=\phi(a).$$

The former of these, for any function ϕ, makes a a function of y only; the latter, instead of being identically satisfied (as it should be) if the integral $z=x+y$ could thus arise, leads to a relation between x and y. Any such relation is excluded. Hence $z=x+y$ is not a particular case of the general integral.

It is clear that constant values of a and b cannot be chosen such that the equation $u=b$ leads to the equation $z=x+y$: hence the integral is not a particular case of the complete integral.

It follows therefore that, while $z=x+y$ is an integral of the equation

$$\{1+(z-x-y)^{\frac{1}{2}}\}p+q=2,$$

it does not belong to any of the three usual classes of integrals: an instance is thus provided in which the general theorem due to Lagrange does not hold.

If the differential equation is rationalised, so that it takes the form

$$z-x-y=\left(\frac{2-q}{p}-1\right)^2,$$

the complete integral is

$$\{z+(a-1)y-b\}^2=4(a+1)^2(z-x-y);$$

and $z=x+y$ is easily seen to be a singular integral. The explanation of the difference is left to the student as an exercise.

Ex. 7. Given the equation

$$Ap^2+Bpq+Cq^2=D,$$

where A, B, C, D are functions of x and y only, investigate the conditions necessary and sufficient to secure that it possesses a complete integral of the form

$$z=au^2+\frac{1}{a}v^2+b,$$

where u and v are functions of x and y, and a, b are constants.

Verify that, if the conditions are satisfied, it also possesses an integral

$$z=uv+b.$$

What is the character of this integral?

TESTS FOR A COMPLETE INTEGRAL.

75. In the preceding investigation, it has been assumed that a complete integral of the differential equation is known, so that it is possible to proceed from that integral to the differential equation, and to that equation alone: and it has been pointed out that an integral, containing the proper number of arbitrary constants, is not necessarily complete. The important limitation is that elimination among the equations, denoted in § 71 by

$$\phi=0,\quad \phi_1=0,\ \ldots,\ \phi_n=0,$$

should lead to one, and to only one, equation.

For this purpose, it is necessary that not all the Jacobians should vanish: if they do vanish, then the elimination of the n quantities $a_1, \ldots, a_n$ will lead to at least two equations.

Again, if all the Jacobians but one, say

$$J\left(\frac{\phi, \phi_2, \ldots, \phi_n}{a_1, \ldots\ldots, a_n}\right),$$

are known to vanish, then either that Jacobian vanishes or else

$$\frac{\partial \phi_1}{\partial a_1} = 0, \ldots, \frac{\partial \phi_1}{\partial a_n} = 0,$$

that is, either that Jacobian vanishes, or ϕ_1 involves none of the constants. The first of these two alternatives is the preceding case. As regards the second alternative, we at once have

$$\phi_1 = 0$$

as an equation. The constants $a_1, \ldots, a_n$ may or may not be eliminable between $\phi = 0, \phi_2 = 0, \ldots, \phi_n = 0$; so that there would be only one equation if they cannot be eliminated, and there would be at least two equations if they can be eliminated. If there is only one equation, the integral is complete; if there is more than one, the integral is not complete.

If a Jacobian, say

$$J\left(\frac{\phi, \phi_2, \ldots, \phi_n}{a_1, \ldots\ldots, a_n}\right),$$

is known not to vanish, then the equations

$$\phi = 0, \quad \phi_2 = 0, \ \ldots, \ \phi_n = 0$$

can be resolved for $a_1, \ldots, a_n$; their values, substituted in $\phi_1 = 0$, if it involves any of them, lead to a single equation; while, if $\phi_1 = 0$ does not involve any of the constants $a_1, \ldots, a_n$, it is itself one equation involving derivatives. We have only a single equation: the integral is complete.

Ex. 1. Consider an integral equation

$$z = (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2;$$

it is easily seen to be a complete integral of the differential equation

$$4z = p_1^2 + p_2^2 + p_3^2,$$

the elimination being immediate.

An integral equation

$$z = (x_1 - a_1)^2 + (x_2 - a_2)^2 + x_3 - a_3$$

leads to a single equation

$$p_3=1,$$

and no other elimination is possible: the integral is complete.

An integral equation*

$$z=\{(x_1-a_1)^2+(x_2-a_2)^2\}^{\frac{1}{2}}+x_3-a_3$$

leads to two equations

$$p_1{}^2+p_2{}^2=1, \qquad p_3=1\,;$$

it is not a complete integral of either equation, nor of an equation such as

$$p_1{}^2+p_2{}^2+(p_3-1)^2=1.$$

Ex. 2. The equation

$$z=ax_1+bx_2+cx_3$$

is a complete integral of the equation

$$x_1p_1+x_2p_2+x_3p_3=z.$$

Another integral, containing three arbitrary constants, is

$$z=\alpha x_1+\beta x_2+\gamma\frac{x_2{}^2}{x_1}.$$

To determine its significance, we equate the values of p_1, p_2, p_3 derived from the two values; and we have

$$a=\alpha-\gamma\frac{x_2{}^2}{x_1{}^2},$$

$$b=\beta+2\gamma\frac{x_2}{x_1},$$

$$c=0,$$

giving variable values for a and b. These variable values are subject to the two equations

$$4\gamma(a-\alpha)+(b-\beta)^2=0, \qquad c=0\,;$$

and these, as two equations connecting the assumed variable magnitudes, shew that

$$z=\alpha x_1+\beta x_2+\gamma\frac{x_2{}^2}{x_1}$$

is not a complete integral of the equation, but is a special case of the general integral derived from the complete integral

$$z=ax_1+bx_2+cx_3.$$

In point of fact, the equation

$$z=\alpha x_1+\beta x_2+\gamma\frac{x_2{}^2}{x_1}$$

leads to two equations

$$x_1p_1+x_2p_2=z, \qquad p_3=0,$$

thus verifying the conclusion that it is not a complete integral of the original equation.

Ex. 3. To illustrate a different aspect of the relations of integrals, consider the equation

$$4z=p_1{}^2+p_2{}^2+p_3{}^2,$$

* This example is given by Goursat, *Leçons*, p. 98.

which occurred in Ex. 1. It possesses an integral

$$z=(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2,$$

which is easily seen to be complete: it possesses an integral

$$z=\frac{(x_1+\alpha_1x_2+\alpha_2x_3+\alpha_3)^2}{1+\alpha_1^2+\alpha_2^2},$$

which also is easily seen to be complete. What is the relation, if any, between the two integrals?

To determine it, we first equate the values of p_1, p_2, p_3 for the two values of z, and resolve the three equations for (say) α_1, α_2, α_3; and we find the three variable values

$$\alpha_1=\frac{x_2-a_2}{x_1-a_1}, \qquad \alpha_2=\frac{x_3-a_3}{x_1-a_1},$$

$$\alpha_3=a_1-a_2\frac{x_2-a_2}{x_1-a_1}-a_3\frac{x_3-a_3}{x_1-a_1}.$$

These are connected by a functional relation

$$a_1+a_2\alpha_1+a_3\alpha_2+\alpha_3=0.$$

If then we construct the general integral to be associated with the complete integral

$$z=(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2,$$

the integral

$$z=\frac{(x_1+\alpha_1x_2+\alpha_2x_3+\alpha_3)^2}{1+\alpha_1^2+\alpha_2^2}$$

is a particular case of that general integral given by the particular equations

$$\left.\begin{aligned} a_1&=-\alpha_1a_2-\alpha_2a_3-\alpha_3\\ z&=(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2\\ 0&=(x_1-a_1)\,\alpha_1-(x_2-a_2)\\ 0&=(x_1-a_1)\,\alpha_2-(x_3-a_3)\end{aligned}\right\},$$

when a_1, a_2, a_3 are eliminated among them.

On the other hand, if we construct the general integral to be associated with the complete integral

$$z=\frac{(x_1+\alpha_1x_2+\alpha_2x_3+\alpha_3)^2}{1+\alpha_1^2+\alpha_2^2},$$

the integral

$$z=(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2$$

is a particular case of that general integral given by the particular equations

$$\left.\begin{aligned} \alpha_3&=-a_2\alpha_1-a_3\alpha_2-a_1\\ z&=\frac{(x_1+\alpha_1x_2+\alpha_2x_3+\alpha_3)^2}{1+\alpha_1^2+\alpha_2^2}\\ 0&=x_2-a_2-\alpha_1\,\frac{x_1+\alpha_1x_2+\alpha_2x_3+\alpha_3}{1+\alpha_1^2+\alpha_2^2}\\ 0&=x_3-a_3-\alpha_2\,\frac{x_1+\alpha_1x_2+\alpha_2x_3+\alpha_3}{1+\alpha_1^2+\alpha_2^2}\end{aligned}\right\},$$

when α_1, α_2, α_3 are eliminated among them.

It thus appears that a single equation, of degree higher than the first, may have quite distinct complete integrals; and that a complete integral may be a particular case of a general integral derived from another complete integral, and, *à fortiori*, may be a particular case of the general integral derived from itself. (See also Ex. 3, § 74.)

Ex. 4. Discuss the character of the integral of

$$(p_1-1)^2+p_2^2+\dots+p_n^2=1,$$

as given by the equation

$$z-a_1=x_1+\{(x_2-a_2)^2+\dots+(x_n-a_n)^2\}^{\frac{1}{2}}.$$

SINGULAR INTEGRALS.

76. We have seen that, when a singular integral of the equation

$$f(x_1, \dots, x_n, z, p_1, \dots, p_n)=0$$

exists, it can be obtained from the complete integral

$$\phi(z, x_1, \dots, x_n, a_1, \dots, a_n)=0,$$

by eliminating $a_1, \dots, a_n$ between the equations

$$\phi=0, \quad \frac{\partial\phi}{\partial a_1}=0, \quad \dots, \quad \frac{\partial\phi}{\partial a_n}=0.$$

But when it exists, it may also be obtained from the differential equation itself: the formal argument is as follows.

The values of $p_1, \dots, p_n$ belonging to any integral given by $\phi=0$ are

$$\frac{\partial\phi}{\partial z}p_r+\frac{\partial\phi}{\partial x_r}=0,$$

for $r=1, \dots, n$; when these are substituted in the differential equation $f=0$, the latter becomes a relation between $z, x_1, \dots, x_n$ and the quantities $a_1, \dots, a_n$ introduced by the derivatives of ϕ. When the value of z given by $\phi=0$ is substituted in this relation, it becomes an identity: for it is thus that the original differential equation is satisfied in connection with $\phi=0$. Hence some value of z given by the changed form of $f=0$ is the same as a value of z given by $\phi=0$; for all such values, the two equations

$$f=0, \quad \phi=0$$

are equivalent to one another, f being transformed by the introduction of the values of $p_1, \dots, p_n$.

Now suppose that the integral is the singular integral, assumed to exist; we know that the equations

$$\frac{\partial \phi}{\partial a_1} = 0, \ \ldots, \ \frac{\partial \phi}{\partial a_n} = 0$$

are satisfied. As the transformed expression for f is equivalent to ϕ for this integral, we must therefore have

$$\frac{\partial f}{\partial a_1} = 0, \ \ldots, \ \frac{\partial f}{\partial a_n} = 0:$$

hence, as the quantities $a_1, \ldots, a_n$ have been introduced into this transformed expression solely through $p_1, \ldots, p_n$, we have

$$\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial a_r} + \frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial a_r} + \ldots + \frac{\partial f}{\partial p_n}\frac{\partial p_n}{\partial a_r} = 0,$$

for $r = 1, \ldots, n$.

These are an aggregate of n equations, linear and homogeneous in the n derivatives of f with regard to p.

They could be satisfied by non-zero values for these derivatives, if

$$J\left(\frac{p_1, \ \ldots, \ p_n}{a_1, \ \ldots, \ a_n}\right) = 0;$$

when this is the case, there exist m relations, where $m \leqslant n-1$, connecting these derivatives of f linearly and homogeneously. As our purpose is the derivation (if possible) of an integral from the differential equation itself without assuming knowledge of the actual form of the complete integral, we shall omit any further discussion of this alternative.

The aggregate of n equations could also be satisfied (and if the preceding alternative were inadmissible, the aggregate could only be satisfied) by

$$\frac{\partial f}{\partial p_1} = 0, \ \ldots, \ \frac{\partial f}{\partial p_n} = 0:$$

and these must coexist with $f = 0$. It may be possible that these $n+1$ equations determine $z, p_1, \ldots, p_n$ as functions of $x_1, \ldots, x_n$; but the value of z so obtained cannot be an integral of the original equation, unless the values of $p_1, \ldots, p_n$ are the same as the values of $\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}$ derived from that value of z. To test this possibility, suppose that the n equations

$$\frac{\partial f}{\partial p_1} = 0, \ \ldots, \ \frac{\partial f}{\partial p_n} = 0$$

can be resolved* with regard to $p_1, \ldots, p_n$, and that the values of $p_1, \ldots, p_n$ thence deduced are substituted in $f=0$; the latter then becomes a relation between $z, x_1, \ldots, x_n$. If the relation provides an integral of the original equation, then

$$p_r = \frac{\partial z}{\partial x_r},$$

for $r = 1, \ldots, n$, the values of $p_1, \ldots, p_n$ being the above values, and the values of $\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}$ being deduced from the integral relation. The latter are given by

$$\frac{\partial f}{\partial x_r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x_r} + \sum_{i=1}^{n}\left\{\frac{\partial f}{\partial p_i}\left(\frac{\partial p_i}{\partial x_r} + \frac{\partial p_i}{\partial z}\frac{\partial z}{\partial x_r}\right)\right\} = 0,$$

for $r = 1, \ldots, n$; hence we must have

$$\frac{\partial f}{\partial x_r} + \frac{\partial f}{\partial z} p_r = 0.$$

Conversely, if this equation is satisfied, and if the initial assumption that the $n+1$ equations determine $z, p_1, \ldots, p_n$ as functions of $x_1, \ldots, x_n$ is justified, we have

$$\frac{\partial z}{\partial x_r} = p_r,$$

for $r = 1, \ldots, n$, provided $\frac{\partial f}{\partial z}$ is not zero. In that case, we have an integral of the differential equation: it is the singular integral. But if the values of $z, p_1, \ldots, p_n$ make $\frac{\partial f}{\partial z}$ vanish, the inference cannot be made: separate investigation is then required and will come later. We thus have the following theorem:

If the equations

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0,$$

$$\frac{\partial f}{\partial p_1} = 0, \ldots, \frac{\partial f}{\partial p_n} = 0, \quad \frac{\partial f}{\partial x_1} + p_1\frac{\partial f}{\partial z} = 0, \ \ldots, \ \frac{\partial f}{\partial x_n} + p_n\frac{\partial f}{\partial z} = 0,$$

are consistent with one another, and if they determine $z, p_1, \ldots, p_n$ as functions of $x_1, \ldots, x_n$, such that $\frac{\partial f}{\partial z}$ does not vanish identically,

* This supposition requires that the Hessian of f does not vanish simultaneously with the $n+1$ quantities $f, \frac{\partial f}{\partial p_1}, \ldots, \frac{\partial f}{\partial p_n}$.

the value of z thus given is an integral of the equation, being the singular integral.

Of course, if the $2n+1$ equations are not consistent with one another, no integral of the differential equation can be found by this avenue.

And it must not be assumed that the locus, given by the elimination of $p_1, \ldots, p_n$ among the equations

$$f=0, \quad \frac{\partial f}{\partial p_1}=0, \ldots, \frac{\partial f}{\partial p_n}=0,$$

is the singular integral: if it exists, it will be included in the locus, but the locus may include other equations which do not provide integrals.

Ex. Discuss in the preceding manner, so as to obtain singular integrals (if any), the equations

(i) $z=p_1x_1+\ldots+p_nx_n+ap_1\ldots p_n$;

(ii) $(ap_1-z)(ap_2-z)(ap_3-z)=a^3p_1p_2p_3$;

(iii) $z=f(p_1, \ldots, p_n)$,

where, in the last equation, f is a polynomial in its arguments.

Exceptional Integrals.

77. Now it may happen that the $2n+1$ equations

$$f=0, \quad \frac{\partial f}{\partial p_1}=0, \ldots, \frac{\partial f}{\partial p_n}=0, \quad \frac{\partial f}{\partial x_1}+p_1\frac{\partial f}{\partial z}=0, \ldots, \frac{\partial f}{\partial x_n}+p_n\frac{\partial f}{\partial z}=0$$

are consistent with one another, but that (contrary to the hypothesis in the preceding theorem) they do not determine all the quantities $z, p_1, \ldots, p_n$ in terms of $x_1, \ldots, x_n$; they may determine only a number of these quantities in terms of the remainder, say

$$p_r=g_r(x_1, \ldots, x_n, z, p_{m+1}, \ldots, p_n),$$

for $r=1, \ldots, m$. When these values are substituted in the above equations, each of them becomes an identity, $z, p_{m+1}, \ldots, p_n$ being regarded as functions of $x_1, \ldots, x_n$. In particular, $f=0$ is an identity; and therefore

$$\frac{\partial f}{\partial x_s}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x_s}+\sum_{r=1}^{m}\left[\frac{\partial f}{\partial p_r}\left\{\frac{\partial g_r}{\partial x_s}+\frac{\partial g_r}{\partial z}\frac{\partial z}{\partial x_s}+\sum_{\mu}\left(\frac{\partial g_r}{\partial p_{m+\mu}}\frac{dp_{m+\mu}}{dx_s}\right)\right\}\right]$$
$$+\sum_{\mu}\frac{\partial f}{\partial p_{m+\mu}}\frac{dp_{m+\mu}}{dx_s}=0.$$

But the equations

$$\frac{\partial f}{\partial p_r} = 0, \quad \frac{\partial f}{\partial p_{m+\mu}} = 0,$$

are satisfied identically, for $r = 1, \ldots, m$, and $\mu = 1, \ldots, n - m$, by the values in question; also

$$\frac{\partial f}{\partial x_s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x_s} = 0,$$

for $s = 1, \ldots, n$: hence, unless $\frac{\partial f}{\partial z}$ vanishes for the values in question, we have

$$p_s = \frac{\partial z}{\partial x_s},$$

for all the values of s.

Thus the set of $2n + 1$ equations may be replaced by a set

$$p_r = g_r(x_1, \ldots, x_n, z, p_{m+1}, \ldots, p_n),$$

for $r = 1, \ldots, m$: and, from their source, we have seen that

$$p_s = \frac{\partial z}{\partial x_s}, \qquad (s = 1, \ldots, n),$$

that is, the quantities p are the derivatives of z. Thus the set of m equations is a complete system: it possesses an integral containing $n - m + 1$ arbitrary constants.

Although such an integral has affinities with the complete integral, it can hardly be claimed as a specialised case of the complete integral: and although it has affinities with the singular integral, it can hardly be claimed as a generalised case of the singular integral. It may be regarded as belonging to the, as yet, unclassified aggregate of special integrals.

Examples will be given later.

Integrals of Equations of first order in two independent variables.

78. After the general discussion for equations in n independent variables, it is unnecessary to enter upon the similar discussion for equations in two independent variables: but the results are so important for the latter set of equations, particularly in connection with the geometry of ordinary space, that they are worthy of separate statement.

Accordingly, an equation of the first order in two independent variables, represented as usual by

$$f(x, y, z, p, q) = 0,$$

possesses a *complete integral* involving two arbitrary constants, which may be represented by

$$\phi(x, y, z, a, b) = 0.$$

In addition, it possesses a *general integral*, obtained by the elimination of a and b between the equations

$$\left.\begin{aligned} \phi(x, y, z, a, b) &= 0 \\ b - \theta(a) &= 0 \\ \frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b}\theta'(a) &= 0 \end{aligned}\right\},$$

where θ is an arbitrary function: frequently, the elimination cannot explicitly be performed, and then the three equations give the general integral.

The differential equation may, but does not necessarily, possess another integral derivable from the complete integral. If the equations

$$\phi = 0, \quad \frac{\partial\phi}{\partial a} = 0, \quad \frac{\partial\phi}{\partial b} = 0,$$

furnish values of z, a, b in terms of x and y, such that z is an integral of the differential equation, then if b can be expressed in terms of a alone, the integral so furnished is a particular case of the general integral: but, if b cannot be expressed in terms of a alone, the integral is a *singular integral.*

Moreover, a differential equation may possess integrals of the unclassified aggregate called *special*; they are not derivable from the complete integral.

Further, if the equation possesses a singular integral, it is given by the equations

$$f = 0, \quad \frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0, \quad \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0,$$

provided these equations are consistent with one another and determine p, q, z as functions of x and y, such as to leave $\dfrac{\partial f}{\partial z}$ different from zero: the value of z so determined is the singular

integral. If the five equations are inconsistent with one another, there is no singular integral. If the five equations are consistent with one another but are equivalent to two equations only, which may be regarded as determining p and q in terms of x, y, z, then the equation

$$dz = p\,dx + q\,dy$$

is exact after the values of p and q have been substituted: a quadrature leads to an equation, involving one arbitrary constant and providing an integral of the equation. Such an integral will be called *special*.

Ex. Examples of the ordinary integrals that occur most frequently in connection with the simplest forms of equations are found freely in text-books.

As an illustration of the integrals here called special, when they arise through the process that, if otherwise favourable, allows the deduction of the singular integral from the equation itself, consider the equation

$$f = (px + qy - z)^2 + \frac{z^2}{x^2+y^2-1} - p^2 - q^2 = 0.$$

The five equations

$$f=0,\quad \frac{\partial f}{\partial p}=0,\quad \frac{\partial f}{\partial q}=0,\quad \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0,\quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0,$$

are satisfied by $z=0$, which is a singular integral. They also are satisfied by

$$p=\frac{xz}{x^2+y^2-1},\quad q=\frac{yz}{x^2+y^2-1};$$

and they then do not determine z in terms of x and y. When these values of p and q are substituted in

$$dz = p\,dx + q\,dy,$$

and the quadrature is effected, we have

$$z^2 = a^2(x^2+y^2-1),$$

where a is an arbitrary constant. It is easy to verify that this value of z satisfies the differential equation, and therefore is an integral.

In order to consider the relation of the integral thus deduced to other integrals of the equation, we use Charpit's method (§ 68) for the solution of the equation. Writing

$$\zeta = px + qy - z,\qquad u = x^2+y^2-1,$$

and equating to $\frac{1}{2}dt$ each of the fractions in Charpit's subsidiary equations these become

$$\frac{dx}{dt}=p-x\zeta,\quad \frac{dy}{dt}=q-y\zeta,$$

$$\frac{dz}{dt}=\frac{z^2}{u}-z\zeta,$$

$$\frac{dp}{dt}=\frac{z}{u}\left(p-\frac{z}{u}x\right),\quad \frac{dq}{dt}=\frac{z}{u}\left(q-\frac{z}{u}y\right),$$

after slight reduction and using the equation $f=0$. Hence

$$\frac{d\zeta}{dt}=\left(\zeta-\frac{z}{u}\right)\frac{z}{u},\quad \frac{du}{dt}=2z-2\zeta u,$$

and therefore

$$\frac{d}{dt}\left(\zeta-\frac{z}{u}\right)=0,$$

that is, one integral of Charpit's equations, involving the derivatives p and q, is given by

$$\zeta-\frac{z}{u}=a,$$

where a is an arbitrary constant. When this is combined with the original equation, and the two equations are resolved for p and q, we find

$$p\,(x^2+y^2)=xv-yau^{\frac{1}{2}},$$

$$q\,(x^2+y^2)=yv+xau^{\frac{1}{2}},$$

where

$$v=a+z+\frac{z}{u};$$

and these are to be substituted in

$$dz=p\,dx+q\,dy,$$

which then becomes an exact equation. We have

$$dz=\tfrac{1}{2}\frac{du}{u+1}\left(a+z\frac{u+1}{u}\right)+\frac{xdy-ydx}{u+1}au^{\frac{1}{2}},$$

that is,

$$d\left(\frac{z}{u^{\frac{1}{2}}}\right)=\tfrac{1}{2}a\frac{du}{(u+1)\,u^{\frac{1}{2}}}+a\frac{xdy-ydx}{x^2+y^2};$$

after a quadrature, we have

$$\frac{z}{u^{\frac{1}{2}}}-\beta=a\left\{\tan^{-1}u^{\frac{1}{2}}+\tan^{-1}\frac{y}{x}\right\},$$

which may be regarded as the complete integral, expressed in a form that is both transcendental and irrational.

Writing

$$x=r\cos\theta,\quad y=r\sin\theta,$$

this complete integral becomes

$$\frac{z}{(r^2-1)^{\frac{1}{2}}}=\beta+a\,[\theta+\tan^{-1}\{(r^2-1)^{\frac{1}{2}}\}].$$

The general integral is expressible in the form

$$\frac{z^2}{r^2-1}=F[\theta+\tan^{-1}\{(r^2-1)^{\frac{1}{2}}\}].$$

The special integral, which was obtained in the form

$$z^2=a^2\,(x^2+y^2-1),$$

can be deduced from the complete integral by assuming $a=0$, $\beta=a$, and rationalising the result: it can also be obtained from the general integral by assuming $F(\xi)=a^2$.

The singular integral $z=0$ can be derived from the complete integral, taken in the form

$$z^2=(r^2-1)\left[\beta+\alpha\theta+a\tan^{-1}\{(r^2-1)^{\frac{1}{2}}\}\right]^2,$$

by the customary process: it can also be deduced as a particular case of the special integral, by the assumption $a=0$.

79. A whole class of equations possessing special integrals of the indicated type can be constructed as follows*. Let

$$f(x, y, z, p, q)=0$$

be an equation which has a singular integral according to the formal Lagrangian theory: the values of z, p, q given by this integral must satisfy the equations

$$\frac{\partial f}{\partial p}=0,\quad \frac{\partial f}{\partial q}=0,\quad \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0,\quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0.$$

Let the first two (or any two) of the last four consistent equations be resolved so as to express p and q in terms of the rest of the variables; and let the result of substituting these expressions for p and q in $f(x, y, z, p, q)$ be $g(x, y, z)$: then

$$g(x, y, z)=0$$

provides the singular integral of the equation

$$f(x, y, z, p, q)=0,$$

on the Lagrangian theory.

The equations

$$\frac{\partial g}{\partial x}+p\frac{\partial g}{\partial z}=0,\quad \frac{\partial g}{\partial y}+q\frac{\partial g}{\partial z}=0,$$

are consistent with the preceding five equations. Moreover, as $g(x, y, z)$ is the value of $f(x, y, z, p, q)$ when the values of p and q given by $\frac{\partial f}{\partial p}=0$ and $\frac{\partial f}{\partial q}=0$ are substituted in $f(x, y, z, p, q)$, we have

$$\begin{aligned}\frac{\partial g}{\partial x}+p\frac{\partial g}{\partial z}&=\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}+\frac{\partial p}{\partial x}\frac{\partial f}{\partial p}+\frac{\partial q}{\partial x}\frac{\partial f}{\partial q}\\&=\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z},\end{aligned}$$

and, similarly,

$$\frac{\partial g}{\partial y}+q\frac{\partial g}{\partial z}=\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z},$$

* The process was suggested to me by a remark in a letter from Prof. Chrystal, dated 18 May, 1896.

both identically; that is, these equations are satisfied identically by values of p and q given by

$$\frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0,$$

when in addition to these two relations, we take

$$f(x, y, z, p, q) = g(x, y, z),$$

where $g(x, y, z)$ is the result of substituting the values of p and q in $f(x, y, z, p, q)$.

Now consider the equation

$$F(x, y, z, p, q) = f(x, y, z, p, q) - g(x, y, z) = 0.$$

If it possesses a singular integral according to Lagrange's formal theory, this integral must be such as to satisfy, not merely $F = 0$, but also

$$\frac{\partial F}{\partial p} = 0, \quad \frac{\partial F}{\partial q} = 0, \quad \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0.$$

The first two of the last four are

$$\frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0;$$

when these are resolved for p and q in terms of x, y, z, the values of p and q are such that

$$f(x, y, z, p, q) = g(x, y, z),$$

that is, the three equations

$$F = 0, \quad \frac{\partial F}{\partial p} = 0, \quad \frac{\partial F}{\partial q} = 0,$$

are equivalent to two equations only, expressing p and q in terms of x, y, z.

Moreover, it appears that when the specified values of p and q are substituted in $f(x, y, z, p, q)$, the latter becomes $g(x, y, z)$: hence, after the preceding explanations and taking account of the source of $g(x, y, z)$, the two equations

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = \frac{\partial g}{\partial x} + p\frac{\partial g}{\partial z},$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = \frac{\partial g}{\partial y} + q\frac{\partial g}{\partial z},$$

are satisfied identically, that is, the equations

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0,$$

are satisfied identically. Hence the five equations

$$F=0, \quad \frac{\partial F}{\partial p}=0, \quad \frac{\partial F}{\partial q}=0, \quad \frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}=0, \quad \frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}=0,$$

are equivalent to two equations only, expressing p and q as functions of x, y, z: the equation

$$F=0$$

has a special integral.

Note. In what precedes, there is a tacit assumption that $F=0$ is irreducible; if, however, $F=0$ can be resolved into distinct equations, the argument is no longer valid.

Ex. 1. Apply the preceding process to construct from the equation

$$z^2(1+p^2+q^2)=\lambda^2\{(x+pz)^2+(y+qz)^2\},$$

which has

$$z^2(1-\lambda^2)=\lambda^2(x^2+y^2)$$

for a singular integral, another equation which has a special integral.

Ex. 2. Can the method be applied to the equation

$$z=px+qy+p^nq^n,$$

for any value of n?

80. The preceding discussion has been concerned with the integrals that are derivable from the complete integral of a partial differential equation; a distinctive property of the complete integral is that the number of parameters which it involves is the same as the number of independent variables. But integral equations, not distinguished by this property, may be propounded for consideration: thus the number of parameters may exceed the number of independent variables; and we have seen how an equation can arise as an integral of a set of simultaneous differential equations, and then the number of parameters involved is less than the number of independent variables.

A very brief discussion is sufficient to deal with an equation

$$\phi(z, x_1, \ldots, x_n, a_1, \ldots, a_m)=0,$$

when $m>n$. It may, of course, be assumed that the m parameters are essential*, that is, are not reducible to a smaller number: the necessary and sufficient test is that the equation

$$\alpha_1\frac{\partial\phi}{\partial a_1}+\ldots+\alpha_m\frac{\partial\phi}{\partial a_m}=0$$

* In the sense adopted in Lie's theory of groups.

is not identically satisfied for any non-zero values of $\alpha_1, \ldots, \alpha_m$ as functions of $a_1, \ldots, a_m$. Forming the equations

$$\phi_r = \frac{\partial \phi}{\partial x_r} + p_r \frac{\partial \phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$, it usually is not possible to eliminate a number of constants greater than n among the $n+1$ equations

$$\phi = 0, \quad \phi_1 = 0, \ldots, \phi_n = 0;$$

so that usually, on proceeding to the eliminant equation, we should find that it contained $m - n$ parameters. If, however, the appropriate Jacobian conditions

$$J\left(\left(\frac{\phi, \phi_1, \ldots, \phi_n}{a_1, a_2, \ldots, a_m}\right)\right) = 0$$

are satisfied for each selection of $n + 1$ parameters from the set of m, then the single equation resulting from elimination contains no parameters. The integral equation is then a special case of the general integral of the partial equation.

Ex. The equation

$$z = ax + by + c\frac{y^2}{x} + g\frac{x^2}{y}$$

leads to

$$xp + yq = z:$$

it is a special form of

$$z = xf\left(\frac{y}{x}\right),$$

which is the general integral.

Classes of Integrals of a Complete System.

81. Coming next to an equation

$$\phi(z, x_1, \ldots, x_n, a_1, \ldots, a_m) = 0,$$

for which $m < n$, we shall assume, as before in § 80, that the parameters $a_1, \ldots, a_m$ are essential. We shall also assume that the m constants can be eliminated between $\phi = 0$ and the n derived equations

$$\phi_r = \frac{\partial \phi}{\partial x_r} + p_r \frac{\partial \phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$, so as to give $n - m + 1$ equations involving $z, x_1, \ldots, x_n, p_1, \ldots, p_n$. And we assume that $\frac{\partial \phi}{\partial z}$ does not vanish in

association with $\phi = 0$. We then have the case, which has already been considered, of a number of simultaneous differential equations; these equations form a complete set, because of their source; and the Jacobian method of integration has shewn how to construct an integral $\phi = 0$ containing m constants that are arbitrary. Having regard to the investigation in the case when $m = n$, which made the derivation of other integrals from $\phi = 0$ possible, we proceed to a similar quest and seek to derive other integrals from $\phi = 0$ in the present case when $m < n$.

We proceed as before. The $n - m + 1$ differential equations are the result of eliminating $a_1, \ldots, a_m$ among the equations

$$\phi = 0, \quad \phi_1 = 0, \ldots, \phi_n = 0.$$

The course of the elimination takes no account of the quality of $a_1, \ldots, a_m$: it will lead to the same result if these quantities be changed in such a way that each of the $n + 1$ equations is unaltered in form. Accordingly, subject to this limitation, we make $a_1, \ldots, a_m$ functions of the independent variables $x_1, \ldots, x_n$; and the limitation requires that the n relations

$$\sum_{i=1}^{m} \frac{\partial \phi}{\partial a_i} \frac{\partial a_i}{\partial x_r} = 0,$$

for $r = 1, \ldots, n$, shall be satisfied, conditions that clearly are sufficient as well as necessary to secure the invariability of form of the $n + 1$ equations. Multiplying the n relations by $dx_1, \ldots, dx_n$ respectively, and adding, we obtain a single relation

$$\sum_{i=1}^{m} \frac{\partial \phi}{\partial a_i} da_i = 0$$

in the differential elements: it is equivalent to the n relations and therefore, when satisfied, it suffices for the present purpose.

This differential relation can be satisfied in various ways.

In the first place, all the quantities $da_1, \ldots, da_m$ may vanish, so that all the quantities $a_1, \ldots, a_m$ are constant. We then resume the original integral: on the analogy of the corresponding integral for a single equation, we call it the *complete integral.*

In the second place, an integral equivalent of the differential relation may consist of μ equations

$$g_1(a_1, \ldots, a_m) = 0, \ldots, g_\mu(a_1, \ldots, a_m) = 0,$$

and of these only. Obviously μ cannot be greater than m. If μ be equal to m, then there are m equations involving m quantities a: each of these quantities is a constant, and so we fall back upon the preceding case. Hence we need consider only values of μ that are less than m. As there are μ integral equations in the complete equivalent of the differential relation, the only relations among the differential elements are

$$dg_1 = 0, \ldots, dg_\mu = 0;$$

and the differential relation

$$\sum_{i=1}^{m} \frac{\partial \phi}{\partial a_i} da_i = 0$$

must be satisfied in virtue of them. Consequently, μ quantities $\lambda_1, \ldots, \lambda_\mu$ must exist such that

$$\sum_{i=1}^{m} \frac{\partial \phi}{\partial a_i} da_i = \lambda_1 dg_1 + \ldots + \lambda_\mu dg_\mu;$$

and therefore

$$\frac{\partial \phi}{\partial a_i} = \lambda_1 \frac{\partial g_1}{\partial a_i} + \ldots + \lambda_\mu \frac{\partial g_\mu}{\partial a_i},$$

for $i = 1, \ldots, m$. These m equations, together with

$$\phi = 0, \quad g_1 = 0, \ldots, g_\mu = 0,$$

make up $m + \mu + 1$ equations: eliminating the m parameters $a_1, \ldots, a_m$ and the μ multipliers $\lambda_1, \ldots, \lambda_\mu$ among them, we obtain a single equation among $z, x_1, \ldots, x_n$. The value of z thus determined is an integral of the original differential equation: as before, we call it a *general integral*.

In the expression of a general integral, the functional forms $g_1, \ldots, g_\mu$ occur; and so there are various classes of general integrals, which arise according to the number of postulated relations. It is clear that

$$0 < \mu < m;$$

and it is customary to describe a general integral, associated with μ forms, as of *class* μ.

The extreme case, when $\mu = m$, has already been mentioned: the integral is then complete. The other extreme case, when $\mu = 0$, will be discussed immediately.

As in the case of a single differential equation, it might be supposed that a class of integral, intermediate between the com-

plete integral and the general integrals, would be obtained by taking

$$a_{i+1} = \text{constant}, \ldots, a_m = \text{constant},$$

and then postulating a number of relations

$$g_1(a_1, \ldots, a_i) = 0, \ldots, g_\tau(a_1, \ldots, a_i) = 0,$$

where $\tau < i$, among the remaining parameters $a_1, \ldots, a_i$. Effectively, these equations amount to $m + \tau - i$ relations among the parameters $a_1, \ldots, a_m$, of which $m - i$ are special in form: the corresponding integral would be a specialised general integral of class $m + \tau - i$.

Moreover, it can be proved, as in the case of a single differential equation, that the most comprehensive general integral is of the first class when the single relation is quite arbitrary: on this account, it is sometimes called *the general integral.*

There is one other mode of securing that the differential relation is satisfied. It is possible that the equations

$$\frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_m} = 0$$

could hold; the differential relation then becomes evanescent, and so it ceases to have any necessary influence upon the organic variations under consideration. The equations

$$\phi = 0, \quad \frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_m} = 0$$

may coexist and may be consistent with one another: if the result of the elimination of $a_1, \ldots, a_m$ among them provides a single equation involving no arbitrary element, and if that single equation determines an integral*, the integral thus furnished is called the *singular integral.*

Ex. 1. The simplest cases arise when there are only two independent variables. Thus let

$$\phi = z - a^2 y + ax = 0;$$

the value of z thus provided satisfies the two equations

$$\left.\begin{array}{l} z = px + p^2 y \\ z = px + qy \end{array}\right\}.$$

* The reason for this limitation is similar to the reason in the former case (§ 73). Even when the process is possible, the locus provided by the eliminant frequently is composite: some of its components, even all the components, may not be integrals of the differential equation but may be loci of singularities on the complete integral.

The elimination of a, between $\phi=0$ and

$$\frac{\partial\phi}{\partial a}=x-2ay=0,$$

leads to an equation

$$x^2+4yz=0.$$

It is easy to verify that the value of z given by this last equation satisfies the two differential equations: accordingly, it is a singular integral.

Geometrically interpreted, the complete integral is a family of planes through the origin touching a cone which is the singular integral.

Ex. 2. Obtain the simplest differential equations satisfied by the value of z given by

$$2z=a^2y^2-ax^2:$$

and prove that they possess a singular integral represented by

$$8y^2z+x^4=0.$$

Ex. 3. Discuss similarly the equation

$$z=ax+a^2y+a^3,$$

obtaining the two simplest differential equations which it satisfies, in the form

$$\left.\begin{aligned} z&=px+qy+pq \\ z&=px+p^2y+p^3 \end{aligned}\right\}.$$

Prove that the equation

$$27z^2+z(18xy-4y^3)-x^2y^2+4x^3=0$$

provides a singular integral.

Shew also that the value of z given by the original equation satisfies the two partial equations

$$\left.\begin{aligned} z&=px+p^2y+p^3 \\ (z-qy)^2&=q(x+q)^2 \end{aligned}\right\}.$$

Is the original equation the complete integral of these two equations?

Ex. 4. Integrate the equations

$$\left.\begin{aligned} z&=p_1x_1+p_2x_2+p_3x_3 \\ p_3&=p_1p_2 \end{aligned}\right\};$$

and shew that they possess a singular integral

$$zx_3+x_1x_2=0.$$

Ex. 5. It has been seen (§ 57) that the simultaneous equations

$$p_1p_2-x_3x_4=0, \qquad p_3p_4-x_1x_2=0$$

possess two complete integrals

$$z=\frac{x_1x_3}{a}+ax_2x_4+b, \qquad z=\frac{x_2x_3}{A}+Ax_1x_4+B.$$

The respective general integrals are evidently given by

$$\left.\begin{aligned} z&=\frac{x_1x_3}{a}+ax_2x_4+\phi(a) \\ 0&=-\frac{x_1x_3}{a^2}+x_2x_4+\phi'(a) \end{aligned}\right\},\quad \left.\begin{aligned} z&=\frac{x_2x_3}{A}+Ax_1x_4+\psi(A) \\ 0&=-\frac{x_2x_3}{A^2}+x_1x_4+\psi'(A) \end{aligned}\right\};$$

and there is no singular integral. There is a question as to the relation of the two complete integrals to one another.

If they can be brought into such relation that one of them can be changed into the other, then (after the preceding discussion) we must have

$$p_1=\frac{x_3}{a}=Ax_4,\quad p_3=\frac{x_1}{a}=\frac{x_2}{A},$$

$$p_2=ax_4=\frac{x_3}{A},\qquad p_4=ax_2=Ax_1,$$

all of which are satisfied by the values

$$a=\left(\frac{x_1x_3}{x_2x_4}\right)^{\frac{1}{2}},\quad A=\left(\frac{x_2x_3}{x_1x_4}\right)^{\frac{1}{2}}.$$

In order that the two values of z may be equal, we must (with these values of a and A) have

$$b=B.$$

The other equations require that the relation

$$\frac{\partial z}{\partial a}da+\frac{\partial z}{\partial b}db=0$$

be satisfied: that this may be the case, we must have

$$db=0,$$

that is, b must be a pure constant.

We thus see that the conditions, necessary to secure that each of the complete integrals can be transformed into the other, are not satisfied: the values of a and b, which have been obtained, do not lead after substitution to the other integral. The complete integrals are distinct from one another: it will be seen, on reference to the construction of the integrals, that they belong to different resolutions of the original system.

But, on the other hand, by the substitution of the values of a and A, both complete integrals lead to a new integral

$$z=2(x_1x_2x_3x_4)^{\frac{1}{2}}+b,$$

which is a particular form of each of the general integrals.

82. The aggregate of integrals, composed of the complete integral, the general integrals, and (when it exists) the singular integral, is widely comprehensive: but for a complete set of differential equations, as for a single equation in the earlier discussion, the aggregate cannot be declared wholly comprehensive. The argument is similar to the argument in the case of a single

equation and therefore hardly needs to be repeated in the present connection.

If any given integral is included in the above aggregate, the determination of its character is easily effected. Let

$$\phi(z, x_1, \ldots, x_n, a_1, \ldots, a_m) = 0$$

be the complete integral of a given complete set of $n - m + 1$ partial differential equations; and let

$$\psi(z, x_1, \ldots, x_n) = 0$$

be any other integral of that set. If $\psi = 0$ is included in the aggregate, it must be possible to assign values (constant or variable) so that the equations are satisfied, and so also that the values of z given by the two equations are the same. If the latter condition be satisfied, the values of $p_1, \ldots, p_n$ must be the same. For $\psi = 0$, they are given by

$$\frac{\partial \psi}{\partial x_r} + \frac{\partial \psi}{\partial z} p_r = 0,$$

for $r = 1, \ldots, n$: and for $\phi = 0$, with values assigned to $a_1, \ldots, a_m$ such that the differential equations still are satisfied, the quantities $p_1, \ldots, p_n$ are given by

$$\frac{\partial \phi}{\partial x_r} + \frac{\partial \phi}{\partial z} p_r = 0,$$

for $r = 1, \ldots, n$. If they are the same for the two integrals, we have

$$\frac{\partial \phi}{\partial x_r} \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x_r} \frac{\partial \phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$. Hence the quantities $a_1, \ldots, a_m$ must be such as to satisfy these n equations and also the condition that the value of z given by $\phi = 0$ is the same as the value of z given by $\psi = 0$.

If these $n + 1$ conditions give constant values for $a_1, \ldots, a_m$, the integral furnished by $\psi = 0$ is a particular case of the complete integral.

If the $n + 1$ conditions express $a_1, \ldots, a_m$ as functions of the variables, such that these functions are connected by a number of relations of the type

$$g(a_1, \ldots, a_m) = 0,$$

the number of these relations being less than m, the integral furnished by $\psi = 0$ is a general integral.

If the $n+1$ conditions express $a_1, \ldots, a_m$ in terms of the variables, the values being unconnected by any functional relation or relations, the integral furnished by $\psi=0$ is a singular integral.

But as $m<n$, we cannot affirm that the $n+1$ equations must certainly determine the quantities $a_1, \ldots, a_m$ without the introduction of relations among the independent variables. In all instances, when $a_1, \ldots, a_m$ are not determined in one or other of the foregoing forms, the integral furnished by $\psi=0$ is not included in the aggregate of integrals associated with $\phi=0$; it belongs to the unclassified set of integrals previously called *special.* In such an event, the retained aggregate of integrals associated with $\phi=0$ is not wholly comprehensive.

Ex. 1. The two equations

$$f_1=p_2-p_3=0$$

$$f_2=x_1p_1+2x_2p_2+2x_3p_3-2\left(z-\frac{x_1^2}{x_2+x_3}\right)^2=0,$$

are a complete set: for

$$[f_1, f_2]=0,$$

in virtue of $f_1=0$.

A complete integral is furnished by

$$\frac{x_2+x_3}{z(x_2+x_3)-x_1^2}=a+b\frac{x_1^2}{x_2+x_3}-\log(x_2+x_3);$$

a general integral is furnished by

$$ye^{\frac{x_2+x_3}{z(x_2+x_3)-x_1^2}}=g\left(\frac{x_1^2}{x_2+x_3}\right),$$

where g is an arbitrary function; and there is no singular integral.

It is easy to verify that the two equations are satisfied by

$$z=\frac{x_1^2}{x_2+x_3};$$

no definite values can be assigned to a and b, and no definite form can be assigned to g, so that this integral can be included in the foregoing aggregate.

Ex. 2. Discuss the character of the integral of

$$p_2-p_3=0, \quad \{1+(z-x_1-x_2-x_3)^{\frac{1}{2}}\}p_1+p_2+p_3=3,$$

which is given by

$$z=x_1+x_2+x_3.$$

Singular Integral of a Complete System.

83. When a complete set of simultaneous equations possesses a singular integral, the knowledge of the property of a single equation in similar circumstances makes it natural to enquire whether the singular integral can be derived from the complete set itself, without the intervention of the complete integral. It is possible to do so in cases when the equations possess (or when, without extension of their significance, they can be transformed so as to possess) a particular form, as will now be proved. Partly owing to the elaboration of the conditions even when the particular form is possessed, and partly owing to the fact that a set of simultaneous equations in one dependent variable has nothing like uniqueness of form, an investigation into the general case will not be pursued.

Suppose that, by appropriate combinations of the members of a complete set of $n - m + 1$ equations, it is possible to deduce one equation (or more than one equation) in an equivalent set which involves m, and not more than m, of the derivatives. As the complete set is assumed to possess a singular integral, we shall further suppose that the equation in question is not resolved with regard to any of those derivatives*; and so we may take the equation in the form

$$f(z,\ x_1,\ \ldots,\ x_n,\ p_1,\ \ldots,\ p_m) = 0,$$

the remaining $n - m$ equations involving $p_{m+1}, \ldots, p_n$, or some of them in each equation, as well as possibly $p_1, \ldots, p_m$. Let the complete integral be denoted by

$$\phi\,(z,\ x_1,\ \ldots,\ x_n,\ a_1,\ \ldots,\ a_m) = 0:$$

the values of the derivatives are given by

$$\phi_r = \frac{\partial\phi}{\partial x_r} + p_r\frac{\partial\phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$; and the complete set of $n - m + 1$ equations results from the elimination of $a_1, \ldots, a_m$ among the $n + 1$ equations

$$\phi = 0, \quad \phi_1 = 0, \ \ldots, \ \phi_n = 0.$$

* It will appear from the analysis that a resolved equation of the indicated type would exclude the existence of a singular integral, because it would lead to impossible conditions.

In particular, the equation $f = 0$ selected from the set must result from the elimination: as it involves only $p_1, \ldots, p_m$ but not $p_{m+1}, \ldots, p_n$, it must result from the elimination of $a_1, \ldots, a_m$ among the $m+1$ equations

$$\phi = 0, \quad \phi_1 = 0, \ldots, \phi_m = 0.$$

If we assume that all the quantities $\phi, \phi_1, \ldots, \phi_m$ are rational and integral, quantities $\lambda, \lambda_1, \ldots, \lambda_m$ will exist such that

$$f = \lambda\phi + \lambda_1\phi_1 + \ldots + \lambda_m\phi_m.$$

Now f does not involve any of the parameters $a_1, \ldots, a_m$: hence, in connection with our integral, we have

$$0 = \lambda\frac{\partial\phi}{\partial a_i} + \lambda_1\frac{\partial\phi_1}{\partial a_i} + \ldots + \lambda_m\frac{\partial\phi_m}{\partial a_i},$$

for $i = 1, \ldots, m$. The singular integral is given, in connection with $\phi = 0$, by

$$\frac{\partial\phi}{\partial a_1} = 0, \ \ldots, \ \frac{\partial\phi}{\partial a_m} = 0:$$

and therefore the singular integral, in connection with $\phi = 0$, requires that the equations

$$0 = \lambda_1\frac{\partial\phi_1}{\partial a_i} + \ldots + \lambda_m\frac{\partial\phi_m}{\partial a_i},$$

for $i = 1, \ldots, m$, be satisfied. These m equations are linear and homogeneous in $\lambda_1, \ldots, \lambda_m$, and the determinant of the coefficients is

$$\frac{\partial(\phi_1, \ldots, \phi_m)}{\partial(a_1, \ldots, a_m)},$$

which does not vanish (the elimination would not be possible if this quantity were to vanish); hence the singular integral requires that

$$\lambda_1 = 0, \ \ldots, \ \lambda_m = 0.$$

Again, f involves $p_1, \ldots, p_m$, which do not occur in ϕ and occur only individually in $\phi_1, \ldots, \phi_m$ respectively: hence, in connection with the integral $\phi = 0$,

$$\frac{\partial f}{\partial p_r} = \lambda_r\frac{\partial\phi}{\partial z},$$

for $r = 1, \ldots, m$. Consequently, the singular integral requires that

$$\frac{\partial f}{\partial p_1} = 0, \ \ldots, \ \frac{\partial f}{\partial p_m} = 0,$$

and the singular integral satisfies

$$f=0:$$

hence it may be possible to obtain the singular integral by the elimination of $p_1, \ldots, p_m$ among these $m+1$ equations.

The equation that results from the elimination is not necessarily an integral of $f=0$: if it is, the relations

$$\frac{\partial f}{\partial x_1}+p_1\frac{\partial f}{\partial z}=0, \ \ldots, \ \frac{\partial f}{\partial x_n}+p_n\frac{\partial f}{\partial z}=0,$$

must be satisfied, when the derived values of $p_1, \ldots, p_n$ are substituted. And in order that it may be an integral of the remaining $n-m$ equations of the system, each of those equations must be satisfied when the values of $p_1, \ldots, p_n, z$ are substituted.

It therefore appears that *when a complete set of $n-m+1$ equations possesses a singular integral, and when an equation can be selected or compounded from the set involving only m of the derivatives in a form*

$$f(z, x_1, \ldots, x_n, p_1, \ldots, p_m)=0,$$

the singular integral may be given by the elimination of $p_1, \ldots, p_m$ between the equations

$$f=0, \quad \frac{\partial f}{\partial p_1}=0, \ \ldots, \ \frac{\partial f}{\partial p_m}=0;$$

the conditions indicated must be satisfied in order that the eliminant may provide an integral; and, when they are satisfied, the eliminant proves the singular integral.

Ex. 1. Consider the system

$$\left.\begin{aligned} z&=px+p^2y \\ z&=px+qy \end{aligned}\right\},$$

in Ex. 1, § 81.

An equation of the required type is furnished by

$$f=p^2y+px-z=0;$$

the associated equation is

$$\frac{\partial f}{\partial p}=2py+x=0.$$

Eliminating p between these equations, we have

$$4yz+x^2=0;$$

the value of z thus given satisfies both equations of the system, and therefore it is a singular integral.

Another equation of the required type is furnished by

$$g=(z-qy)^2-x^2q=0;$$

the associated equation is

$$\frac{\partial g}{\partial q}=-2y(z-qy)-x^2=0.$$

Eliminating q between these equations, we have

$$x^2(x^2+4yz)=0.$$

The equation $x=0$ does not provide an integral. The equation $x^2+4yz=0$, as before, does provide an integral which accordingly is the singular integral.

Ex. 2. Obtain all the integrals of the system

$$\left.\begin{aligned} 2z&=x_1p_1+x_2p_2+x_3p_3 \\ 0&=p_1p_2x_3-2x_1x_2p_3 \end{aligned}\right\};$$

and shew that the singular integral can be deduced from the differential equations.

CHAPTER VI.

THE METHOD OF CHARACTERISTICS FOR EQUATIONS IN TWO INDEPENDENT VARIABLES: GEOMETRICAL RELATIONS OF THE VARIOUS INTEGRALS.

FOR the material of this chapter and the next, which are limited to equations in two independent variables, reference should be made in the first place to Cauchy's discussion as given in the section of his *Exercices d'analyse et de physique mathématique* (quoted in § 84), and to the exposition of Cauchy's method given by Mansion, in his treatise already quoted (p. 100).

A considerable portion of the chapter is devoted to the geometrical interpretation of the analysis, particularly to the interpretation of results by the geometry of ordinary space. For this portion, reference should be made to Darboux's memoir, *Mém. des Sav. Étrang.*, t. XXVII (1880), dealing with the singular solutions of partial equations of the first order; ample use has been made of the memoir. Reference may also be made to Monge's treatise, quoted in § 97; to Goursat's treatise, (quoted on p. 55), particularly chapter IX which is based upon Darboux's memoir; and especially to a memoir by Lie, *Math. Ann.*, t. V (1872), pp. 145—256.

84. It has been seen that, in the method of Charpit as applied to any equation of the first order in two independent variables, and in the method of Jacobi for any equation in n independent variables, the first step towards the solution of the equation consists in the construction of an integral of a simultaneous system of ordinary equations.

As introduced by these methods, the system of ordinary equations is subsidiary to the integration of a homogeneous linear partial equation of the first order; there are, however, other ways in which they can arise. Two of these will now be expounded; in presentation, and in significance, they seem distinct from one another, but they will be found to be fundamentally the same. Partly for the sake of simplicity, and partly because of the associated

geometry, we shall begin with equations that involve only two independent variables; the discussion of equations involving n independent variables can be made briefer, after the explanations in the simplest case.

The method adopted* by Cauchy for the construction of the equations was originally propounded† by Ampère; it is based upon a change of independent variables, chosen so as to simplify relations. Let these be changed from x and y to x and u; then y, z, p, q can be regarded as functions of x and u, and, whatever be the differential equation, we have

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x},$$

$$\frac{\partial z}{\partial u} = \quad q\frac{\partial y}{\partial u}.$$

As u and x are independent variables, it follows that

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right),$$

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\right);$$

substituting in the former relation and using the latter, we have

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial q}{\partial u}\frac{\partial y}{\partial x}.$$

When the proper values of y, z, p, q as functions of x and u are substituted in a differential equation

$$f(x, y, z, p, q) = 0,$$

it must become an identity; hence, writing

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial f}{\partial z}, \ \frac{\partial f}{\partial p}, \ \frac{\partial f}{\partial q} = X, Y, Z, P, Q,$$

respectively, we have

$$X + Y\frac{\partial y}{\partial x} + Z\frac{\partial z}{\partial x} + P\frac{\partial p}{\partial x} + Q\frac{\partial q}{\partial x} = 0,$$

$$Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u} = 0.$$

* *Exercices d'analyse et de physique mathématique*, t. II, pp. 238—272. This is dated 1841; but the memoir, which contained a first exposition of Cauchy's theory, was published in 1819.

† In his memoir of 1814, to which references will be given subsequently.

Inserting the above values of $\frac{\partial z}{\partial u}$ and $\frac{\partial p}{\partial u}$ in the second of these equations, it becomes

$$\frac{\partial y}{\partial u}\left(Y + qZ + P\frac{\partial q}{\partial x}\right) + \frac{\partial q}{\partial u}\left(Q - P\frac{\partial y}{\partial x}\right) = 0.$$

As the variable u is thus far at our disposal let it be chosen, if possible, so that

$$Q - P\frac{\partial y}{\partial x} = 0;$$

then, as y cannot be a function of x alone so that $\frac{\partial y}{\partial u}$ cannot be zero, and as $\frac{\partial q}{\partial u}$ cannot be a permanent infinity, we have

$$Y + qZ + P\frac{\partial q}{\partial x} = 0.$$

Inserting in the first of the two derived equations the above value of $\frac{\partial z}{\partial x}$, and the values of $\frac{\partial y}{\partial x}$ and $\frac{\partial q}{\partial x}$ given by the equations just obtained, we find

$$X + pZ + P\frac{\partial p}{\partial x} = 0.$$

Hence there are four equations involving derivatives of y, z, p, q with regard to x alone: they can be taken in the form

$$P\frac{\partial y}{\partial x} = Q, \qquad P\frac{\partial p}{\partial x} = -(X + pZ),$$

$$P\frac{\partial z}{\partial x} = pP + qQ, \qquad P\frac{\partial q}{\partial x} = -(Y + qZ).$$

These do not contain derivatives with regard to u nor do they contain u itself; hence, if we take these equations in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ} = \frac{dp}{-(X + pZ)} = \frac{dq}{-(Y + qZ)},$$

and obtain their integrals, the arbitrary elements that arise in those integrals will be functions of u in the most general case. So far as these equations are concerned, the arbitrary elements may be

made arbitrary functions of u: but other equations must be satisfied, viz.

$$f(x, y, z, p, q) = 0, \quad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u},$$

and these may impose limitations*.

It will be noted that the above set of ordinary equations is the same as the set in Charpit's method (§ 68).

85. To satisfy the requirements, we proceed from the known theory of ordinary equations. If P is not an identical zero for our problem, that is, if it does not vanish for all values of five arguments x, y, z, p, q, tied by a condition

$$f(x, y, z, p, q) = 0,$$

then values x_0, y_0, z_0, p_0, q_0 can be assigned to the arguments such that P does not vanish, provided

$$f(x_0, y_0, z_0, p_0, q_0) = 0.$$

Further, suppose that P, Q, X, Y, Z are regular functions of x, y, z, p, q in the vicinity of these values. Then it is known, by Cauchy's theorem† on the integrals of ordinary equations, that a unique system of integrals exists; they give y, z, p, q as regular functions of x and these acquire values y_0, z_0, p_0, q_0 when $x = x_0$.

If P is an identical zero in the sense explained, then we consider Q. If Q is not similarly an identical zero, we proceed as above making y the independent variable for the ordinary system: and with the same hypotheses, we obtain a set of integrals.

If Q is an identical zero in the sense explained, then we should proceed to consider $X + pZ$, making p the independent variable; and if that were to fail, we should consider $Y + qZ$, making q the independent variable. We should obtain a set of integrals save only in the case, when all the equations

$$P = 0, \quad Q = 0, \quad X + pZ = 0, \quad Y + qZ = 0$$

are satisfied for values of x, y, z, p, q, which obey the relation

$$f(x, y, z, p, q) = 0.$$

* The equation

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial y}{\partial x}\frac{\partial q}{\partial u}$$

imposes no additional limitation: it is satisfied in virtue of the equations retained, being a mere deduction from them.

† See vol. II of this work, ch. II.

This case is obviously exceptional: it provides what has already been recognised as the singular integral, and consequently it will be set aside from the present investigation. It may or may not occur: when it does occur, its relation to other integrals will be considered subsequently.

If then, setting this case aside, we take y_0, z_0, p_0, q_0 as functions of u satisfying the relation

$$f(x_0, y_0, z_0, p_0, q_0) = 0,$$

we have a set of integrals of the system of ordinary equations; and these are further to satisfy the equations

$$f(x, y, z, p, q) = 0, \qquad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u}.$$

Let

$$E = \frac{\partial z}{\partial u} - q\frac{\partial y}{\partial u},$$

so that E is to be zero: thus

$$\frac{\partial E}{\partial x} = \frac{\partial^2 z}{\partial x \partial u} - q\frac{\partial^2 y}{\partial x \partial u} - \frac{\partial q}{\partial x}\frac{\partial y}{\partial u}.$$

But the quantities already obtained satisfy the equation

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x},$$

and therefore

$$\frac{\partial^2 z}{\partial x \partial u} = \frac{\partial p}{\partial u} + q\frac{\partial^2 y}{\partial x \partial u} + \frac{\partial q}{\partial u}\frac{\partial y}{\partial x}.$$

Hence

$$\frac{\partial E}{\partial x} = \frac{\partial p}{\partial u} + \frac{\partial q}{\partial u}\frac{\partial y}{\partial x} - \frac{\partial y}{\partial u}\frac{\partial q}{\partial x}.$$

Now

$$\frac{\partial y}{\partial x} = \frac{Q}{P}, \qquad \frac{\partial q}{\partial x} = -\frac{Y + qZ}{P},$$

so that

$$\frac{\partial E}{\partial x} = \frac{1}{P}\left(Y\frac{\partial y}{\partial u} + qZ\frac{\partial y}{\partial u} + P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u}\right).$$

Our quantities are required to satisfy $f(x, y, z, p, q) = 0$, and therefore

$$Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u} = 0;$$

hence

$$\frac{\partial E}{\partial x} = \frac{1}{P}Z\left(q\frac{\partial y}{\partial u} - \frac{\partial z}{\partial u}\right)$$

$$= -E\frac{Z}{P}.$$

Now after the earlier explanations, we may assume that P is not identically zero, that it does not vanish for the values x_0, y_0, z_0, p_0, q_0 of the variables, and that Z and P are regular functions of the variables in the vicinity of those values: hence, if

$$I = \int_{x_0}^{x} \frac{Z}{P}\, dx,$$

the quantity I is regular in the vicinity of the arguments, and it vanishes when $x = x_0$. If we denote by E_0 the value of E for the values x_0, y_0, z_0, p_0, q_0 of the variables, then

$$E = E_0 e^{-I}.$$

Now E is to vanish, and e^{-I} does not vanish: hence we must have

$$E_0 = 0,$$

that is, we must have

$$\frac{\partial z_0}{\partial u} - q_0 \frac{\partial y_0}{\partial u} = 0.$$

86. This equation can be satisfied in two ways. First, suppose that y_0 is not independent of u: as u has not hitherto been made precise, we take

$$y_0 = u.$$

Let $z_0 = \phi(u)$, where ϕ is an arbitrary function of u; the equation will be satisfied if

$$q_0 = \phi'(u);$$

and the value of p_0 is then determined by

$$f(x_0, y_0, z_0, p_0, q_0) = 0.$$

With these values, the equation $E = 0$ is satisfied.

In the second place, suppose that y_0 is independent of u: as it is a value of y when $x = x_0$, we take it to be an arbitrary constant. The equation is then satisfied if $\frac{\partial z_0}{\partial u}$ vanishes, that is, if z_0 is similarly an arbitrary constant. Then q_0 can be any function of u; and p_0 is given as a function of u satisfying the relation

$$f(x_0, y_0, z_0, p_0, q_0) = 0.$$

With these values, the equation $E = 0$ is satisfied.

Now the integrals of the system of ordinary equations are

$$\begin{aligned} y &= y(x, x_0, z_0, p_0, q_0), \\ z &= z(x, x_0, z_0, p_0, q_0), \\ p &= p(x, x_0, z_0, p_0, q_0), \\ q &= q(x, x_0, z_0, p_0, q_0). \end{aligned}$$

In the former of the above cases, when u is eliminated between the first two equations, a relation is obtained involving x, y, z and an arbitrary function; if it gives an integral of the original equation, the integral so given is *general*. In the latter of the cases, when p_0 and q_0 are eliminated between the first two equations and $f(x_0, y_0, z_0, p_0, q_0) = 0$, a relation is obtained involving x, y, z and two arbitrary constants: if it gives an integral of the original equation, the integral so given is *complete*.

It therefore remains to prove that the relations thus obtained actually satisfy the equation

$$f(x, y, z, p, q) = 0.$$

Now y and z, obtained above as functions of x and u, satisfy the equations

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x}, \quad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u};$$

hence, when u is eliminated so as to give a single relation between x, y, z, the quantities p and q are the derivatives of z with regard to x and y respectively. Further, the values of x, y, z, p, q are such that the equation

$$X + Y\frac{\partial y}{\partial x} + Z\frac{\partial z}{\partial x} + P\frac{\partial p}{\partial x} + Q\frac{\partial q}{\partial x} = 0$$

is satisfied identically, that is,

$$\frac{\partial f}{\partial x} = 0.$$

Also, as $E = 0$, we have

$$q\frac{\partial y}{\partial u} - \frac{\partial z}{\partial u} = 0:$$

and as $\frac{\partial E}{\partial x} = 0$, we have seen that

$$Y\frac{\partial y}{\partial u} + qZ\frac{\partial y}{\partial u} + P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u} = 0,$$

that is,

$$Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u} = 0,$$

or

$$\frac{\partial f}{\partial u} = 0.$$

Hence f is constant: its value when $x = x_0$ is zero: hence its value is zero always, that is, the quantities so obtained satisfy the equation

We thus obtain the *general integral* and the *complete integral* in the respective cases. In the course of the proof, it was seen how the *singular integral* could arise exceptionally: when it arose, it was recognised: and the analysis proceeded with the alternatives. Moreover, it was assumed that X, Y, Z, P, Q are regular functions; if this is not true, the results are not necessarily applicable to the equation and then integrals may arise which are *special integrals*, though these are not the only special integrals that may arise*.

Darboux's modification of Cauchy's method.

87. The preceding exposition is substantially due to Cauchy: a modification in the treatment of the ordinary equations has been introduced† by Darboux, which has the further advantage of permitting an easier discussion of singularities. The equations are taken in the form

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{pP+qQ}=\frac{dp}{-(X+pZ)}=\frac{dq}{-(Y+qZ)}=dt,$$

and they are regarded as determining the five variables x, y, z, p, q in terms of t, the arbitrary quantities that occur being made functions of a variable u, as before. The establishment of the results is simpler than in the Cauchy treatment.

We assume that, when‡ $t=0$, the five variables acquire values x_0, y_0, z_0, p_0, q_0, subject to the relation

$$f(x_0, y_0, z_0, p_0, q_0)=0,$$

and that, in the vicinity of these initial values, the functions P, Q, $pP+qQ$, $X+pZ$, $Y+qZ$ are regular. Then, by the theory of systems of ordinary equations, a unique set of integrals exists, being regular functions of t and acquiring the assigned initial values when $t=0$: let them be

$$\begin{aligned}
x &= x\,(t, x_0, y_0, z_0, p_0, q_0),\\
y &= y\,(t, x_0, y_0, z_0, p_0, q_0),\\
z &= z\,(t, x_0, y_0, z_0, p_0, q_0),\\
p &= p\,(t, x_0, y_0, z_0, p_0, q_0),\\
q &= q\,(t, x_0, y_0, z_0, p_0, q_0).
\end{aligned}$$

* See, for instance, § 34.

† "Mémoire sur les solutions singulières...premier ordre," *Mém. de l'Inst. de France*, t. XXVII (1880), § 24.

‡ No generality is gained by taking t_0 as an initial value for t: the only difference is that $t-t_0$ comes in place of t.

Moreover, we have

$$\frac{df}{dt} = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} + P\frac{dp}{dt} + Q\frac{dq}{dt}$$
$$= 0,$$

on substitution: thus

$$\begin{aligned} f(x, y, z, p, q) &= \text{a quantity independent of } t \\ &= \text{its value when } t = 0 \\ &= f(x_0, y_0, z_0, p_0, q_0) \\ &= 0, \end{aligned}$$

so that the above values are connected by this relation.

If this is to be equivalent to the original differential equation, then p and q must be equal to the derivatives of z with regard to x and y: the requirement is met as follows. Let the quantities x_0, y_0, z_0, p_0, q_0 be functions of u, a new parametric variable, so that x, y, z, p, q are now functions of t and u. So far as variations with regard to t are concerned, the system of equations gives

$$\frac{\partial z}{\partial t} = p\frac{\partial x}{\partial t} + q\frac{\partial y}{\partial t}.$$

Let

$$E = \frac{\partial z}{\partial u} - p\frac{\partial x}{\partial u} - q\frac{\partial y}{\partial u};$$

then

$$\frac{\partial^2 z}{\partial t \partial u} = p\frac{\partial^2 x}{\partial t \partial u} + q\frac{\partial^2 y}{\partial t \partial u} + \frac{\partial p}{\partial u}\frac{\partial x}{\partial t} + \frac{\partial q}{\partial u}\frac{\partial y}{\partial t},$$

and

$$\begin{aligned} \frac{\partial E}{\partial t} &= \frac{\partial^2 z}{\partial t \partial u} - p\frac{\partial^2 x}{\partial t \partial u} - q\frac{\partial^2 y}{\partial t \partial u} - \frac{\partial p}{\partial t}\frac{\partial x}{\partial u} - \frac{\partial q}{\partial t}\frac{\partial y}{\partial u} \\ &= \frac{\partial p}{\partial u}\frac{\partial x}{\partial t} + \frac{\partial q}{\partial u}\frac{\partial y}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial x}{\partial u} - \frac{\partial q}{\partial t}\frac{\partial y}{\partial u} \\ &= P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u} + (X + pZ)\frac{\partial x}{\partial u} + (Y + qZ)\frac{\partial y}{\partial u}, \end{aligned}$$

on substituting from the equations. But

$$X\frac{\partial x}{\partial u} + Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + P\frac{\partial p}{\partial u} + Q\frac{\partial q}{\partial u} = 0,$$

because the values of x, y, z, p, q satisfy the equation

$$f(x, y, z, p, q) = 0;$$

hence

$$\frac{\partial E}{\partial t} = -Z\left(\frac{\partial z}{\partial u} - p\frac{\partial x}{\partial u} - q\frac{\partial y}{\partial u}\right)$$
$$= -ZE,$$

so that

$$E = E_0 e^{-\int_0^t Z dt},$$

where E_0 denotes the value of E when $t=0$. Now Z is a regular function of t in the vicinity of $t=0$, and therefore $\int_0^t Z dt$ is finite. Hence it is necessary and sufficient that the equation

$$E_0 = 0$$

should be satisfied in order that E may vanish, that is,

$$\frac{\partial z_0}{\partial u} = p_0 \frac{\partial x_0}{\partial u} + q_0 \frac{\partial y_0}{\partial u}.$$

88. This equation may be satisfied in three distinct ways. In the first of these ways, both x_0 and y_0 involve u; in that case, let

$$x_0 = f(u), \quad y_0 = g(u), \quad z_0 = h(u),$$

so that the relation

$$h'(u) = p_0 f'(u) + q_0 g'(u)$$

becomes an equation which, with $f(x_0, y_0, z_0, p_0, q_0) = 0$, determines p_0 and q_0. In the second of the ways, only one of the variables x_0 and y_0 involves u, say y_0: in that case, let $x_0 = \alpha$, $y_0 = u$, $z_0 = g(u)$; then

$$g'(u) = q_0,$$

and the equation $f(x_0, y_0, z_0, p_0, q_0) = 0$ determines p_0. In the third of the ways, neither of the variables x_0 and y_0 involves u: in that case, let $x_0 = \alpha$, $y_0 = \beta$: then z_0 does not involve u, and we may write $z_0 = \gamma$, where α, β, γ are constants, and $f(\alpha, \beta, \gamma, p_0, q_0) = 0$.

In all these cases, we have $E_0 = 0$ and therefore $E = 0$, that is,

$$\frac{\partial z}{\partial u} = p\frac{\partial x}{\partial u} + q\frac{\partial y}{\partial u};$$

and we had

$$\frac{\partial z}{\partial t} = p\frac{\partial x}{\partial t} + q\frac{\partial y}{\partial t}.$$

These relations shew that, when z is expressed in terms of x and y consistently with the relations obtained, its derivatives with regard

to x and y are p and q. Moreover, we can prove, exactly as in § 86, that the equation

$$f(x, y, z, p, q) = 0$$

is satisfied: thus we have an integral of the original equation.

Consider the three cases in turn: for all of them, we have to make the respective substitutions in

$$x = x(t, x_0, y_0, z_0, p_0, q_0),$$
$$y = y(t, x_0, y_0, z_0, p_0, q_0),$$
$$z = z(t, x_0, y_0, z_0, p_0, q_0).$$

In the first case, we eliminate t and u between the three equations: there results a single relation between x, y, z. The assigned initial conditions are such that $x = x_0$, $y = y_0$, $z = z_0$, functions of a variable parameter u: or getting rid of u by elimination, we can say that, when some relation

$$\phi(x, y) = 0$$

is satisfied, then z becomes some function of x and y. No limitation except regularity has been imposed upon the functions: thus z becomes an assigned arbitrary function of x and y, when these are connected by any assigned relation $\phi(x, y) = 0$. The integral is *general*.

In the second case, for the initial conditions, we have $x = x_0 = a$, $y = y_0 = u$, $z = z_0 = g(u)$, that is, when $x = a$, $z = g(y)$. Here g can be an arbitrary function. We have a special form of the first case, obtained by writing $\phi(x, y) = x - a$. The integral is *general*.

In the third case, we have to eliminate three quantities t, p_0, q_0 between the equations

$$x = x(t, \alpha, \beta, \gamma, p_0, q_0),$$
$$y = y(t, \alpha, \beta, \gamma, p_0, q_0),$$
$$z = z(t, \alpha, \beta, \gamma, p_0, q_0),$$
$$0 = f(\alpha, \beta, \gamma, p_0, q_0).$$

The resulting equation involves α, β, γ arising as values of x, y, z. One of these may be taken as an initial value: the other two may be arbitrary. When they are quite arbitrary, the integral is *complete*.

As regards the derivation of the result, it is to be noted that the values of p_0 and q_0 are given by

$$f(x_0, y_0, z_0, p_0, q_0) = 0,$$

$$p_0 \frac{\partial x_0}{\partial u} + q_0 \frac{\partial y_0}{\partial u} = \frac{\partial z_0}{\partial u};$$

if f be a regular function of its arguments, the two equations can be resolved for p_0 and q_0, provided the magnitude

$$\frac{\partial f}{\partial p_0} \frac{\partial y_0}{\partial u} - \frac{\partial f}{\partial q_0} \frac{\partial x_0}{\partial u}$$

does not vanish. We may assume this to be the case for the first two of the three alternatives, because we have excluded the hypothesis that P_0 and Q_0 are zero: it does not arise for the third alternative.

Again, the equations for x and y may be resolved for t and u in the vicinity of $t = 0$ and $u = u_0$, provided

$$\frac{\partial x}{\partial t} \frac{\partial y}{\partial u} - \frac{\partial y}{\partial t} \frac{\partial x}{\partial u}$$

does not vanish there, that is, provided

$$P_0 \frac{\partial y_0}{\partial u} - Q_0 \frac{\partial x_0}{\partial u}$$

does not vanish: this is the above magnitude, assumed not to vanish. The values of t and u so obtained are substituted in the expression for z, which becomes the integral.

It might of course happen that $P_0 = 0$, $Q_0 = 0$, without $X_0 + p_0 Z_0$ and $Y_0 + q_0 Z_0$ vanishing: the possibility is discussed later (Chapter VII, § 109) and will be seen to provide a singularity.

89. The preceding analysis and argument are valid in establishing this result, save in one set of circumstances controlling the hypotheses adopted. It has been assumed that P, Q, $X + pZ$, $Y + qZ$ are regular functions of their arguments in the vicinity of the values x_0, y_0, z_0, p_0, q_0; but if, for all sets of values satisfying the necessary relation

$$f(x_0, y_0, z_0, p_0, q_0) = 0,$$

it should happen that

$$P = 0, \quad Q = 0, \quad X + pZ = 0, \quad Y + qZ = 0,$$

then the only regular integrals of the system of ordinary equations are

$$x = x_0, \quad y = y_0, \quad z = z_0, \quad p = p_0, \quad q = q_0,$$

and the results can no longer be established.

The case, as before, is usually that of the *singular integral*: it is put on one side, for it admits of other treatment as follows. The equations

$$f = 0, \quad P = 0, \quad Q = 0, \quad X + pZ = 0, \quad Y + qZ = 0$$

coexist. If it is possible to eliminate p and q between them, there will result an equation between x, y, z, which is the singular integral. If it is not possible to eliminate p and q, then, f being supposed irreducible, they will serve to determine p and q: the values of p and q are substituted in

$$dz = p\,dx + q\,dy,$$

and quadrature leads to an integral involving one arbitrary constant: it may (§ 78) be regarded as a specialised or particular form of the complete integral.

Further, if P, Q, $X + pZ$, $Y + qZ$ are not regular functions of their arguments, the inference as regards the ordinary equations cannot be made and so the result cannot be established. Special integrals can thus arise: but it is not the only source of such integrals.

90. One other exceptional case requires special mention, viz. that in which

$$pP + qQ = 0,$$

though neither P nor Q vanishes: it is a case which occurs when f is homogeneous in p and q of any degree. An integral of the ordinary equations is then

$$\begin{aligned} z &= \text{quantity independent of } t \\ &= z_0; \end{aligned}$$

and if it should happen that we are dealing with conditions leading to a complete integral, it is clear that the equation $z = z_0$ cannot be used for purposes of elimination.

As in corresponding difficulties (§§ 58, 59), we use a Legendrian transformation, introducing a new variable z' by the relation

$$z' = z - px,$$

and assigning other associated variables in the form

$$y' = y, \quad q' = q, \quad x' = -p, \quad p' = x.$$

The quantity $p'P' + q'Q'$ which occurs in the modified system is obtained from

$$x(X + pZ) + qQ,$$

by making the above substitutions; this does not vanish in general, and so the process can be applied to the modified equation: the integral of the original equation can be deduced as before.

Note 1. Both in Cauchy's method and in Darboux's modification, an integral has been obtained which has been declared a general integral. Its expression certainly contains an arbitrary function in the least restricted case; but this property is not, of itself, sufficient to secure the character in the customary form. A general integral is given by the elimination of a between the equations

$$g\{x, y, z, a, f(a)\} = 0,$$

$$\frac{\partial g}{\partial a} + \frac{\partial g}{\partial f} f'(a) = 0,$$

that is, one of the equations is the derivative of the other with regard to the parameter which is to be eliminated. Of course, when the elimination can be actually achieved, this relation between the two equations disappears; usually, however, the expression of the general integral must be left in this form.

In particular instances, the result can be verified by bringing the last two equations into an equivalent form which exhibits the relation characteristic of the general integral.

Note 2. The general integral that has thus been obtained is the integral specified in Cauchy's existence-theorem.

Taking the simpler form, we have an integral such that $y = y_0$ and $z = z_0 = \phi(y_0)$, when $x = x_0$, where ϕ can be any function subject to the conditions involved in the existence-theorem for ordinary differential equations: in other words, the integral is such that z acquires a value $\phi(y)$, when $x = x_0$. The conditions have relation to the regularity of the coefficients in the ordinary equations and of the integrals of those equations; they are the same as those set out in Cauchy's existence-theorem, and so need not be repeated here.

Of course, it must not be assumed that, if all the conditions required in the proof of the theorem are not satisfied, the integral does not exist: in such a case, we merely are not in a position to affirm its existence.

Ex. 1. An example is given by Cauchy*: he discusses the equation

$$pq - xy = 0.$$

The initial values must be such that

$$p_0 q_0 - x_0 y_0 = 0.$$

The subsidiary system of ordinary equations is

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq} = \frac{dp}{y} = \frac{dq}{x};$$

and a unique system of regular integrals satisfying all the conditions is given by

$$\begin{aligned} y &= y_0 \left\{1 + \tfrac{1}{2}\frac{x^2 - x_0^2}{q_0^2} - \tfrac{1}{8}\frac{(x^2 - x_0^2)^2}{q_0^2} - \dots\right\} \\ &= y_0 \left\{1 + \frac{x^2 - x_0^2}{q_0^2}\right\}^{\frac{1}{2}}, \\ z &= z_0 + \frac{y_0}{q_0}(x^2 - x_0^2), \\ p &= x\frac{y_0}{q_0}, \\ q &= q_0 \left\{1 + \frac{x^2 - x_0^2}{q_0^2}\right\}^{\frac{1}{2}}, \end{aligned}$$

the same branch of the irrational quantity being taken in q as in y. We have

$$(z - z_0) q_0 = y_0 (x^2 - x_0^2),$$
$$(z - z_0)^2 = (y^2 - y_0^2)(x^2 - x_0^2);$$

when we take, in accordance with the preceding theory,

$$y_0 = u, \quad z_0 = \phi(u), \quad q_0 = \phi'(u_0),$$

we have a general integral given by the two equations: and when, also in accordance with the preceding theory, we take

$$y_0 = a, \quad z_0 = \beta,$$

where a and β are arbitrary constants, the second equation becomes

$$(z - \beta)^2 = (y - a)^2 (x - x_0)^2,$$

giving rise to a complete integral. And there is no singular integral.

If the subsidiary equations are treated by Darboux's method, they can be taken in the form†

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq} = \frac{dp}{y} = \frac{dq}{x} = \frac{dt}{2pq\,(=2xy)}.$$

* *L.c.*, t. II, p. 249.

† The new variable t is at our disposal, and any modification tending to simplify the equations may be adopted; accordingly, $\dfrac{dt}{2pq}$ is chosen instead of dt, as the common value of the fractions in the subsidiary equations.

Integrals of these, which give x_0, y_0, z_0, p_0, q_0 as the values of x, y, z, p, q, when $t=0$, are

$$x^2 = x_0^2 + t\,\frac{q_0}{y_0},$$

$$y^2 = y_0^2 + t\,\frac{y_0}{q_0},$$

$$z = z_0 + t,$$

$$p^2 = p_0^2 + t\,\frac{y_0}{q_0},$$

$$q^2 = q_0^2 + t\,\frac{q_0}{y_0}.$$

The first three equations give

$$(z - z_0)\,q_0 = y_0\,(x^2 - x_0^2),$$
$$(z - z_0)^2 = (y^2 - y_0^2)\,(x^2 - x_0^2):$$

the assumptions

$$y_0 = u, \quad z_0 = \phi(u), \quad q_0 = \phi'(u),$$

lead to the general integral as before, on the elimination of u: the assumptions

$$y_0 = a, \quad z_0 = \beta,$$

lead to the complete integral as contained in the second of the two equations.

Ex. 2. Discuss similarly the equation

$$2xz - px^2 + qxy + q^2x = 0,$$

obtaining its complete integral and its general integral. Are there any limitations upon the initial conditions caused by the form of the equation?

(Mansion.)

Ex. 3. Integrate the equation

$$pqy - pz + aq = 0,$$

where a is a constant. (Mansion.)

Ex. 4. As a slight variation in the details of working in any particular question when a Cauchy integral is required, we obtain the integral of

$$xzp + yzq = xy,$$

which is such that z becomes $\phi(y)$, when $x = x_0$.

The ordinary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{pxz + qyz} = \ldots;$$

both y and z can be expressed in terms of x, on using the original equation, by integrals

$$\frac{y}{x} = \text{constant}, \quad z^2 - xy = \text{constant}.$$

We take $y = u$ and $z = \phi(u)$, when $x = x_0$: thus

$$\frac{y}{x} = \frac{u}{x_0}, \quad z^2 - xy = \{\phi(u)\}^2 - x_0 u.$$

Eliminating u, the relation is

$$z^2 - xy = \left\{\phi\left(\frac{x_0}{x}y\right)\right\}^2 - \frac{x_0{}^2}{x}y,$$

which gives the integral in question.

In this case, there is no limitation upon the form of the function ϕ.

Geometry of the Integrals.

91. It is found convenient, particularly for equations involving two independent variables, to associate geometrical considerations with the analysis. A slight use of this association has already been made (§ 21), by way of interpreting an integral of an equation: we shall now proceed to greater detail, particularly in order to indicate the significance of the various equations, to illustrate the relations of different integrals to one another, and to discuss some at least of the singularities which have received no more than passing mention in the preceding sections. The differential equation, when there are two independent variables, is taken to be

$$f(x, y, z, p, q) = 0;$$

its complete integral is taken to be

$$\phi(x, y, z, a, b) = 0,$$

where a and b are arbitrary constants; and other integrals can be deduced by methods already explained.

The equation $\phi = 0$ is the equation of a double family of surfaces. All these surfaces are such that $f = 0$ is satisfied; consequently, they are said to satisfy the differential equation. Also, $f = 0$ is given by the elimination of a and b between the equations

$$\phi = 0, \quad \frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0, \quad \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0.$$

Through any point of space x, y, z, there passes a simple infinitude of the surfaces. The tangent planes at the point can be represented by the equation

$$z' - z = p(x' - x) + q(y' - y),$$

where x', y', z' are current coordinates; and the normals to the surfaces are given by the equations

$$\frac{x' - x}{p} = \frac{y' - y}{q} = \frac{z' - z}{-1}.$$

All these normals lie on the cone

$$f\left(x, y, z, -\frac{x'-x}{z'-z}, -\frac{y'-y}{z'-z}\right) = 0\,;$$

this cone of normals will be denoted by N. The reciprocal of the cone of normals is the envelope of all the tangent planes at the point to the surfaces satisfying the differential equation: this cone, the envelope of the tangent planes, will be denoted by T.

The equation of the cone T can be simply constructed as follows. The tangent plane is

$$z' - z = p\,(x' - x) + q\,(y' - y)\,;$$

we require the envelope of this plane, subject to the condition $f(x, y, z, p, q) = 0$, and therefore we have

$$0 = (x' - x)\,dp + (y' - y)\,dq,$$

$$0 = \frac{\partial f}{\partial p}\,dp + \frac{\partial f}{\partial q}\,dq,$$

so that the equation of the cone T is given by the elimination of p and q between the equations

$$\left.\begin{array}{c} z' - z = p\,(x' - x) + q\,(y' - y) \\ \dfrac{x'-x}{\dfrac{\partial f}{\partial p}} = \dfrac{y'-y}{\dfrac{\partial f}{\partial q}} \\ f(x, y, z, p, q) = 0 \end{array}\right\}.$$

The result will obviously be of the form

$$F\left(x, y, z, \frac{z'-z}{x'-x}, \frac{y'-y}{x'-x}\right) = 0.$$

Moreover, the equations of the generator of T, which lies in the tangent plane in question, are

$$\frac{x'-x}{\dfrac{\partial f}{\partial p}} = \frac{y'-y}{\dfrac{\partial f}{\partial q}} = \frac{z'-z}{p\dfrac{\partial f}{\partial p} + q\dfrac{\partial f}{\partial q}}\,;$$

this generator is the line along which the tangent plane touches the cone T, and it obviously is perpendicular to the corresponding normal on the reciprocal cone N.

Again, take any plane

$$z' = \alpha x' + \beta y' + \gamma$$

in space: if this plane touches an integral surface

$$\phi\,(x, y, z, a, b) = 0,$$

the point of contact is given by

$$\phi = 0, \quad \frac{\partial \phi}{\partial x} + \alpha \frac{\partial \phi}{\partial z} = 0, \quad \frac{\partial \phi}{\partial y} + \beta \frac{\partial \phi}{\partial z} = 0.$$

The coordinates of the point of contact satisfy the equation

$$f(x, y, z, \alpha, \beta) = 0,$$

whatever a and b may be: and, of course, as the point lies in the plane, we have

$$z = \alpha x + \beta y + \gamma.$$

But the two equations

$$z = \alpha x + \beta y + \gamma, \quad f(x, y, z, \alpha, \beta) = 0,$$

determine a plane curve: owing to its source, it is the locus of points of contact of integral surfaces with the assumed plane. This curve lying in the plane will be denoted by C.

Moreover, as this plane touches a surface at the point x, y, z, it touches the cone T which is associated with the point; we may therefore regard the curve C as the locus of those points in the plane at which the plane touches the associated cones T. And, conversely, a cone T associated with a point is the envelope of those planes whose curves C pass through the point.

Consequently, the integral surfaces which satisfy the differential equation can be regarded in two ways. On the one hand, all those which pass through a given point have their tangent planes enveloped by a cone T: on the other hand, all those which touch a given plane have their points of contact with the plane lying upon a curve C in the plane. Each of these is obviously deducible from the other by reciprocal polars.

Ex. 1. Shew that if the curve C is a degenerate curve, composed of a number of straight lines, and if the (Legendre) transformation

$$\zeta = px + qy - z$$

is applied to the partial differential equation, the transformed differential equation is linear.

(Goursat.)

Ex. 2. Shew that, if integral surfaces be given by an equation

$$x^2 + y^2 + z^2 = 2ax + 2by + 2cz,$$

where

$$(a\alpha + b\beta + c\gamma - 1)^2 = (a^2 + b^2 + c^2)(\alpha^2 + \beta^2 + \gamma^2),$$

α, β, γ being given constants, and a, b, c arbitrary constants subject to this condition, the curves C are circles.

(Goursat.)

CHARACTERISTICS: THEIR PROPERTIES.

92. Now consider a general integral as deduced from a complete integral; it is given by the equations

$$\phi = \phi\{x, y, z, a, g(a)\} = 0,$$

$$\frac{d\phi}{da} = \frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial g} g'(a) = 0,$$

where $g(a)$ is an arbitrary function: the surface, represented by the general integral, is obtained by eliminating a between the two equations. The equation $\phi = 0$ is one of the surfaces included in the complete integral; the equation $\frac{d\phi}{da} = 0$ then determines a curve on the surface $\phi = 0$, being in fact the intersection of $\phi = 0$ with the surface that arises from a consecutive value of a; the curve thus given, for any value of a, is called a *characteristic* of the surface $\phi = 0$. The general integral, arising from the elimination of a between $\phi = 0$ and $\frac{d\phi}{da} = 0$, is thus the locus of the characteristics of the surfaces $\phi = 0$, which arise for any one function g and for all values of a.

On any surface represented by a complete integral, there is an infinitude of characteristics: they arise because $g(a)$ is an arbitrary function which can be assigned in an infinitude of ways. Through any ordinary point on such a surface, there passes certainly one characteristic: for, at that point, there are two independent equations to determine a and $g(a)$. Moreover, through any ordinary point there passes only one characteristic in general: because the two equations, $\phi = 0$ and $\frac{d\phi}{da} = 0$, in general give unique values for the ratios $dx : dy : dz$.

At any point on a characteristic of $\phi = 0$, the curve is touched by the tangent plane there: it is also touched there by the tangent plane at that point to the consecutive surface, because it lies on that surface. Hence the tangent line to the characteristic is the intersection of the tangent planes to two consecutive surfaces through the point, that is, it coincides with the generator of the cone T along which the cone touches the tangent plane to $\phi = 0$.

The general integral touches the surface $\phi=0$ along the characteristic. On the surface $\phi=0$, the tangent to the surface is determined by the values of p and q which are given by

$$\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}=0,\quad \frac{\partial\phi}{\partial y}+q\frac{\partial\phi}{\partial z}=0.$$

At that point on the general integral, the value of a is given, in terms of x, y, z, by $\dfrac{d\phi}{da}=0$: when this value of a is substituted in $\phi=0$, the values of p and q determining the tangent plane to the general integral at the point are given by

$$\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}+\frac{d\phi}{da}\frac{da}{dx}=0,\quad \frac{\partial\phi}{\partial y}+q\frac{\partial\phi}{\partial z}+\frac{d\phi}{da}\frac{da}{dy}=0,$$

that is, they are the same as for the tangent plane to $\phi=0$, because $\dfrac{d\phi}{da}=0$. As the tangent planes are the same at every point, the general integral touches the surface $\phi=0$ everywhere along the characteristic.

Further, if two integral surfaces touch at a point, they touch everywhere along the characteristic through the point: for each of them touches the general integral everywhere along the characteristic.

Again, general integrals arising from the assumption of different forms of arbitrary function represent different surfaces, being the loci of the characteristics: it is natural to enquire whether two different surfaces representing general integrals have any characteristics in common. It is obvious that the equations

$$\phi(x, y, z, a, b)=0,\quad b=g(a),\quad \frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial b}g'(a)=0,$$

$$\phi(x, y, z, a, c)=0,\quad c=h(a),\quad \frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial c}h'(a)=0,$$

will represent the same curve on the two different surfaces for each value of a, which satisfies the relations

$$g(a)=h(a),\quad g'(a)=h'(a);$$

and that, if there are m such values of a, the two surfaces will have m characteristics in common.

Now pass to the limit when these m characteristics coincide:

then the common value of a is a root of multiple order $m+1$ of the equation

$$g(a)=h(a),$$

so that we have

$$g(a)=h(a),\quad g'(a)=h'(a),\ \ldots,\ g^{(m)}(a)=h^{(m)}(a).$$

Then along this common characteristic the two surfaces representing the general integral have contact of order m. The derivatives of z, up to and including those of order m, belonging to the surface

$$\phi(x, y, z, a, b)=0,\quad b=g(a),\quad \frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial b}g'(a)=0,$$

involve derivatives of $g(a)$ of order not higher than m; and similarly for the derivatives of z, given by

$$\phi(x, y, z, a, c)=0,\quad c=h(a),\quad \frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial c}h'(a)=0,$$

derivatives of $h(a)$ of order not higher than m occur. Owing to the relations between $g(a)$ and $h(a)$, the derivatives of z at a common point are the same for the two surfaces up to order m; and therefore the two surfaces have contact of order m along the characteristic.

It is an immediate corollary that, if two integral surfaces have contact of order m at any point, they have contact of that order along the whole characteristic through the point.

Ex. 1. These properties are used by Darboux to determine the integral surface or surfaces which pass through any given curve K.

We may assume that K does not lie on the singular integral, if any: otherwise, a single surface is at once given; and other surfaces will occur among those which represent complete integrals or general integrals.

We may also assume that K does not lie on a complete integral: otherwise, a single surface is at once given; and, if K be a characteristic, an infinitude of surfaces satisfies the required condition.

Accordingly, take any point P on the curve K and draw the tangent PQ to the curve at the point P: through the line PQ draw a plane to touch the cone T associated with the point. Then, for the point, we have values of x, y, z, p, q, the two latter being given by the tangent plane: by means of the integrals of the equations of the characteristics, we construct the complete integral having these for initial values. This complete integral touches the curve K at the point P. Through this point P draw the characteristic on the complete integral; it touches the curve K at the point.

Now let P travel along the curve: for each successive position, we have a characteristic; the locus of these characteristics is the integral surface required, and (being such a locus) it is a general integral. It can also be obtained as the envelope of the complete integrals touching K.

If only a single tangent plane can be drawn through PQ to the cone T, the general integral thus obtained constitutes the whole of the surface required. If several tangent planes can be drawn, the general integral consists of a corresponding number of sheets.

Ex. 2. Shew that, if the curve K lies on a surface representing a complete integral though it is not a characteristic on that surface, no other integral surface can be drawn through K.

(Darboux.)

93. It is convenient to consider the developable surface circumscribed to an integral surface along a characteristic: it is called the *characteristic developable.*

To obtain the simplest properties, consider less particularly any two surfaces S and S': let a plane touch them in A and A' respectively; then the developable surface circumscribed to S and S' is the envelope of such planes. A generator of the developable is the intersection of two consecutive planes: hence AA' is a generator, because a plane, consecutive to the supposed plane, touches S in A and S' in A'.

Take a plane P and, in that plane, the curve C which is the locus of its points of contact with integral surfaces. Suppose that the two preceding surfaces S and S' touch the plane, and let their points of contact be Q and Q', being points on C; then QQ' is a generator of the developable circumscribed to S and S'. Now let S and S' be consecutive surfaces: the circumscribed developable becomes the characteristic developable circumscribed along the characteristic which is the ultimate intersection of S and S': the points Q and Q' coincide, and the line QQ' becomes the tangent to C. Hence the generator of the characteristic developable through a point on the characteristic is the tangent to C at that point.

Moreover, the cone T, being the envelope of the tangent planes at the point to all the integral surfaces through the point, touches P at the point: the generator along which it touches P is, as seen above, the tangent to the characteristic line.

Take the characteristic, being the intersection of two consecutive surfaces, and draw the tangent planes along it: the envelope of these is the characteristic developable, and the

generator of this developable at any point is the tangent there to the curve C. Accordingly, at any point on the characteristic line, draw the tangent: construct the plane which touches the cone T along this tangent: in this plane, obtain the curve C: the tangent to C at the point is the generator of the characteristic developable through the point, being the intersection of two consecutive tangent planes to the surface along the characteristic.

94. On the basis of these geometrical properties we can obtain the differential equations of the characteristics: these will, of course, be a set of ordinary equations, because each characteristic is a curve and therefore admits of only one independent variable.

The tangent line to the characteristic at any point x, y, z, is the generator of the cone T lying in the tangent plane; the equations of this generator were obtained (§ 91) in the form

$$\frac{x'-x}{\dfrac{\partial f}{\partial p}} = \frac{y'-y}{\dfrac{\partial f}{\partial q}} = \frac{z'-z}{p\dfrac{\partial f}{\partial p} + q\dfrac{\partial f}{\partial q}},$$

and therefore the direction of the tangent to the characteristic satisfies the equations

$$\frac{dx}{\dfrac{\partial f}{\partial p}} = \frac{dy}{\dfrac{\partial f}{\partial q}} = \frac{dz}{p\dfrac{\partial f}{\partial p} + q\dfrac{\partial f}{\partial q}} = du,$$

say. Next, the curve C in the plane

$$z'-z = p(x'-x) + q(y'-y)$$

satisfies the equation

$$f(x', y', z', p, q) = 0,$$

where p and q define the plane: hence the tangent to C at the point is given by

$$\delta z = p\,\delta x + q\,\delta y,$$

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0;$$

so that

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\delta x + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\delta z = 0.$$

This direction is to be the same as the intersection of the tangent plane at the point with the tangent plane at a consecutive point

on the characteristic: at this consecutive point, the tangent plane is

$$z' - (z + dz) = (p + dp)\{x' - (x + dx)\} + (q + dq)\{y' - (y + dy)\},$$

and therefore, along the intersection, we have

$$-dz = (x' - x)\,dp - p\,dx + (y' - y)\,dq - q\,dy.$$

But, along the characteristic,

$$dz = p\,dx + q\,dy;$$

and therefore

$$(x' - x)\,dp + (y' - y)\,dq = 0,$$

that is,

$$\delta x\,dp + \delta y\,dq = 0.$$

Hence

$$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = dv,$$

say. Also, the variations along the characteristic must be subject to the equation

$$f(x, y, z, p, q) = 0,$$

and therefore

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial p}\,dp + \frac{\partial f}{\partial q}\,dq = 0.$$

Substituting, we find

$$du + dv = 0;$$

consequently, the equations of the characteristic are

$$\frac{dx}{\dfrac{\partial f}{\partial p}} = \frac{dy}{\dfrac{\partial f}{\partial q}} = \frac{dz}{p\dfrac{\partial f}{\partial p} + q\dfrac{\partial f}{\partial q}} = \frac{dp}{-\left(\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}\right)} = \frac{dq}{-\left(\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}\right)}.$$

But these are the equations which occur in Cauchy's method of integration, whether in its original form or in Darboux's presentation: accordingly, it is usually described as the *method of characteristics.*

The integration of these equations gives an integrated form of the equations of the characteristic, coupled with the assignment of initial values satisfying

$$f(x, y, z, p, q) = 0, \quad dz = p\,dx + q\,dy;$$

these initial values are regarded as functions of a variable u: when this variable is eliminated between the equations

$$\left.\begin{array}{l} y = y(x, x_0, y_0, z_0, p_0, q_0) \\ z = z(x, x_0, y_0, z_0, p_0, q_0) \end{array}\right\},$$

the result is a single equation representing a surface which, whether the integral be general or be complete, is a locus of characteristics.

Ex. Denoting any integral of the partial equation

$$f(x, y, z, p, q) = 0$$

by z, the derivatives of x, y, z, with respect to a variable t by x', y', z', and any arbitrary function of t by T, verify that the conditions, necessary and sufficient to secure a stationary (zero) value for the integral

$$\int_{t_0}^{t} T(px' + qy' - z')\, dt,$$

are the equations of the characteristics of the partial differential equation.

Apply this method to deduce the condition that two equations

$$f(x, y, z, p, q) = 0, \quad g(x, y, z, p, q) = 0,$$

may possess a common integral. (Hilbert.)

95. In the preceding investigation of the equations of the characteristics, they have been associated analytically with the original partial differential equation: it is desirable to associate them also with the integral of the differential equation.

The differential equation $f(x, y, z, p, q) = 0$ is the result of eliminating a and b between the equations

$$\left.\begin{array}{ll} \phi = \phi(x, y, z, a, b) & = 0 \\ \phi_1 = \dfrac{\partial \phi}{\partial x} + p\dfrac{\partial \phi}{\partial z} & = 0 \\ \phi_2 = \dfrac{\partial \phi}{\partial y} + q\dfrac{\partial \phi}{\partial z} & = 0 \end{array}\right\};$$

and $f = 0$ is the only equation which results from that elimination. The only independent relations, that connect differential elements dx, dy, dz, dp, dq, are

$$d\phi = 0, \quad d\phi_1 = 0, \quad d\phi_2 = 0;$$

and $df = 0$ is a relation connecting these differential elements: hence quantities λ, μ, ρ, free from differential elements, must exist such that the relation

$$df = \lambda d\phi + \mu d\phi_1 + \rho d\phi_2$$

is satisfied*, and therefore we have

$$f = \lambda\phi + \mu\phi_1 + \rho\phi_2,$$

because f, ϕ, ϕ_1, ϕ_2 vanish together.

As a and b do not occur in f and do occur in ϕ, ϕ_1, ϕ_2, we have

$$\lambda\frac{\partial\phi}{\partial a} + \mu\frac{\partial\phi_1}{\partial a} + \rho\frac{\partial\phi_2}{\partial a} = 0,$$

$$\lambda\frac{\partial\phi}{\partial b} + \mu\frac{\partial\phi_1}{\partial b} + \rho\frac{\partial\phi_2}{\partial b} = 0,$$

parts such as $\phi\dfrac{\partial\lambda}{\partial a} + \phi_1\dfrac{\partial\mu}{\partial a} + \phi_2\dfrac{\partial\rho}{\partial a}$ vanishing, because of the integral equations under consideration. Moreover, p occurs in ϕ_1, but not in ϕ or ϕ_2: and q occurs in ϕ_2, but not in ϕ or ϕ_1; hence

$$\frac{\partial f}{\partial p} = \mu\frac{\partial\phi}{\partial z}, \qquad \frac{\partial f}{\partial q} = \rho\frac{\partial\phi}{\partial z}.$$

Again,

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial\phi}{\partial x} + \mu\left(\frac{\partial^2\phi}{\partial x^2} + p\frac{\partial^2\phi}{\partial x\partial z}\right) + \rho\left(\frac{\partial^2\phi}{\partial x\partial y} + q\frac{\partial^2\phi}{\partial x\partial z}\right),$$

$$\frac{\partial f}{\partial y} = \lambda\frac{\partial\phi}{\partial y} + \mu\left(\frac{\partial^2\phi}{\partial x\partial y} + p\frac{\partial^2\phi}{\partial y\partial z}\right) + \rho\left(\frac{\partial^2\phi}{\partial y^2} + q\frac{\partial^2\phi}{\partial y\partial z}\right),$$

$$\frac{\partial f}{\partial z} = \lambda\frac{\partial\phi}{\partial z} + \mu\left(\frac{\partial^2\phi}{\partial x\partial z} + p\frac{\partial^2\phi}{\partial z^2}\right) + \rho\left(\frac{\partial^2\phi}{\partial y\partial z} + q\frac{\partial^2\phi}{\partial z^2}\right);$$

hence

$$\begin{aligned}\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = \mu&\left(\frac{\partial^2\phi}{\partial x^2} + 2p\frac{\partial^2\phi}{\partial x\partial z} + p^2\frac{\partial^2\phi}{\partial z^2}\right)\\ &+ \rho\left(\frac{\partial^2\phi}{\partial x\partial y} + q\frac{\partial^2\phi}{\partial x\partial z} + p\frac{\partial^2\phi}{\partial y\partial z} + pq\frac{\partial^2\phi}{\partial z^2}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = \mu&\left(\frac{\partial^2\phi}{\partial x\partial y} + q\frac{\partial^2\phi}{\partial x\partial z} + p\frac{\partial^2\phi}{\partial y\partial z} + pq\frac{\partial^2\phi}{\partial z^2}\right)\\ &+ \rho\left(\frac{\partial^2\phi}{\partial y^2} + 2q\frac{\partial^2\phi}{\partial y\partial z} + q^2\frac{\partial^2\phi}{\partial z^2}\right).\end{aligned}$$

Now the integral equations of the characteristics are

$$\phi = \phi(x, y, z, a, b) = 0, \quad b = g(a),$$

$$\psi = \frac{\partial\phi}{\partial a} + b'\frac{\partial\phi}{\partial b} = 0, \quad b' = g'(a).$$

* The equation

$$f = \lambda\phi + \mu\phi_1 + \rho\phi_2$$

can also be obtained by the ordinary theory of elimination.

We have, from foregoing equations,

$$\lambda\left(\frac{\partial\phi}{\partial a}+b'\frac{\partial\phi}{\partial b}\right)+\mu\left(\frac{\partial\phi_1}{\partial a}+b'\frac{\partial\phi_1}{\partial b}\right)+\rho\left(\frac{\partial\phi_2}{\partial a}+b'\frac{\partial\phi_2}{\partial b}\right)=0,$$

and therefore, along any characteristic,

$$\mu\left(\frac{\partial\phi_1}{\partial a}+b'\frac{\partial\phi_1}{\partial b}\right)+\rho\left(\frac{\partial\phi_2}{\partial a}+b'\frac{\partial\phi_2}{\partial b}\right)=0,$$

that is,

$$\mu\left(\frac{\partial\psi}{\partial x}+p\frac{\partial\psi}{\partial z}\right)+\rho\left(\frac{\partial\psi}{\partial y}+q\frac{\partial\psi}{\partial z}\right)=0.$$

But as $\psi=0$ permanently along the characteristics, we have

$$\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial z}dz=0,$$

that is,

$$\left(\frac{\partial\psi}{\partial x}+p\frac{\partial\psi}{\partial z}\right)dx+\left(\frac{\partial\psi}{\partial y}+q\frac{\partial\psi}{\partial z}\right)dy=0,$$

and therefore

$$\frac{dx}{\mu}=\frac{dy}{\rho}=u,$$

say, where dx and dy are elements of a characteristic. Consequently,

$$\begin{aligned}u\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)&=\left(\frac{\partial^2\phi}{\partial x^2}+2p\frac{\partial^2\phi}{\partial x\partial z}+p^2\frac{\partial^2\phi}{\partial z^2}\right)dx\\&\quad+\left(\frac{\partial^2\phi}{\partial x\partial y}+q\frac{\partial^2\phi}{\partial x\partial z}+p\frac{\partial^2\phi}{\partial y\partial z}+pq\frac{\partial^2\phi}{\partial z^2}\right)dy\\&=d\left(\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}\right)-\frac{\partial\phi}{\partial z}dp\\&=-\frac{\partial\phi}{\partial z}dp,\end{aligned}$$

because the relation $\phi_1=\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}=0$ is satisfied in connection with our equations; and similarly

$$u\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)=-\frac{\partial\phi}{\partial z}dq.$$

Also

$$u\frac{\partial f}{\partial p}=\frac{\partial\phi}{\partial z}dx,\qquad u\frac{\partial f}{\partial q}=\frac{\partial\phi}{\partial z}dy.$$

Hence, gathering together the various results, we have

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}} = \frac{dp}{-\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{-\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)}.$$

These equations are satisfied along the characteristics; and they have been derived from the integral equations.

Ex. As an illustration of the elimination process, consider the equation

$$\phi = z - ax^2 - by^2 - ab = 0:$$

then

$$\phi_1 = p - 2ax = 0, \quad \phi_2 = q - 2by = 0,$$

and so

$$f = z - \tfrac{1}{2}px - \tfrac{1}{2}qy - \frac{pq}{4xy} = 0.$$

Now it is easy to verify that

$$\begin{aligned} z - \tfrac{1}{2}px - \tfrac{1}{2}qy - \frac{pq}{4xy} \\ &= z - ax^2 - by^2 - ab - \tfrac{1}{2}\left(\frac{b}{x} + x\right)(p - 2ax) \\ &\qquad - \tfrac{1}{2}\left(\frac{a}{y} + y\right)(q - 2by) - \frac{1}{4xy}(p - 2ax)(q - 2by), \end{aligned}$$

identically: and therefore

$$f = \phi - \tfrac{1}{2}\left(\frac{b}{x} + x\right)\phi_1 - \tfrac{1}{2}\left(\frac{a}{y} + y\right)\phi_2 - \frac{1}{4xy}\phi_1\phi_2.$$

Thus

$$-\tfrac{1}{2}x - \frac{q}{4xy} = \frac{\partial f}{\partial p} = -\tfrac{1}{2}\left(\frac{b}{x} + x\right) - \frac{1}{4xy}\phi_2,$$

$$-\tfrac{1}{2}y - \frac{p}{4xy} = \frac{\partial f}{\partial q} = -\tfrac{1}{2}\left(\frac{a}{y} + y\right) - \frac{1}{4xy}\phi_1,$$

which are satisfied in virtue of the equations $\phi_1 = 0$, $\phi_2 = 0$: these, together with $\phi = 0$, lead to $f = 0$.

96. In these discussions the surface, which is connected with a complete integral, has been given by a single equation

$$\phi(x, y, z, a, b) = 0.$$

In the general theory of surfaces, it often is convenient to have a parametric representation, whereby x, y, z are expressed in terms of two independent parameters; when this mode is adopted for the representation of the complete integral, we should have equations of the type

$$x = h(u, v, a, b), \quad y = k(u, v, a, b), \quad z = l(u, v, a, b).$$

The single equation of the complete surface would be obtained by the elimination of u and v between these three equations; hence, taking

$$\phi(x, y, z, a, b) = \alpha(x-h) + \beta(y-k) + \gamma(z-l),$$

we have

$$\frac{\partial\phi}{\partial x} = \alpha, \quad \frac{\partial\phi}{\partial y} = \beta, \quad \frac{\partial\phi}{\partial z} = \gamma,$$

$$0 = \alpha\frac{\partial h}{\partial u} + \beta\frac{\partial k}{\partial u} + \gamma\frac{\partial l}{\partial u},$$

$$0 = \alpha\frac{\partial h}{\partial v} + \beta\frac{\partial k}{\partial v} + \gamma\frac{\partial l}{\partial v},$$

$$-\frac{\partial\phi}{\partial a} = \alpha\frac{\partial h}{\partial a} + \beta\frac{\partial k}{\partial a} + \gamma\frac{\partial l}{\partial a},$$

$$-\frac{\partial\phi}{\partial b} = \alpha\frac{\partial h}{\partial b} + \beta\frac{\partial k}{\partial b} + \gamma\frac{\partial l}{\partial b}.$$

The characteristic is given by

$$\frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b}f'(a) = 0, \quad b = f(a);$$

hence

$$\alpha\left(\frac{\partial h}{\partial a} + \frac{\partial h}{\partial b}b'\right) + \beta\left(\frac{\partial k}{\partial a} + \frac{\partial k}{\partial b}b'\right) + \gamma\left(\frac{\partial l}{\partial a} + \frac{\partial l}{\partial b}b'\right) = 0.$$

Accordingly, the equations of the characteristic are

$$\left.\begin{array}{c} x = h, \quad y = k, \quad z = l \\ b = f(a) \\ \dfrac{\partial(h, k, l)}{\partial(u, v, a)} + \dfrac{\partial(h, k, l)}{\partial(u, v, b)}f'(a) = 0 \end{array}\right\};$$

the form earlier considered would be given by the elimination of u, v, b among these five equations.

As regards p and q, we have

$$dz = p\,dx + q\,dy$$

for all variations; hence p and q are given by the equations

$$\left.\begin{array}{l} \dfrac{\partial l}{\partial u} = p\dfrac{\partial h}{\partial u} + q\dfrac{\partial k}{\partial u} \\ \dfrac{\partial l}{\partial v} = p\dfrac{\partial h}{\partial v} + q\dfrac{\partial k}{\partial v} \end{array}\right\}.$$

The differential equation

$$f(x, y, z, p, q) = 0$$

results from the elimination of u, v, a, b between these two equations and

$$x = h, \quad y = k, \quad z = l.$$

Taking

$$f(x, y, z, p, q) = A\,(x-h) + B\,(y-k) + C\,(z-l)$$
$$+ D\left(p\frac{\partial h}{\partial u} + q\frac{\partial k}{\partial u} - \frac{\partial l}{\partial u}\right) + E\left(p\frac{\partial h}{\partial v} + q\frac{\partial k}{\partial v} - \frac{\partial l}{\partial v}\right),$$

and noting the explicit occurrences of variables, as well as the disappearances of parameters and constants, we have

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial y} = B, \quad \frac{\partial f}{\partial z} = C,$$
$$\frac{\partial f}{\partial p} = D\frac{\partial h}{\partial u} + E\frac{\partial h}{\partial v},$$
$$\frac{\partial f}{\partial q} = D\frac{\partial k}{\partial u} + E\frac{\partial k}{\partial v};$$

$$0 = -A\frac{\partial h}{\partial u} - B\frac{\partial k}{\partial u} - C\frac{\partial l}{\partial u} + D\left(p\frac{\partial^2 h}{\partial u^2} + q\frac{\partial^2 k}{\partial u^2} - \frac{\partial^2 l}{\partial u^2}\right)$$
$$+ E\left(p\frac{\partial^2 h}{\partial u\partial v} + q\frac{\partial^2 k}{\partial u\partial v} - \frac{\partial^2 l}{\partial u\partial v}\right),$$

$$0 = -A\frac{\partial h}{\partial v} - B\frac{\partial k}{\partial v} - C\frac{\partial l}{\partial v} + D\left(p\frac{\partial^2 h}{\partial u\partial v} + q\frac{\partial^2 k}{\partial u\partial v} - \frac{\partial^2 l}{\partial u\partial v}\right)$$
$$+ E\left(p\frac{\partial^2 h}{\partial v^2} + q\frac{\partial^2 k}{\partial v^2} - \frac{\partial^2 l}{\partial v^2}\right),$$

$$0 = -A\frac{\partial h}{\partial a} - B\frac{\partial k}{\partial a} - C\frac{\partial l}{\partial a} + D\left(p\frac{\partial^2 h}{\partial u\partial a} + q\frac{\partial^2 k}{\partial u\partial a} - \frac{\partial^2 l}{\partial u\partial a}\right)$$
$$+ E\left(p\frac{\partial^2 h}{\partial v\partial a} + q\frac{\partial^2 k}{\partial v\partial a} - \frac{\partial^2 l}{\partial v\partial a}\right),$$

$$0 = -A\frac{\partial h}{\partial b} - B\frac{\partial k}{\partial b} - C\frac{\partial l}{\partial b} + D\left(p\frac{\partial^2 h}{\partial u\partial b} + q\frac{\partial^2 k}{\partial u\partial b} - \frac{\partial^2 l}{\partial u\partial b}\right)$$
$$+ E\left(p\frac{\partial^2 h}{\partial v\partial b} + q\frac{\partial^2 k}{\partial v\partial b} - \frac{\partial^2 l}{\partial v\partial b}\right).$$

The last four equations determine the ratios $A : B : C : D : E$. The equations of the characteristic, being

$$\frac{dx}{D\dfrac{\partial h}{\partial u} + E\dfrac{\partial h}{\partial v}} = \frac{dy}{D\dfrac{\partial k}{\partial u} + E\dfrac{\partial k}{\partial v}} = \frac{dp}{-(A+pC)} = \frac{dq}{-(B+qC)}$$
$$= \frac{dz}{D\dfrac{\partial l}{\partial u} + E\dfrac{\partial l}{\partial v}},$$

on substituting for $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$, take the simple form

$$\frac{du}{D} = \frac{dv}{E} = \frac{dp}{-(A+pC)} = \frac{dq}{-(B+qC)},$$

when the values of x, y, z are inserted. While these are the general equations of the characteristic, it is clear that the direction of the characteristic at any point on the surface is given by

$$\frac{du}{D} = \frac{dv}{E},$$

where the quantity $D:E$ is given by the preceding equations and where the magnitudes p and q, which occur in that quantity, are

$$\frac{p}{\frac{\partial(l, k)}{\partial(u, v)}} = \frac{q}{\frac{\partial(h, l)}{\partial(u, v)}} = \frac{1}{\frac{\partial(h, k)}{\partial(u, v)}}.$$

Edge of Regression: Integral Curves.

97. Returning now to a complete integral surface

$$\phi(x, y, z, a, b) = 0,$$

we have the characteristics given by

$$\phi = 0, \quad b = g(a), \quad \frac{d\phi}{da} = \frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b} g'(a) = 0.$$

As has been seen, these equations represent the general integral surface when a and b are eliminated, this surface being the locus of the characteristics. When a and b are not eliminated, the equations represent the characteristics which lie on this general integral surface.

Now take a characteristic on the general integral determined by a value a, and a neighbouring characteristic determined by a value $a + \delta a$, where δa is infinitesimal: the equations of the former are

$$\phi = 0, \quad \frac{d\phi}{da} = 0,$$

and those of the latter can be taken

$$\phi + \frac{d\phi}{da}\delta a = 0, \quad \frac{d\phi}{da} + \frac{d^2\phi}{da^2}\delta a = 0,$$

keeping δa sufficiently small; hence the ultimate intersection of two consecutive characteristics is given by

$$\phi = 0, \quad \frac{d\phi}{da} = 0, \quad \frac{d^2\phi}{da^2} = 0.$$

The three equations determine the point of intersection, giving its coordinates as functions of a. When a is eliminated between the three equations, the two resulting equations give a curve, which is the locus of the ultimate intersections of the characteristics. This curve lies on the surface

$$\phi = 0, \quad \frac{d\phi}{da} = 0,$$

which represents the general integral; on the analogy of developable surfaces, Monge called* it the *edge of regression* of the general integral surface.

This locus may be regarded as the envelope of the characteristics on the general integral: for it touches a characteristic at its point of ultimate intersection with its neighbour. To verify this statement, we note that the curve passes through the point: for it is the locus of such points. Further, to obtain its tangent at the point, we assume that, for it, a is determined as a function of x, y, z by means of the equation

$$\frac{d^2\phi}{da^2} = 0,$$

and that the value of a is substituted in the other two equations: the values of $dx : dy : dz$, derived from them, are then given by

$$d\phi + \frac{d\phi}{da}\, da = 0, \quad d\frac{d\phi}{da} + \frac{d^2\phi}{da^2}\, da = 0,$$

where

$$d\phi = \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y}\, dy + \frac{\partial\phi}{\partial z}\, dz,$$

and so for the others: that is, the values are given by

$$d\phi = 0, \quad d\frac{d\phi}{da} = 0,$$

which are the equations determining the ratios $dx : dy : dz$ for the characteristic at the point: hence the two curves touch at the point. Thus all the characteristics on the general integral touch the edge of regression.

* *Application de l'Analyse à la Géométrie*, § VI: the last edition was edited by Liouville in 1850.

Ex. The simplest example of all arises when the complete integral is a double family of planes

$$z = ax + by + f(a, b).$$

The general integral is of the form

$$\left.\begin{aligned} z &= x\theta(a) + y\phi(a) + \psi(a) \\ 0 &= x\theta'(a) + y\phi'(a) + \psi'(a) \end{aligned}\right\},$$

being a developable surface, for it is the envelope of a plane whose equation contains one parameter. The characteristics are the generators: they are the intersections of consecutive planes, and each is the curve of contact of a plane with the general integral. The envelope of the generators is the edge of regression of the developable surface.

98. We have seen that the tangent to the characteristic at any point P is a generator of the cone T associated with P; hence, as the equation of this cone is

$$F\left(x, y, z, \frac{z' - z}{x' - x}, \frac{y' - y}{x' - x}\right) = 0,$$

P being the point x, y, z, the tangents to the characteristic, and therefore the characteristic itself at the point, satisfy the equation

$$F\left(x, y, z, \frac{dz}{dx}, \frac{dy}{dx}\right) = 0.$$

Curves satisfying this equation are called *integral curves.*

It is clear that characteristics are included among these integral curves: it is equally clear that they are not the most general integral curves, because $F = 0$ is only a single equation involving two unknown quantities and one of these can be assumed arbitrarily, the equation then determining the other. Properties sufficient to distinguish characteristics among integral curves have already (§§ 92, 93) been given.

The edge of regression of the general integral is easily seen to be an integral curve: for, at any point on it, the tangent is the same as that of the characteristic which touches it there. The latter at the point satisfies $F = 0$: hence, also, the equations of the edge of regression satisfy $F = 0$.

Conversely, every integral curve can be obtained as an edge of regression. Owing to the equation, the tangents to the integral curves through the point are the generators of the cone associated with the point. Taking any integral curve, its tangent is a generator of the cone, and the direction of the generator deter-

mines a characteristic through the point, which characteristic accordingly touches the integral curve at the point: moreover, it is the only characteristic which can be drawn through the point. Now it has been proved (Ex. 1, § 92) that a general integral passing through a curve is generated as the locus of the characteristics drawn through the points of the curve: and the result is not affected by the angle of intersection between the curve and the characteristic. Hence in the present case, taking the aggregate of the characteristics tangent to the integral curve, we have a general integral surface: on that surface, the integral curve is the envelope of the characteristics and therefore is an edge of regression.

Hence we may take

$$\phi = 0, \quad \frac{d\phi}{da} = 0, \quad \frac{d^2\phi}{da^2} = 0,$$

as the comprehensive integral curve satisfying the equation $F = 0$.

The contact relations of the various integral surfaces with one another will be discussed later: it is worth noting the contact relations of these integral curves with the surface $\phi = 0$.

At a point along the integral curve determined by the relations

$$\phi = 0, \quad \frac{d\phi}{da} = 0, \quad \frac{d^2\phi}{da^2} = 0,$$

we have

$$d\phi + \frac{d\phi}{da} da = 0,$$

$$d\frac{d\phi}{da} + \frac{d^2\phi}{da^2} da = 0,$$

that is,

$$d\phi = 0, \quad d\frac{d\phi}{da} = 0.$$

Again, because $d\phi = 0$, we have

$$d(d\phi) = 0,$$

that is,

$$d^2\phi + \left(d\frac{d\phi}{da}\right) da = 0,$$

and therefore

$$d^2\phi = 0.$$

Consequently, along the integral curve, we have

$$\phi = 0, \quad d\phi = 0, \quad d^2\phi = 0:$$

shewing that the integral surface $\phi = 0$ is cut in three consecutive points by the integral curve. Thus the edge of regression of the general integral has contact of the second order with the complete integral surface from which it originates.

The edge of regression of a general integral is given by the equations

$$\phi = 0, \quad \frac{d\phi}{da} = 0, \quad \frac{d^2\phi}{da^2} = 0;$$

the quantity b is any function of a, and the equations involve b', b'', the first and the second derivatives of b. As there thus arises a curve associated with any assumed function, there thus will arise an infinitude of such curves associated with all forms of the function.

Let those curves among this infinitude be selected which are such that

$$\frac{d^3\phi}{da^3} = 0;$$

that such curves do, in general, exist can be seen as follows. For variations along any edge of regression, the ratios $dx : dy : dz$ satisfy the equations

$$d\phi + \frac{d\phi}{da}\,da = 0,$$

$$d\,\frac{d\phi}{da} + \frac{d^2\phi}{da^2}\,da = 0,$$

$$d\,\frac{d^2\phi}{da^2} + \frac{d^3\phi}{da^3}\,da = 0;$$

hence, if the edge of regression be such that

$$\frac{d^3\phi}{da^3} = 0,$$

its direction at the point is given by

$$d\phi = 0, \quad d\,\frac{d\phi}{da} = 0, \quad d\,\frac{d^2\phi}{da^2} = 0.$$

Between these three equations and the equations

$$\phi = 0, \quad \frac{d\phi}{da} = 0, \quad \frac{d^2\phi}{da^2} = 0,$$

we eliminate a, b, b', b'', and we have two resulting equations involving $dx : dy : dz$. The two differential equations (which will

be algebraical in form when ϕ is algebraical) define curves: so that curves of the specified type do exist in general. Further, along such a curve x, y, z, $dx : dy : dz$ are expressible in terms of a parametric variable, say t: hence at the point, the foregoing equations, when resolved, will express both a and b in terms of t and so, on the elimination of t between the expressions, will give b in terms of a.

To find the order of contact of this curve with the integral surface from which it originates, we proceed as before. Along the curve, we have

$$d\phi = 0, \quad d\frac{d\phi}{da} = 0, \quad d\frac{d^2\phi}{da^2} = 0.$$

Also, since $d\phi = 0$, we have

$$d\,(d\phi) = 0,$$

that is,

$$d^2\phi + \left(d\frac{d\phi}{da}\right) da = 0,$$

and therefore

$$d^2\phi = 0.$$

And, since $d\dfrac{d\phi}{da} = 0$, we have

$$d\left(d\frac{d\phi}{da}\right) = 0,$$

that is,

$$d^2\frac{d\phi}{da} + \left(d\frac{d^2\phi}{da^2}\right) da = 0,$$

so that

$$d^2\frac{d\phi}{da} = 0.$$

And, since $d^2\phi = 0$, we have

$$d\,(d^2\phi) = 0,$$

that is,

$$d^3\phi + \left(d^2\frac{d\phi}{da}\right) da = 0,$$

so that

$$d^3\phi = 0.$$

Hence along the curve in question, we have

$$\phi = 0, \quad d\phi = 0, \quad d^2\phi = 0, \quad d^3\phi = 0,$$

at the point: consequently, the curve in question meets the integral surface in four consecutive points: that is, the curve has contact of the third order with the surface.

Ex. 1. The equation of a sphere, that is absolutely unrestricted, contains four arbitrary independent constants: hence, when it is made subject to two independent conditions, the equation will contain two arbitrary constants and may be regarded as giving the complete integral of some partial differential equation.

To obtain a general integral, we select a family of these spheres and construct their envelope. The characteristics, being the intersections of consecutive spheres, are circles: each sphere touches the envelope surface along a circle. The edge of regression on the envelope surface, which is the general integral, is itself the envelope of the characteristic circles; and the earlier investigation shewed that, where the sphere meets this edge of regression, it meets the edge in three consecutive points.

But there is one general integral for which the contact is closer. By an appropriate choice of a functional relation between the two constants in the complete integral, we obtain the envelope of one selected family of spheres: each sphere, where it meets the edge of regression on this envelope surface, meets it in four consecutive points, and therefore is its osculating sphere. The characteristics are the circles which are the intersections of consecutive spheres: hence, for this general integral, they are the osculating circles of the edge of regression.

Ex. 2. As a particular instance of the last example, construct the various equations of the complete integral, the general integral, and the selected general integral, when the edge of regression is a regular helix; and obtain the partial differential equation satisfied by the integrals.

Ex. 3. In the explanations in the general theory, all the surfaces and curves concerned are unrestricted in properties; it has been assumed that the various contacts are possible.

Consider, in particular, a plane. Its equation involves three independent constants when completely unrestricted: if the plane be subject to one condition, two independent constants will remain in the equation, which then can be regarded as giving the complete integral of a partial differential equation. To obtain a general integral, we make one of the constants an arbitrary function of the other and proceed to obtain the envelope of the planes so selected. As their equation involves one arbitrary parameter, this envelope is a developable surface: the characteristics are the generators, being the intersections of consecutive planes; and each plane osculates the edge of regression of the surface.

But it is not possible to select a general integral so as to have closer contact between the plane and its edge of regression: because not more than three consecutive points of a curve can lie in a plane, unless at a singular point, or unless the curve be a plane curve.

In this case, the selected general integral of § 98 does not occur.

99. In discussing the selected general integral, and its edge of regression with which a complete integral has triple contact, the complete integral is supposed known; and the equations of the

edge of regression in question are deduced from that of the complete integral. It is, however, possible to deduce equations of the particular edge of regression from the original differential equation.

For the purpose it is sufficient to note that, at a point on the particular edge of regression, there is triple contact between the curve and the complete integral surface; and therefore, when the equations

$$f(x, y, z, p, q) = 0, \quad dz = p\,dx + q\,dy,$$

are regarded as determining p and q in terms of x, y, z, dx, dy, dz, (supposed known, as belonging to the required curve), they must provide a triple root. Hence, on writing

$$z' = \frac{dz}{dx}, \quad y' = \frac{dy}{dx},$$

the equation

$$f(x, y, z, z' - qy', q) = 0$$

must provide a triple root; so that

$$\frac{\partial f}{\partial q} - y' \frac{\partial f}{\partial p} = 0,$$

$$\frac{\partial^2 f}{\partial q^2} - 2y' \frac{\partial^2 f}{\partial p \partial q} + y'^2 \frac{\partial^2 f}{\partial p^2} = 0.$$

The elimination of q, between the last two equations and

$$f(x, y, z, z' - qy', q) = 0,$$

leads to two equations which are the (ordinary differential) equations of the curve in question.

It is clear that, if the equation is rational and integral in p and q, it must be of at least the third degree if the curve in question is to arise.

Ex. 1. Prove that, if such a curve exists, it touches the edges of regression of the cones T and that its tangents are perpendicular to the planes of inflexion of the cones N.

(Darboux.)

Ex. 2. Discuss the various loci, indicated in the preceding sections, to be associated with the partial differential equation, the complete integral of which is

$$(1 - a^2)\,x + (1 + a^2)\,kz + 2ay + b = 0,$$

where a and b are arbitrary constants, and k is a pure constant.

(Goursat.)

Discuss also the integrals of the equation which has

$$\left.\begin{array}{r} ax^2+by^2+cz^2=1 \\ a^3+b^3+c^3=1 \end{array}\right\}$$

for its complete primitive.

Ex. 3. It has been proved that every integral curve (§ 98) can be represented as an edge of regression of a general integral: it has also been proved that an integral surface touching an edge of regression has contact of the second order: hence every integral curve, touching an integral surface, has contact of the second order with that surface.

Verify this proposition directly from the equations.

(Lie.)

Lie's Classification of Equations.

100. In the discussion of the characteristics, regard has been paid chiefly to their association with the surface represented by the general integral: but they can be considered also in their association with the surface represented by the complete integral. It is in this connection that Lie has considered them*, in particular, classifying partial differential equations according to the nature of the characteristics as curves upon the complete integral surface. Some of his results can be obtained very simply as follows.

Among the various curves that can be drawn upon a surface, three of the most important classes are asymptotic lines (being the lines touched by the principal tangents of the surface at successive points), lines of curvature, and geodesics; and, accordingly, Lie investigates those partial differential equations of the first order, the surface integrals of which have characteristics belonging to one of these three classes of curves upon the surface.

(i) The directions of the asymptotic lines upon a surface at any point, being the directions of the principal tangents at the point, are given by

$$pdx+qdy=dz,$$
$$rdx^2+2sdxdy+tdy^2=0,$$

in the usual notation: as

$$dp=rdx+sdy, \quad dq=sdx+tdy,$$

the latter equation is

$$dpdx+dqdy=0.$$

* *Math. Ann.* t. v (1872), pp. 188—200.

If these are the characteristics of the surface integral of an equation

$$f(x, y, z, p, q) = 0,$$

they must accord with the ordinary differential equations of the characteristics (§ 94). Hence

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial q} = 0;$$

and this condition is sufficient, as well as necessary, to secure the property. Accordingly, any function f, which satisfies this condition, will lead to a differential equation possessing the property that *the characteristics are asymptotic lines upon the integral surface.*

Ex. Verify that a complete integral of the foregoing equation, which must be satisfied by f, is

$$f = ax + by - \frac{ac+b}{cp+q}z + g(cp+q) + h,$$

where a, b, c, g, h are arbitrary constants: obtain other integrals of that equation: and discuss the surface integrals of the equation $f=0$ for the respective forms of f.

(ii) The directions of the lines of curvature at any point of a surface are given by

$$pdx + qdy = dz,$$

$$(dx + pdz)\,dq = (dy + q\,dz)\,dp.$$

If these are the characteristics of the surface integral of an equation

$$f(x, y, z, p, q) = 0,$$

they must accord with the ordinary differential equations of the characteristics; hence (§ 94)

$$\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\left\{\frac{\partial f}{\partial p} + p\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\right\}$$
$$= \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\left\{\frac{\partial f}{\partial q} + q\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\right\},$$

that is,

$$\frac{\partial f}{\partial x}\left\{\frac{\partial f}{\partial q} + q\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\right\} - \frac{\partial f}{\partial y}\left\{\frac{\partial f}{\partial p} + p\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\right\}$$
$$+ \frac{\partial f}{\partial z}\left(p\frac{\partial f}{\partial q} - q\frac{\partial f}{\partial p}\right) = 0;$$

and this condition is sufficient, as well as necessary, to secure the property. Accordingly, any function f, which satisfies this condition, will lead to a differential equation possessing the property that *the characteristics are lines of curvature upon the integral surface.*

(iii) The determination of equations such that the integral surfaces have geodesics for their characteristics is, at first sight, a problem leading to equations of the second order. Geodesics are given by the equations

$$\frac{1}{p}\frac{d^2x}{ds^2}=\frac{1}{q}\frac{d^2y}{ds^2}=-\frac{d^2z}{ds^2};$$

and these can be replaced by the equations

$$\frac{dz}{dt}=p\frac{dx}{dt}+q\frac{dy}{dt},$$

$$(1+p^2+q^2)\left(\frac{d^2x}{dt^2}\frac{dy}{dt}-\frac{d^2y}{dt^2}\frac{dx}{dt}\right)$$
$$=\left(q\frac{dx}{dt}-p\frac{dy}{dt}\right)\left(\frac{dp}{dt}\frac{dx}{dt}+\frac{dq}{dt}\frac{dy}{dt}\right),$$

where t is any variable. When the partial differential equation is

$$f(x, y, z, p, q)=0,$$

the characteristics are given by

$$\frac{dx}{dt}=\frac{\partial f}{\partial p},\quad \frac{dy}{dt}=\frac{\partial f}{\partial q},\quad \frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q},$$

$$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z},\quad \frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}.$$

The first of the two equations for the geodesics is satisfied identically; when substitution is effected in the other equation, it becomes a partial differential equation of the second order—a result to be expected, when the curvature property of the geodesic is used.

But an equation of the first order can be obtained, by using the known properties of geodesic parallels and their orthogonal geodesics*. The element of arc upon the surface is given by

$$ds^2=(1+p^2)\,dx^2+2pq\,dx\,dy+(1+q^2)\,dy^2;$$

* Darboux, *Théorie générale des surfaces*, t. II, pp. 424 et seq.

if θ be an integral of the equation

$$(1+q^2)\left(\frac{\partial\theta}{\partial x}\right)^2 - 2pq\,\frac{\partial\theta}{\partial x}\,\frac{\partial\theta}{\partial y} + (1+p^2)\left(\frac{\partial\theta}{\partial y}\right)^2 = 1+p^2+q^2,$$

containing a non-additive constant a, the geodesics are given by

$$\frac{\partial\theta}{\partial a} = \text{constant}.$$

On the basis of this property, the equation of the required surfaces can be constructed.

Let $\phi(x, y, z)$ denote any function of x, y, z: and let the (unknown) value of z of the surface be supposed substituted in ϕ, so that it becomes the foregoing function θ; then

$$\frac{\partial\theta}{\partial x} = \frac{\partial\phi}{\partial x} + p\,\frac{\partial\phi}{\partial z}, \quad \frac{\partial\theta}{\partial y} = \frac{\partial\phi}{\partial y} + q\,\frac{\partial\phi}{\partial z}.$$

When these values are substituted in the above equation, it becomes

$$(1+q^2)\left(\frac{\partial\phi}{\partial x}\right)^2 + (1+p^2)\left(\frac{\partial\phi}{\partial y}\right)^2 + (p^2+q^2)\left(\frac{\partial\phi}{\partial z}\right)^2$$
$$+\,2p\,\frac{\partial\phi}{\partial x}\,\frac{\partial\phi}{\partial z} + 2q\,\frac{\partial\phi}{\partial y}\,\frac{\partial\phi}{\partial z} - 2pq\,\frac{\partial\phi}{\partial x}\,\frac{\partial\phi}{\partial y} = 1+p^2+q^2,$$

or, what is an equivalent form,

$$\left(p\,\frac{\partial\phi}{\partial x} + q\,\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial z}\right)^2 = (1+p^2+q^2)\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 - 1\right\}.$$

When a value of z is obtained so as to satisfy this equation, and when it is substituted in the assumed function $\phi(x, y, z)$, the latter becomes a quantity θ, such that $\theta = \text{constant}$ gives a family of parallel curves. The orthogonal geodesics on the surface are given by

$$\frac{\partial\theta}{\partial a} = \text{constant}$$
$$= 0,$$

say, for one geodesic, where a is a non-additive constant in θ. Now

$$\phi(x, y, z) = \theta(x, y, a),$$

and the quantity a has entered only through z, as given by the integral of the foregoing equation: hence

$$\frac{\partial\phi}{\partial z}\,\frac{dz}{da} = \frac{\partial\theta}{\partial a} = 0,$$

so that, as $\frac{\partial \phi}{\partial z}$ is not zero, we have

$$\frac{dz}{da} = 0;$$

and this gives a geodesic on the surface. But

$$z = \text{function } (x, y, a),$$

$$\frac{dz}{da} = 0,$$

are the equations of the characteristic: hence *the surfaces, obtained by integrating the equation*

$$\left(p\frac{\partial \phi}{\partial x} + q\frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial z}\right)^2 = (1 + p^2 + q^2)\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 - 1\right\},$$

where $\phi(x, y, z)$ is any function of x, y, z, have geodesics for their characteristics.*

Ex. 1. Shew that, if the normals to a surface touch the sphere $x^2 + y^2 + z^2 = 1$, its equation satisfies the partial differential equation

$$(x^2 + y^2 + z^2 - 1)(p^2 + q^2 + 1) = (z - px - qy)^2;$$

integrate this equation, discussing the characteristics and the edge of regression. (Monge.)

Ex. 2. If the characteristics of a non-linear equation are straight lines, the equation is of the form

$$z = px + qy + f(p, q).$$

(Goursat.)

Ex. 3. Shew that the characteristics of the equation

$$p + a = (q + b) f(x, y, z)$$

are curves in parallel planes; and indicate how to form the equation of surfaces whose characteristics are plane curves.

* These results are due to Lie, who gives other properties in his memoir quoted on p. 244. The method of establishment differs from Lie's, which is based upon properties of complexes of lines and curves.

CHAPTER VII.

SINGULAR INTEGRALS AND THEIR GEOMETRICAL PROPERTIES: SINGULARITIES OF THE CHARACTERISTICS.

THE authorities used in the construction of this chapter have been quoted at the beginning of the preceding chapter.

101. We now proceed to consider the relations of the Singular Integral (which will be assumed to exist) with the various integral surfaces and curves that have been discussed. That Singular Integral is given by the equation, which results from the elimination* of a and b between the equations

$$\phi(x, y, z, a, b) = 0, \quad \frac{\partial\phi}{\partial a} = 0, \quad \frac{\partial\phi}{\partial b} = 0,$$

or by part of that resulting equation; the value of z must be such that the differential equation is satisfied. In that case, the resulting equation (or the part of the resulting equation) represents the envelope which then is possessed by the family of complete integral surfaces: as the values of p and q are the same

* It is to be borne in mind that the singular integral is assumed to exist, so that it will arise as indicated in the text. The result of the elimination in general is not to give an integral of the equation; the eliminant contains the locus of conical points (if any), the locus of double lines (if any), and other loci, which are not integrals of the differential equation. For a discussion of such matters, which are not our concern at this stage, reference may be made to a memoir by M. J. M. Hill, *Phil. Trans.* A (1892), pp. 141—278.

Moreover, it is assumed that the elimination is possible, so that the three equations are independent of one another. This condition, however, is not always satisfied; and it is easy to construct exceptions of the type

$$\phi = \lambda\frac{\partial\phi}{\partial a} + \mu\frac{\partial\phi}{\partial b},$$

which would make the three equations equivalent to two only. Again, such matters are not our present concern: we assume that the elimination is possible, and that it leads to the singular integral.

at any point, common to the envelope and the complete integral surface, the latter surface is touched by the envelope at a common point. Let the envelope be denoted by E.

The three equations are equivalent to the equation of E and to values (or to sets of values) of a and b: by each such set of values of a and b, there is given a point on E where it is touched by $\phi = 0$.

Take any point P on E, and let the values of a and b at P be denoted by a_0 and b_0. A characteristic through P is given by

$$\phi(x, y, z, a_0, b_0) = 0, \quad b_0 = f(a_0),$$

$$\frac{\partial \phi}{\partial a_0} + f'(a_0)\frac{\partial \phi}{\partial b_0} = 0;$$

but these equations are satisfied, whatever be the form $f(a_0)$, because the equations

$$\frac{\partial \phi}{\partial a_0} = 0, \quad \frac{\partial \phi}{\partial b_0} = 0$$

are satisfied at the point; hence an infinitude of characteristics passes through any point on the envelope. Moreover, all these characteristics through P touch E there: for they pass through P as a point on the complete surface which touches E at the point.

Take any two curves PT and PT' on E passing through P; let T and T' denote points consecutive to P on those curves respectively; also let $a_0 + da_0$, $b_0 + db_0$ be the values of a and b for T, and $a_0 + \delta a_0$, $b_0 + \delta b_0$ be their values for T'. Then along PT, the values at P of the differential elements dx, dy, dz are given by

$$0 = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \frac{\partial \phi}{\partial a_0}da_0 + \frac{\partial \phi}{\partial b_0}db_0,$$

$$0 = \frac{\partial^2 \phi}{\partial a_0 \partial x}dx + \frac{\partial^2 \phi}{\partial a_0 \partial y}dy + \frac{\partial^2 \phi}{\partial a_0 \partial z}dz + \frac{\partial^2 \phi}{\partial a_0^2}da_0 + \frac{\partial^2 \phi}{\partial a_0 \partial b_0}db_0,$$

$$0 = \frac{\partial^2 \phi}{\partial b_0 \partial x}dx + \frac{\partial^2 \phi}{\partial b_0 \partial y}dy + \frac{\partial^2 \phi}{\partial b_0 \partial z}dz + \frac{\partial^2 \phi}{\partial a_0 \partial b_0}da_0 + \frac{\partial^2 \phi}{\partial b_0^2}db_0;$$

and the last two terms in the first equation vanish. Now the complete integral touching E at T' cuts the complete integral touching E at P in the characteristic whose equations are

$$\phi = 0, \quad \frac{\partial \phi}{\partial a_0} + \frac{\delta b_0}{\delta a_0}\frac{\partial \phi}{\partial b_0} = 0;$$

and the direction of the tangent to this characteristic is given by the two equations

$$0 = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz,$$

$$0 = \frac{\partial^2 \phi}{\partial a_0 \partial x} dx + \frac{\partial^2 \phi}{\partial a_0 \partial y} dy + \frac{\partial^2 \phi}{\partial a_0 \partial z} dz$$
$$+ \frac{\delta b_0}{\delta a_0}\left(\frac{\partial^2 \phi}{\partial b_0 \partial x} dx + \frac{\partial^2 \phi}{\partial b_0 \partial y} dy + \frac{\partial^2 \phi}{\partial b_0 \partial z} dz\right).$$

If then this characteristic touches PT at P, the last two equations giving the ratios $dx : dy : dz$ must be satisfied by the values of dx, dy, dz belonging to PT: hence

$$\frac{\partial^2 \phi}{\partial a_0^2} da_0 + \frac{\partial^2 \phi}{\partial a_0 \partial b_0} db_0 + \frac{\delta b_0}{\delta a_0}\left(\frac{\partial^2 \phi}{\partial a_0 \partial b_0} da_0 + \frac{\partial^2 \phi}{\partial b_0^2} db_0\right) = 0,$$

that is,

$$\frac{\partial^2 \phi}{\partial a_0^2} da_0 \delta a_0 + \frac{\partial^2 \phi}{\partial a_0 \partial b_0}(db_0 \delta a_0 + da_0 \delta b_0) + \frac{\partial^2 \phi}{\partial b_0^2} db_0 \delta b_0 = 0.$$

Moreover, this relation is symmetrical between the two sets of differential elements that are associated with T and T' respectively. Hence we have Darboux's theorem*:

If we take any direction PT' through a point P on E which represents the singular integral, and if PT be the direction of the characteristic which is the intersection of the complete integrals touching E at consecutive points P and T', then PT' is the direction of the characteristic which is the intersection of the complete integrals touching E at consecutive points P and T.

Characteristics, in directions such as PT and PT' at P, may be called *conjugate*: obviously, any characteristic has a conjugate.

When a characteristic coincides with its conjugate, so that it may be called *self-conjugate*, its direction at the point on the envelope is given by

$$\frac{\partial^2 \phi}{\partial a_0^2} da_0^2 + 2\frac{\partial^2 \phi}{\partial a_0 \partial b_0} da_0 db_0 + \frac{\partial^2 \phi}{\partial b_0^2} db_0^2 = 0.$$

Hence, in general, there are two sets of curves upon the envelope such that, at every point upon each of them, the tangent characteristic is self-conjugate; such curves may be called *asym-*

* *L.c.*, p. 60.

ptotic curves. The two curves through any point touch one another, if the equation

$$\frac{\partial^2\phi}{\partial a_0^2}\frac{\partial^2\phi}{\partial b_0^2}=\left(\frac{\partial^2\phi}{\partial a_0\partial b_0}\right)^2$$

is satisfied at the point. If the equation is satisfied everywhere upon the envelope, then there is only one asymptotic curve through a point, and there is only a single set of such curves upon the surface.

These results are the analogue of the results in the ordinary theory of surfaces: the singular integral corresponds to a surface, the complete integrals correspond to the tangent planes, the characteristics to the tangent lines, conjugate directions to conjugate directions, and asymptotic curves to asymptotic curves.

102. These properties of the singular integral have been derived from the complete integral with its associated curves and surfaces: they can be used to bring the singular integral into relation with the original partial equation

$$f(x, y, z, p, q)=0.$$

We have seen that, through any point on the singular integral surface, there passes an infinitude of tangent characteristics: the direction therefore of a characteristic through such a point, as given by

$$\frac{dx}{\dfrac{\partial f}{\partial p}}=\frac{dy}{\dfrac{\partial f}{\partial q}}=\frac{dz}{p\dfrac{\partial f}{\partial p}+q\dfrac{\partial f}{\partial q}},$$

must be indeterminate, and so

$$\frac{\partial f}{\partial p}=0,\quad \frac{\partial f}{\partial q}=0.$$

Moreover, the equation must be identically satisfied when the values of z, p, q belonging to the singular integral at the point are substituted; hence

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}+r\frac{\partial f}{\partial p}+s\frac{\partial f}{\partial q}=0,$$

$$\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}+s\frac{\partial f}{\partial p}+t\frac{\partial f}{\partial q}=0,$$

that is, in connection with the former equations, we must have

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0,\quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0.$$

We thus have the former aggregate of equations: every singular integral must satisfy the equations

$$f=0,\quad \frac{\partial f}{\partial p}=0,\quad \frac{\partial f}{\partial q}=0,\quad \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0,\quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0.$$

It does not follow that these equations definitely determine a singular integral. We have already indicated one class of exceptions, when the five equations are satisfied in virtue of two only, so that p and q cannot be eliminated among them. But, as pointed* out by Darboux, there may be other exceptions. Thus, suppose the five equations do determine z, p, q as functions of x and y; then, along the surface expressing z in terms of x and y, the equation

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial p}dp+\frac{\partial f}{\partial q}dq=0$$

is satisfied, that is, in virtue of the five equations, we must have

$$\frac{\partial f}{\partial z}(dz-p\,dx-q\,dy)=0.$$

If $\frac{\partial f}{\partial z}$ does not vanish, by virtue of the values of z, p, q in terms of x and y, then

$$dz-p\,dx-q\,dy=0;$$

the relation between z, x, y is then a singular integral. But if $\frac{\partial f}{\partial z}$ does vanish by virtue of these values, we are not justified in making the inference; the general case then requires separate consideration, though in any particular instance the test of satisfying the equation can be applied immediately.

Ex. 1. Prove that the normal to the singular integral surface is a double line on the cone N of normals at the point. (Darboux.)

Ex. 2. Obtain the partial differential equation of the first order which has

$$(z+ax)(z+by)=x^2+y^2$$

for a complete integral. Does it possess a singular integral?

Discuss the characteristics: and, in particular, prove that there is a rectilinear edge of regression of the type indicated in § 97.

[The edge of regression in question is given by

$$\alpha x=\beta y=\gamma z,$$

* *L.c.*, p. 67; see also § 76, *ante*.

where

$$\gamma^2(a^2+\beta^2)+4\beta\gamma=0,$$

and the relation between a and b is

$$(a\gamma+a)(b\gamma+\beta)+4=0.]$$

Ex. 3. Discuss the integrals of the equation

$$(z-\tfrac{1}{2}p^2)^2-\tfrac{1}{3}q^3=0;$$

and indicate the character of the integral

$$z=\tfrac{1}{2}(x-a)^2. \qquad \text{(Darboux.)}$$

Ex. 4. If an equation of the first order possesses singular integrals and can be expressed in a form

$$au^m+bv^n+cw^p=0,$$

where m, n, p are integers, the singular integrals satisfy the equations

$$u=0, \quad v=0, \quad w=0. \qquad \text{(Darboux.)}$$

103. The relation between the surfaces represented by the general integral and the singular integral respectively can be indicated simply. In the equations of the singular integral, which are

$$\phi(x, y, z, a, b)=0, \quad \frac{\partial\phi}{\partial a}=0, \quad \frac{\partial\phi}{\partial b}=0,$$

the quantities a, b are functions of x, y, z: if, therefore, we take

$$b=f(a),$$

where f is any function, we are selecting a curve through a point on the singular integral. At every point on this curve, the equations

$$\phi=0, \quad b=f(a), \quad \frac{\partial\phi}{\partial a}+f'(a)\frac{\partial\phi}{\partial b}=0,$$

are satisfied: as these are the equations of the general integral, the curve lies upon the surface represented by that integral. At any point on this curve common to the two surfaces, the values of p and q are the same, being given by

$$\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}=0, \quad \frac{\partial\phi}{\partial y}+q\frac{\partial\phi}{\partial z}=0,$$

together with the other equations: hence the two surfaces touch along a curve.

It may happen that the equation

$$b=f(a)$$

determines not a single curve alone but a number of curves on the singular integral: in that case, each of the curves is common to the singular integral and the appropriate general integral determined in connection with the above equation, and the two surfaces touch one another at every point on each of the curves. Hence the general integral and the singular integral touch one another along a curve or curves: or, what is the same property, through any curve on the surface, represented by the singular integral, there passes the surface, represented by the general integral which is the envelope of the complete integrals that touch the singular integral along the curve.

Order of Contact of the Singular Integral with the General Integral and the Complete Integral.

104. We have seen that the singular integral, when it exists, is the envelope of the complete integrals, each of which touches it at one or more points: it also touches the general integrals along a curve or curves. We have to consider the order of the contact, a matter already (§§ 92, 98) discussed for characteristics and complete integrals.

The assumed singular integral is given by the single equation which results from the elimination of a and b between

$$\phi(x, y, z, a, b) = 0, \quad \frac{\partial \phi}{\partial a} = 0, \quad \frac{\partial \phi}{\partial b} = 0;$$

let this single equation be supposed resolved with regard to z, so that it has the form

$$z = \psi(x, y).$$

We introduce a new dependent variable ζ, defined by the equation

$$\zeta = z - \psi(x, y);$$

the complete integral now is

$$\phi(x, y, \zeta + \psi, a, b) = 0.$$

Derivatives of the first order (and of all orders) with regard to a and b are the same as before: hence, when the elimination of a and b is performed between the equations

$$\phi = 0, \quad \frac{\partial \phi}{\partial a} = 0, \quad \frac{\partial \phi}{\partial b} = 0,$$

and the eliminant is resolved with regard to ζ, the resolved equation of the singular integral is

$$\zeta = 0.$$

Now it may be that, in both forms, there are several branches of the singular integral leading to resolved equations; in that case, one of them is certainly $\zeta = 0$ in the second form; and though this will not simultaneously represent all branches of the singular integral, it will suffice for the discussion of the order of contact at the point. Accordingly, without loss of generality for the immediate purpose, we may take the singular integral in the form

$$z = 0.$$

The values of p and q, in general, are given by

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0, \quad \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0:$$

for our present purpose, $z = 0$, so that $p = 0$, $q = 0$: and therefore

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0$$

Thus we have, at all points of the singular integral taken in the form $z = 0$, the five equations

$$\phi = 0, \quad \frac{\partial \phi}{\partial a} = 0, \quad \frac{\partial \phi}{\partial b} = 0, \quad \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0.$$

But we do not have $\dfrac{\partial \phi}{\partial z} = 0$, in addition: for the point would then be a singularity, conical or otherwise, on the complete integral: and circumstances would be exceedingly special if such a singularity of the complete integrals were to lie on the envelope of those integrals.

Take, first, a general integral in the form

$$\phi(x, y, z, a, b) = 0, \quad b = f(a),$$

$$\frac{d\phi}{da} = \frac{\partial \phi}{\partial a} + f'(a)\frac{\partial \phi}{\partial b} = 0:$$

it touches the singular integral $z = 0$ at one point and along a curve through the point. Consider variations in the vicinity of any such point. On the singular integral $z = 0$, they will be represented by dx and dy; the variations da and db, as determined for the singular integral by dx and dy, are not at once required for the present

purpose. On the general integral, they will be represented by dz, da, and by the same values of dx and dy as for the singular integral: and the order of contact of the two surfaces at the point will be measured by the order of dz, expressed in terms of the small quantities dx and dy. Thus, for the general integral, we have

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz+\frac{d\phi}{da}da$$
$$+\tfrac{1}{2}\left\{\frac{\partial^2\phi}{\partial x^2}dx^2+\ldots\right\}=0,$$

where the unexpressed terms are the aggregate of terms bilinear in dx, dy, dz, da. But at the point of contact of the singular integral and the general integral, we have

$$\frac{\partial\phi}{\partial x}=0,\quad \frac{\partial\phi}{\partial y}=0,\quad \frac{d\phi}{da}=0,$$

so that dz is at least of the second order of small quantities; hence dz is given, accurately to the second order of small quantities inclusive, by the equation

$$\frac{\partial\phi}{\partial z}dz+\tfrac{1}{2}\left\{\frac{\partial^2\phi}{\partial x^2}dx^2+2\frac{\partial^2\phi}{\partial x\partial y}dx\,dy+\frac{\partial^2\phi}{\partial y^2}dy^2\right\}$$
$$+\tfrac{1}{2}\left\{2\frac{\partial}{\partial x}\left(\frac{d\phi}{da}\right)da\,dx+2\frac{\partial}{\partial y}\left(\frac{d\phi}{da}\right)da\,dy+\frac{d^2\phi}{da^2}da^2\right\}=0.$$

But, because $\dfrac{d\phi}{da}=0$ for the general integral, we have

$$\frac{\partial}{\partial x}\left(\frac{d\phi}{da}\right)dx+\frac{\partial}{\partial y}\left(\frac{d\phi}{da}\right)dy+\frac{d^2\phi}{da^2}da=0,$$

accurately to the first order of small quantities: thus

$$-2\frac{\partial\phi}{\partial z}dz=\frac{\partial^2\phi}{\partial x^2}dx^2+2\frac{\partial^2\phi}{\partial x\partial y}dx\,dy+\frac{\partial^2\phi}{\partial y^2}dy^2-\frac{d^2\phi}{da^2}da^2$$
$$=\frac{\partial^2\phi}{\partial x^2}dx^2+2\frac{\partial^2\phi}{\partial x\partial y}dx\,dy+\frac{\partial^2\phi}{\partial y^2}dy^2$$
$$-\frac{1}{\dfrac{d^2\phi}{da^2}}\left\{\frac{\partial}{\partial x}\left(\frac{d\phi}{da}\right)dx+\frac{\partial}{\partial y}\left(\frac{d\phi}{da}\right)dy\right\}^2,$$

accurately to the second order inclusive.

When the contact between the surfaces is of the first order, dz is of the second order in the small quantities dx and dy. The

right-hand side of the above expression for dz does not vanish for all values of dx and dy, except under special conditions; and therefore *the contact between the general integral and the singular integral at any point is usually of the first order.* There are apparently two directions, given by

$$t = \frac{dy}{dx},$$

where t satisfies the quadratic

$$\frac{d^2\phi}{da^2}\left(\frac{\partial^2\phi}{\partial x^2} + 2t\,\frac{\partial^2\phi}{\partial x\partial y} + t^2\,\frac{\partial^2\phi}{\partial y^2}\right) = \left\{\frac{\partial}{\partial x}\left(\frac{d\phi}{da}\right) + t\,\frac{\partial}{\partial y}\left(\frac{d\phi}{da}\right)\right\}^2,$$

along which dz is of the third order; but these two directions will later* be proved to be the same, a property that can be verified by means of the analysis that follows.

105. When the contact between the surfaces is of the second order, dz is of the third order in the small quantities dx and dy; and therefore the right-hand side of the expression obtained for dz, being generally accurate up to the second order, must vanish up to that order for all values of dx and dy. That this may be the case, we must have

$$\frac{d^2\phi}{da^2}\,\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2\phi}{\partial a\partial x} + b'\,\frac{\partial^2\phi}{\partial b\partial x}\right)^2,$$

$$\frac{d^2\phi}{da^2}\,\frac{\partial^2\phi}{\partial x\partial y} = \left(\frac{\partial^2\phi}{\partial a\partial x} + b'\,\frac{\partial^2\phi}{\partial b\partial x}\right)\left(\frac{\partial^2\phi}{\partial a\partial y} + b'\,\frac{\partial^2\phi}{\partial b\partial y}\right),$$

$$\frac{d^2\phi}{da^2}\,\frac{\partial^2\phi}{\partial y^2} = \left(\frac{\partial^2\phi}{\partial a\partial y} + b'\,\frac{\partial^2\phi}{\partial b\partial y}\right)^2;$$

so that, if we take

$$\frac{\partial^2\phi}{\partial x^2} = \mu\,\frac{\partial^2\phi}{\partial x\partial y},$$

we must have

$$\frac{\partial^2\phi}{\partial x\partial y} = \mu\,\frac{\partial^2\phi}{\partial y^2},$$

and also

$$\frac{\partial^2\phi}{\partial a\partial x} + b'\,\frac{\partial^2\phi}{\partial b\partial x} = \mu\,\frac{\partial^2\phi}{\partial a\partial y} + b'\mu\,\frac{\partial^2\phi}{\partial b\partial y}.$$

We can prove that the last equation is satisfied in virtue of the other two, as follows.

* See § 125.

Consider the variations along the singular integral in the immediate vicinity of the point of contact; the quantities dx and dy determine variations da and db along the singular integral, and conversely: and because $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$, these are subject to the relations

$$\frac{\partial^2 \phi}{\partial x^2} dx + \frac{\partial^2 \phi}{\partial x \partial y} dy + \frac{\partial^2 \phi}{\partial a \partial x} da + \frac{\partial^2 \phi}{\partial b \partial x} db = 0,$$

$$\frac{\partial^2 \phi}{\partial x \partial y} dx + \frac{\partial^2 \phi}{\partial y^2} dy + \frac{\partial^2 \phi}{\partial a \partial y} da + \frac{\partial^2 \phi}{\partial b \partial y} db = 0.$$

Multiplying the second equation by μ and subtracting from the first, we have

$$\left(\frac{\partial^2 \phi}{\partial a \partial x} - \mu \frac{\partial^2 \phi}{\partial a \partial y}\right) da + \left(\frac{\partial^2 \phi}{\partial b \partial x} - \mu \frac{\partial^2 \phi}{\partial b \partial y}\right) db = 0.$$

In order to take all directions through the point, we must keep dx and dy independent of one another; and therefore da and db are independent of one another, so that

$$\frac{\partial^2 \phi}{\partial a \partial x} - \mu \frac{\partial^2 \phi}{\partial a \partial y} = 0,$$

$$\frac{\partial^2 \phi}{\partial b \partial x} - \mu \frac{\partial^2 \phi}{\partial b \partial y} = 0.$$

Hence the equation

$$\frac{\partial^2 \phi}{\partial a \partial x} + b' \frac{\partial^2 \phi}{\partial b \partial x} = \mu \frac{\partial^2 \phi}{\partial a \partial y} + b' \mu \frac{\partial^2 \phi}{\partial b \partial y}$$

is satisfied for all values of b': and this holds in virtue of the equations

$$\frac{\partial^2 \phi}{\partial x^2} = \mu \frac{\partial^2 \phi}{\partial x \partial y}, \quad \frac{\partial^2 \phi}{\partial x \partial y} = \mu \frac{\partial^2 \phi}{\partial y^2}.$$

Consequently, *if one general integral has contact of the second order with the singular integral, all general integrals have contact of that order with the singular integral.*

Ex. 1. Prove that, if the equation

$$\frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2$$

is satisfied at any point on the singular integral, then the equation

$$\frac{\partial^2 \phi}{\partial a^2} \frac{\partial^2 \phi}{\partial b^2} = \left(\frac{\partial^2 \phi}{\partial a \partial b}\right)^2$$

is also satisfied: and conversely. (Darboux.)

Ex. 2. Discuss the preceding proposition in the text when $\frac{\partial^2 \phi}{\partial x \partial y}$ vanishes.

106. Now consider the order of contact between the complete integral and the singular integral at a point. In the vicinity of the point, variations along the complete integral are given by dx, dy, dz alone; for a and b are constants along the complete integral. As its equation is

$$\phi(x, y, z, a, b) = 0,$$

those variations are given by

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \tfrac{1}{2}\left\{\frac{\partial^2\phi}{\partial x^2}dx^2 + \dots\right\} = 0.$$

But $\dfrac{\partial\phi}{\partial x} = 0$ and $\dfrac{\partial\phi}{\partial y} = 0$ at the point in question: hence, accurately to the second order inclusive,

$$-2\frac{\partial\phi}{\partial z}dz = \frac{\partial^2\phi}{\partial x^2}dx^2 + 2\frac{\partial^2\phi}{\partial x\partial y}dx\,dy + \frac{\partial^2\phi}{\partial y^2}dy^2.$$

The expression on the right-hand side does not in general vanish for all values of dx and dy, except under very special conditions: hence *the contact between the complete integral and the singular integral is usually of the first order.* There are however two directions given by

$$\frac{dy}{dx} = t,$$

where t satisfies the quadratic

$$\frac{\partial^2\phi}{\partial x^2} + 2t\frac{\partial^2\phi}{\partial x\partial y} + t^2\frac{\partial^2\phi}{\partial y^2} = 0,$$

along which dz is of the third order; the point is usually a double point on the curve of intersection of the two surfaces, and these directions are the tangents to this curve at the point.

Special interest attaches to the case when these two directions coincide: the two surfaces are then said to *osculate.* In that case, we have

$$\frac{\partial^2\phi}{\partial x^2} = \mu\frac{\partial^2\phi}{\partial x\partial y}, \quad \frac{\partial^2\phi}{\partial x\partial y} = \mu\frac{\partial^2\phi}{\partial y^2};$$

and conversely, if these conditions hold, the two surfaces osculate. Taking account of the earlier result relating to general integrals, we have the theorem: *if the complete integrals do not osculate the*

singular integral, then no integral has contact of the second order with the singular integral: but if the complete integrals do osculate the singular integral, then all the general integrals have contact of the second order with the singular integral.

Ex. Prove that, if the complete integral osculates the singular integral, all the characteristics passing through the point of contact have the same tangent.

Illustrate this property by reference to the spheres that osculate a surface. (Darboux.)

SINGULARITIES ON THE CHARACTERISTICS.

107. In the preceding discussion of the surfaces and curves associated with the integrals of the differential equation

$$f(x, y, z, p, q) = 0,$$

we have been concerned mainly with regions that are devoid of singularities for those surfaces and curves; the effect of possible singularities must now be considered. Also, it will be convenient to consider at the same time the exceptional cases of the preceding investigations that were noted (§§ 85, 89) but not discussed. We shall proceed, as before, from the characteristics.

It may be assumed that the equation $f = 0$ has been transformed so as to be free from irrationalities; and we shall discount the loss of generality in assuming, as will be done, that f is a regular function of its arguments. Take the tangent plane to any integral surface as the plane $z = 0$, and the point of contact for origin: then at that point $p = 0$, $q = 0$; in the vicinity of the point, f is a regular function of x, y, z, p, q, which vanishes at the point. The equations of the characteristic are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ} = \frac{dp}{-(X + pZ)} = \frac{dq}{-(Y + qZ)} = dt;$$

we propose to discuss the form of the characteristic in the vicinity of the origin, according to the varieties of form of $f = 0$.

If P vanishes but not Q, the axes of x and y can be changed to another set such that neither the new P nor the new Q shall vanish at the point: similarly, if either $X + pZ$ or $Y + qZ$ should vanish but not both, a change can be made such that neither of

the quantities in the new position shall vanish. Also, taking account of earlier exceptions, we thus have the following cases:

I. All the denominators are different from zero at the point:

II. The quantities P and Q vanish at the point, but not the others:

III. The quantities $X+pZ$ and $Y+qZ$ vanish at the point, but not the others:

IV. All the denominators vanish at the point, but Z is not zero:

V. All the quantities X, Y, Z, P, Q vanish.

Of these, the first is included in order to make the set complete: it is the assumption that was made in the earlier investigations and, as there, it will be found to constitute the origin an ordinary point. The second has been left over from § 88, and the fifth from § 78; and the fourth has given a singular integral, if such an integral exists. Let

$$f = ax + by + cz + gp + hq + \dots .$$

108. *Case I.* As $X+pZ$, $Y+qZ$, P, Q do not vanish at the origin, a, b, g, h are not zero. Hence, assuming that t vanishes at the origin, we have, in the immediate vicinity,

$$\frac{dx}{g+\dots} = \frac{dy}{h+\dots} = \frac{-dp}{a+\dots} = \frac{-dq}{b+\dots} = dt,$$

$$dz = p\,dx + q\,dy,$$

and therefore

$$x = gt + \dots, \qquad y = ht + \dots,$$

$$p = -at + \dots, \quad q = -bt + \dots,$$

$$dz = -(ag+bh)\,t\,dt + \dots,$$

so that

$$z = -\tfrac{1}{2}(ag+bh)\,t^2 + \dots .$$

Hence, in the vicinity of the origin, the characteristic is given by

$$y = \frac{h}{g}x + \dots,$$

$$z = -\frac{ag+bh}{2g^2}x^2 + \dots;$$

the origin is an ordinary point on the characteristic, which touches the plane $z=0$ there. This, as already indicated, is the former result.

109. *Case II.* As P and Q vanish at the origin, we have

$$g=0, \quad h=0.$$

Let

$$f = ax + by + cz + (a'x + b'y + c'z)p + (a''x + b''y + c''z)q \\ + \tfrac{1}{2}(\alpha p^2 + 2\beta pq + \gamma q^2) + \phi(x, y, z, p, q),$$

where $\phi(x, y, z, p, q)$ contains terms of the second degree in x, y, z alone, and all other terms in all the quantities of higher degree in the aggregate. Two of the equations of the characteristic now are

$$\frac{-dp}{a+\dots} = \frac{-dq}{b+\dots} = dt,$$

so that

$$p = -at + \dots, \quad q = -bt + \dots,$$

in the vicinity of the origin; and the other equations are

$$\frac{dx}{dt} = a'x + b'y + c'z + \alpha p + \beta q + \dots,$$

$$\frac{dy}{dt} = a''x + b''y + c''z + \beta p + \gamma q + \dots,$$

$$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt}.$$

(A) First, suppose that neither of the quantities $\alpha a + \beta b$ and $\beta a + \gamma b$ vanishes; this is the most general case as it involves no restricting relation between constants. Then we have

$$x = -\tfrac{1}{2}(\alpha a + \beta b)t^2 + \dots, \quad y = -\tfrac{1}{2}(\beta a + \gamma b)t^2 + \dots,$$

$$\frac{dz}{dt} = (\alpha a^2 + 2\beta ab + \gamma b^2)t^2 + \dots,$$

so that

$$z = \tfrac{1}{3}(\alpha a^2 + 2\beta ab + \gamma b^2)t^3 + \dots;$$

and therefore, in the immediate vicinity of the origin, the characteristic is given by

$$y = \mu x + \dots, \quad z = \mu' x^{\frac{3}{2}} + \dots,$$

where μ and μ' are determinate constants. The origin is a cusp on the characteristic; the tangent at the cusp lies in the plane $z=0$, along the line $y=\mu x$.

It thus appears that, when $f=0$, $P=0$, $Q=0$, while $X+pZ$ and $Y+qZ$ are not zero, and when they allow the elimination of p and q between the three equations (which is the most general

case, as not involving relations among the equations), the surface given by the eliminant is a locus of cusps of the characteristics. Hence, *in general, the surface obtained by eliminating p and q between the equations*

$$f=0,\quad P=0,\quad Q=0,$$

is a locus of cusps of the characteristics; through every point of the surface there passes a characteristic having a cusp at that point.

If it should happen that $\alpha a^2+2\beta ab+\gamma b^2$ vanishes though $\alpha a+\beta b$ and $\beta a+\gamma b$ do not vanish, the only difference is that, in the vicinity of the origin,

$$y=\mu x+\dots,\quad z=\lambda x^2+\lambda_1 x^{\frac{5}{2}}+\dots;$$

the origin is still a singularity on the curve in general, of an order higher than a cusp.

Ex. Prove that the tangent plane of the characteristic developable is given by

$$z+atx+bty+\tfrac{1}{6}(\alpha a^2+2\beta ab+\gamma b^2)\,t^3=0,$$

keeping the most important terms, and that therefore the point is an ordinary point for the developable. (Darboux.)

(B) Next, suppose that one (but not both) of the two quantities $\alpha a+\beta b$, $\beta a+\gamma b$ vanishes: let

$$\beta a+\gamma b=0.$$

Proceeding as before, we have

$$\begin{aligned} x&=-\tfrac{1}{2}(\alpha a+\beta b)\,t^2+\dots,\\ y&=\lambda t^3+\dots,\\ z&=\tfrac{1}{3}(\alpha a+\beta b)\,at^3+\dots; \end{aligned}$$

and therefore, in the vicinity of the origin, we have

$$y=\mu z+\dots,\quad z=\mu x^{\frac{3}{2}}+\dots.$$

The origin is a cusp on the characteristic; the tangent to the cusp is the axis of y.

If $\lambda=0$, or if $a=0$, or if both λ and a vanish, the origin is still a singularity on the curve of an order higher than a cusp: the curve still touches the plane of z at the origin.

(C) Lastly, suppose that

$$\alpha a+\beta b=0,\quad \beta a+\gamma b=0:$$

then, in the most general case, that is subject to these conditions, we find

$$x = \lambda_1 t^3 + \ldots, \quad y = \lambda_2 t^3 + \ldots, \quad z = \lambda_3 t^4 + \ldots,$$

so that, in the vicinity of the origin,

$$y = \mu x + \ldots, \quad z = \mu x^{\frac{4}{3}} + \ldots.$$

The origin is a singularity on the characteristic: and the tangent to the characteristic lies in the plane of $z = 0$ along the line $y = \mu x$.

Similarly, for other special relations among the constants, we obtain a corresponding result: in every case, the equation obtained by eliminating p and q between $f = 0$, $P = 0$, $Q = 0$, when $X + pZ$ and $Y + qZ$ do not vanish, is a locus of singularities on the characteristics.

110. *Case III.* As $X + pZ$ and $Y + qZ$ vanish at the origin, we have

$$a = 0, \quad b = 0;$$

but P and Q do not vanish there, so that g and h are different from zero. Hence

$$\begin{aligned} f = cz + gp + hq + (a'x + b'y + c'z)\,p + (a''x + b''y + c''z)\,q \\ + \tfrac{1}{2}(\alpha p^2 + 2\beta pq + \gamma q^2) + \tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2) + \psi(x, y, z, p, q), \end{aligned}$$

where ψ contains terms of the third and higher orders. The equations are

$$\frac{dx}{dt} = g + \ldots, \quad \frac{dy}{dt} = h + \ldots,$$

so that

$$x = gt + \ldots, \quad y = ht + \ldots;$$

also

$$-\frac{dp}{dt} = (c + a')\,p + a''q + Ax + By + \ldots,$$

$$-\frac{dq}{dt} = b'p + (b'' + c)\,q + Bx + Cy + \ldots,$$

so that

$$p = -\tfrac{1}{2}(Ag + Bh)\,t^2 + \ldots,$$

$$q = -\tfrac{1}{2}(Bg + Ch)\,t^2 + \ldots.$$

Hence

$$\begin{aligned} \frac{dz}{dt} &= p\frac{dx}{dt} + q\frac{dy}{dt} \\ &= -\tfrac{1}{2}(Ag^2 + 2Bgh + Ch^2)\,t^2 + \ldots, \end{aligned}$$

and so

$$z = -\tfrac{1}{6}(Ag^2 + 2Bgh + Ch^2)\,t^3 + \ldots.$$

Consequently the characteristic, in the vicinity of the origin, is given by

$$y = \mu x + \ldots, \quad z = \rho x^3 + \ldots;$$

the point is an inflexion on the characteristic, and the inflexional tangent to the characteristic lies in the plane of z along the line $y = \mu x$.

Ex. Prove, by means of reciprocal polars applied to Case II or otherwise, that the surface, obtained by eliminating p and q between the equations

$$f=0, \quad X+pZ=0, \quad Y+qZ=0,$$

is a locus such that, at every point on it, the characteristic developable of the equation $f=0$ has a singular plane. Sketch the characteristic developable in the vicinity of the point. (Darboux.)

111. *Case IV.* Here we have

$$f=0, \quad P=0, \quad Q=0, \quad X+pZ=0, \quad Y+qZ=0,$$

while Z is not zero. These are five equations, which involve five quantities x, y, z, p, q: hence if f, already supposed regular, be a polynomial function of its arguments, there would in general be only a limited number of sets of values of the five variables, and therefore only a limited number of such points.

Taking the origin to be such a point, we assume (as before) that x, y, z, p, q all vanish there: hence

$$a=0, \quad b=0, \quad g=0, \quad h=0,$$

and the equations of the characteristic, in the immediate vicinity of the origin, are

$$\frac{dx}{dt} = a'x + b'y + c'z + \alpha p + \beta q + \ldots,$$

$$\frac{dy}{dt} = a''x + b''y + c''z + \beta p + \gamma q + \ldots,$$

$$\frac{dp}{dt} = Ax + By \qquad + (c+a')\,p + a''q + \ldots,$$

$$\frac{dq}{dt} = Bx + Cy \qquad + b'p + (b''+c)\,q + \ldots,$$

$$\frac{dz}{dt} = p\,\frac{dx}{dt} + q\,\frac{dy}{dt}.$$

As the only regular integrals of these equations, which vanish when $t=0$, are

$$x=0, \quad y=0, \quad z=0, \quad p=0, \quad q=0,$$

it is simpler to make x the independent variable of the characteristic and omit t completely. It is clear that z is of a higher order of small quantities than x or y in the vicinity of the origin; hence, retaining only the most important terms within this vicinity, the equations may be taken in the form

$$\frac{dy}{dx}=\frac{a''x+b''y+\beta p+\gamma q}{a'x+b'y+\alpha p+\beta q},$$

$$\frac{dp}{dx}=\frac{Ax+By+(c+a')\,p+a''q}{a'x+b'y+\alpha p+\beta q},$$

$$\frac{dq}{dx}=\frac{Bx+Cy+b'p+(b''+c)\,q}{a'x+b'y+\alpha p+\beta q}.$$

Then* there are integrals of these equations of the form

$$y=x\,(\rho+\eta),$$
$$p=x\,(\sigma+\pi),$$
$$q=x\,(\tau+\kappa),$$

where ρ, σ, τ are constants, and η, π, κ are functions of x that vanish with x. As regards the constants, they are given by the equations

$$\rho=\frac{a''+b''\rho+\beta\sigma+\gamma\tau}{a'+b'\rho+\alpha\sigma+\beta\tau},$$

$$\sigma=\frac{A+B\rho+(c+a')\,\sigma+a''\tau}{a'+b'\rho+\alpha\sigma+\beta\tau},$$

$$\tau=\frac{B+C\rho+b'\sigma+(b''+c)\,\tau}{a'+b'\rho+\alpha\sigma+\beta\tau}.$$

Writing

$$\theta=a'+b'\rho+\alpha\sigma+\beta\tau,$$

we have

$$\rho\theta=a''+b''\rho+\beta\sigma+\gamma\tau,$$
$$\sigma\theta=A+B\rho+(c+a')\,\sigma+a''\tau,$$
$$\tau\theta=B+C\rho+b'\sigma+(b''+c)\,\tau.$$

Hence θ is a root of the equation

$$\begin{vmatrix} a'-\theta, & b', & \alpha, & \beta \\ a'', & b''-\theta, & \beta, & \gamma \\ A, & B, & c+a'-\theta, & a'' \\ B, & C, & b', & b''+c-\theta \end{vmatrix}=0;$$

* See Part III of this treatise, chapters XI, XII.

when θ is known, any three of the earlier equations determine ρ, σ, τ.

For the quantities η, π, κ, we have

$$\rho + \eta + x\frac{d\eta}{dx} = \frac{\rho\theta + b''\eta + \beta\pi + \gamma\kappa}{\theta + b'\eta + \alpha\pi + \beta\kappa},$$

so that

$$\begin{aligned} x\frac{d\eta}{dx} &= -\eta + \frac{(b''-b'\rho)\eta + (\beta - \alpha\rho)\pi + (\gamma - \beta\rho)\kappa}{\theta + b'\eta + \alpha\pi + \beta\kappa} \\ &= \frac{1}{\theta}\{(b'' - b'\rho - \theta)\eta + (\beta - \alpha\rho)\pi + (\gamma - \beta\rho)\kappa\}, \end{aligned}$$

to the order of quantities retained; and similarly

$$x\frac{\partial\pi}{\partial x} = \frac{1}{\theta}\{(B - b'\sigma)\eta + (c + a' - \alpha\sigma - \theta)\pi + (a'' - \beta\sigma)\kappa\},$$

$$x\frac{\partial\kappa}{\partial x} = \frac{1}{\theta}\{(C - b'\tau)\eta + (b' - \alpha\tau)\pi + (b'' + c - \beta\tau - \theta)\kappa\};$$

and the quantities η, π, κ are to vanish with x. The characters of these functions depend upon the roots of the critical cubic

$$\begin{vmatrix} b''-b'\rho-\theta-\mu, & \beta-\alpha\rho, & \gamma-\beta\rho \\ B-b'\sigma, & c+a'-\alpha\sigma-\theta-\mu, & a''-\beta\sigma \\ C-b'\tau, & b'-\alpha\tau, & b''+c-\beta\tau-\theta-\mu \end{vmatrix} = 0$$

in μ: and these can be expressed in terms of the roots of the preceding quartic.

Let θ_1, θ_2, θ_3, θ_4 be the roots of that quartic: of these, suppose that θ_1 is the root of the quartic chosen, and that the corresponding values of ρ, σ, τ have been obtained. Multiply the columns of the quartic determinant by 1, ρ, σ, τ and subtract the sum of the last three from the first: the determinant is

$$\begin{vmatrix} \theta_1-\theta, & b', & \alpha, & \beta \\ \rho(\theta_1-\theta), & b''-\theta, & \beta, & \gamma \\ \sigma(\theta_1-\theta), & B, & c+a'-\theta, & a'' \\ \tau(\theta_1-\theta), & C, & b', & b''+c-\theta \end{vmatrix}.$$

Multiply the first row by ρ, σ, τ in turn and subtract the products from the second row, the third row, and the fourth row in respective succession: the determinant is

$$\begin{vmatrix} \theta_1-\theta, & b', & \alpha, & \beta \\ 0, & b''-b'\rho-\theta, & \beta-\alpha\rho, & \gamma-\beta\rho \\ 0, & B-b'\sigma, & c+a'-\alpha\sigma-\theta, & a''-\beta\sigma \\ 0, & C-b'\tau, & b'-\alpha\tau, & b''+c-\beta\tau-\theta \end{vmatrix},$$

and it is equal to the determinant in the quartic equation. Hence the roots of

$$\begin{vmatrix} b''-b'\rho-\theta, & \beta-\alpha\rho &, & \gamma-\beta\rho \\ B-b'\sigma \ , & c+a'-\alpha\sigma-\theta, & & a''-\beta\sigma \\ C-b'\tau \ , & b'-\alpha\tau &, & b''+c-\beta\tau-\theta \end{vmatrix}=0$$

are $\theta_2, \theta_3, \theta_4$. But the cubic in μ is

$$\begin{vmatrix} b''-b'\rho-\theta_1-\mu, & \beta-\alpha\rho &, & \gamma-\beta\rho \\ B-b'\sigma \quad , & c+a'-\alpha\sigma-\theta_1-\mu, & & a''-\beta\sigma \\ C-b'\tau \quad , & b'-\alpha\tau &, & b''+c-\beta\tau-\theta_1-\mu \end{vmatrix}=0;$$

hence the roots are

$$\mu+\theta_1=\theta_2, \theta_3, \theta_4,$$

that is, the roots of the cubic in μ are

$$\theta_2-\theta_1, \quad \theta_3-\theta_1, \quad \theta_4-\theta_1,$$

where θ_1 is the root of the quartic selected for the case under consideration.

We shall denote these roots by μ_1, μ_2, μ_3.

112. Of the various sub-cases, it will be sufficient to mention some of the more important.

(i) Let the three roots of the critical cubic be unequal to one another, no one of them being a positive integer; then the preceding equations possess integrals, expressing η, π, κ as unique regular functions of x which vanish with x. For this characteristic, we have

$$y=(\rho+\eta)x,$$
$$z=\tfrac{1}{2}(\sigma+\rho\tau)x^2+\ldots;$$

the point in question is an ordinary point on the curve, which has its tangent lying in the plane $z=0$ along the line $y=\rho x$.

Further integrals may be possessed by the equations, on the preceding assumption as to the roots: but the integrals are not regular functions of x.

If the real parts of μ_1, μ_2, μ_3 are positive and are such that no one of the quantities

$$(m_1-1)\mu_1+m_2\mu_2+m_3\mu_3+m_4,$$
$$m_1\mu_1+(m_2-1)\mu_2+m_3\mu_3+m_4,$$
$$m_1\mu_1+m_2\mu_2+(m_3-1)\mu_3+m_4,$$

vanishes for positive integer values m_1, m_2, m_3, m_4, such that $m_1+m_2+m_3+m_4 \geqslant 2$, the equations possess a triple infinitude of non-regular integrals that vanish with x: these integrals are regular functions of x, x^{μ_1}, x^{μ_2}, x^{μ_3}. There is thus a triple infinitude of curves through the point: it is easily seen to be a singularity on each of them.

If the real parts of μ_1 and μ_2 be positive, if that of μ_3 be negative, and if no one of the quantities

$$(m_1-1)\mu_1+m_2\mu_2+m_3, \quad m_1\mu_1+(m_2-1)\mu_2+m_3$$

vanishes for positive integer values m_1, m_2, m_3, such that

$$m_1+m_2+m_3 \geqslant 2,$$

the equations possess a double infinitude of non-regular integrals that vanish with x: these integrals are regular functions of x, x^{μ_1}, x^{μ_2}. There is then a double infinitude of curves through the origin: it is easily seen to be a singularity on each of them.

If the real part of μ_1 be positive, and if the real parts of μ_2 and μ_3 be negative, the equations possess a single infinitude of non-regular integrals that vanish with x; these integrals are regular functions of x and x^{μ_1}. There is then a single infinitude of curves through the origin; the point is easily seen to be a singularity on each of them.

If the real parts of μ_1, μ_2, μ_3 be each negative, the regular integrals first indicated are the only integrals of the equation which vanish with x. As already proved, the origin is then an ordinary point upon the sole characteristic, which passes through it touching the plane $z=0$ at the point.

(ii) Let the three roots of the critical cubic be unequal to one another, and let one (but only one) of them, say μ_1, be a positive integer.

Unless a particular condition among the coefficients in the equations be satisfied, the equations possess no regular integrals that vanish with x. When that condition is not satisfied, they possess a simple infinitude of non-regular integrals that vanish with x, being regular functions of x and $x \log x$, if the real parts of μ_2 and μ_3 are negative: they possess a double infinitude of non-regular integrals that vanish with x, being regular functions of x, $x \log x$, and x^{μ_2}, if the real part of μ_2 is positive and the real part of μ_3 is negative; they possess a triple infinitude of non-regular

integrals that vanish with x, being regular functions of x, $x\log x$, x^{μ_2}, x^{μ_3}, if the real parts of μ_2 and μ_3 are positive. Every curve passing through the point has the origin for a singularity.

When the particular condition among the coefficients in the equation is satisfied, the equations possess a simple infinitude of regular integrals that vanish with x: each of these gives a characteristic touching the plane $z=0$ along the same tangent $y=\rho x$ in that plane, and the point of contact is an ordinary point for each curve in the infinitude. Further, when the condition is satisfied, the equations possess either a double infinitude or a single infinitude of non-regular integrals according as the real parts of μ_2 and μ_3, or of only one of them, are positive, these integrals being regular functions of x, x^{μ_2} and x^{μ_3}, or of x and either x^{μ_2} or x^{μ_3}, according to the respective cases; for each of these curves, the origin is a singularity. But if the real parts of μ_2 and μ_3 are negative, the equations are devoid of non-regular integrals.

And so for other cases: the results depend upon the characters of the integrals of the equations for η, π, κ: and these characters are known by the critical conditions* as regards the roots μ_1, μ_2, μ_3. Regular integrals make the origin an ordinary point on the corresponding characteristics: non-regular integrals make the origin a singularity on the corresponding characteristics.

113. At the beginning of the discussion, it was pointed out that the five equations

$$f=0,\quad P=0,\quad Q=0,\quad X+pZ=0,\quad Y+qZ=0$$

will, in general, give a finite number of determinations of sets of values of the variables involved: so that, in general, there will be a finite number of points in space at which the characteristics must be considered for the present purpose. The preceding discussion is typical of the discussion to be effected at each such point for any particular equation of the assumed character.

But the five equations may be of such a form, or they may be so related, that they do not determine a set of values or a limited

* These are set out for equations of the form in question in n variables, in § 187, chapter XII, vol. III, of this treatise. They were proved in full detail for all the cases, that arise in a system of two equations, in my memoir, *On the integrals of systems of differential equations*, published in the Stokes 1899 Commemoration volume (vol. XVIII, 1900) of the Transactions of the Cambridge Philosophical Society.

number of sets of values for the five variables involved. In that event, there are three possible alternatives, always on the adopted hypothesis that the equation $f=0$ is irreducible:

(a) they may determine four of the variables in terms of the remaining one, say, y, z, p, q, in terms of x; there then is a curve-locus in space, and values of p and q are associated with each point on the curve:

(b) they may determine three of the variables in terms of the remaining two, say z, p, q in terms of x, y; there then is a surface-locus in space, and values of p and q are associated with each point on the surface:

(c) they may determine two of the variables in terms of the remaining three, say p, q in terms of x, y, z; values of p and q are then associated with each point of space.

The third of these alternatives has already been discussed (§§ 78, 79): substitution of p and q in

$$dz = p\,dx + q\,dy,$$

followed by a quadrature, leads to a surface integral of the equation. A characteristic through any point on such a surface lies in the surface. The second of these alternatives leads to a singular integral of the equation, unless $Z=0$; we shall come to the consideration of the characteristics through a point on the surface after the discussion of the first alternative, which can be discussed briefly after the earlier analysis.

114. Suppose, then, that the five equations

$$f=0, \quad P=0, \quad Q=0, \quad X+pZ=0, \quad Y+qZ=0$$

determine a curve-locus

$$y=u(x), \quad z=v(x),$$

together with values of p and q as functions of x in the form

$$p=\pi(x), \quad q=\kappa(x).$$

Take any point on this locus, say $x=x_0$; transfer the origin to that point, and let

$$\pi(x_0)=\pi_0, \quad \kappa(x_0)=\kappa_0.$$

Then the form of f must be such that the five equations give

$$y=0, \quad z=0, \quad p=\pi_0, \quad q=\kappa_0,$$

when $x=0$. As may easily be verified, we have

$$\begin{aligned} f = c(z - x\pi_0 - y\kappa_0) &+ \text{higher powers of } x, y, z \\ &+ (p-\pi_0)\, R(x, y, z, p-\pi_0, q-\kappa_0) \\ &+ (q-\kappa_0)\, S(x, y, z, p-\pi_0, q-\kappa_0), \end{aligned}$$

where R and S are regular functions of their arguments: these functions must vanish when $x=0$, $y=0$, $z=0$, $p=\pi_0$, $q=\kappa_0$: and there must be limitations on the form of f sufficient to make the five equations equivalent to four only, though the precise form of the limitation is not necessary for the present purpose. What is required is the nature of this point as a point on the characteristic.

The point is the origin: as usual, we take the tangent plane to the integral surface to be $z=0$, so that $p=0$, $q=0$ at the point on the characteristic. Let

$$\begin{aligned} \lambda_0 &= R(0, 0, 0, -\pi_0, -\kappa_0) - \pi_0 R_0' - \kappa_0 S_0', \\ \mu_0 &= S(0, 0, 0, -\pi_0, -\kappa_0) - \pi_0 R_0'' - \kappa_0 S_0'', \end{aligned}$$

where R_0', R_0'', S_0', S_0'' are the values of $\dfrac{\partial R}{\partial p}, \dfrac{\partial R}{\partial q}, \dfrac{\partial S}{\partial p}, \dfrac{\partial S}{\partial q}$, when x, y, z, p, q all are made zero: then, in the vicinity of the origin along the characteristic, we have

$$\frac{dx}{dt} = \lambda_0 + \dots, \quad \frac{dy}{dt} = \mu_0 + \dots,$$

as two of the equations. Also, let R_1, R_2, S_1, S_2 denote the values of $\dfrac{\partial R}{\partial x}, \dfrac{\partial R}{\partial y}, \dfrac{\partial S}{\partial x}, \dfrac{\partial S}{\partial y}$, when x, y, z, p, q all vanish: then two other equations of the characteristic are

$$-\frac{dp}{dt} = -c\pi_0 - \pi_0 R_1 - \kappa_0 S_1 = \rho_0, \text{ say},$$

$$-\frac{dq}{dt} = -c\kappa_0 - \pi_0 R_2 - \kappa_0 S_2 = \sigma_0, \text{ say},$$

in the immediate vicinity of the origin. Hence in that vicinity,

$$\begin{aligned} x = \lambda_0 t + \dots, &\quad y = \mu_0 t + \dots, \\ z = -\tfrac{1}{2}&(\rho_0\lambda_0 + \mu_0\sigma_0)\, t^2 + \dots \end{aligned}$$

are the most important terms in the equations of the characteristic. It is clear that, unless λ_0 and μ_0 both vanish, the origin is an ordinary point on the characteristic, and that its tangent lies in the plane $z=0$ along the line $\lambda_0 y - \mu_0 x = 0$.

The inference cannot be made if $\lambda_0 = 0$ and $\mu_0 = 0$. The further consideration, in this event, will not be undertaken; in the complicated analysis that would be necessary, particularly as regards the forms of the integrals of the differential equations, the details would be substantially similar to those which have been given for the case when the five equations determine sets of values for the five variables.

115. Coming to the remaining alternative, in which the five equations determine z, p, q, as functions of x and y, and assuming (as has been assumed throughout) that Z does not vanish for such relations, we know (§ 78) that the equations define a singular integral; and our quest is the examination of the characteristics at and near any point on this integral surface.

Let the singular integral be given by $z = \theta(x, y)$: then, when a new variable ζ is defined by the relation $z - \theta(x, y) = \zeta$, the singular integral will be given by $\zeta = 0$, as in § 104. Hence we may take the plane $z = 0$ as the singular integral. Moreover, Z does not vanish on account of $z = 0$ and it does not vanish identically: consequently, we may take the differential equation in a resolved form

$$f = z - \phi(x, y, p, q) = 0.$$

Take any point on the singular integral

$$z = 0,$$

and make it the origin; moreover, as z is steadily zero along the singular integral, the associated values of p and q are

$$p = 0, \quad q = 0.$$

The singular integral, thus defined by

$$z = 0, \quad p = 0, \quad q = 0,$$

is to satisfy the equations

$$f = 0, \quad \frac{\partial \phi}{\partial p} = 0, \quad \frac{\partial \phi}{\partial q} = 0, \quad \frac{\partial \phi}{\partial x} - p = 0, \quad \frac{\partial \phi}{\partial y} - q = 0,$$

that is, $\phi, \dfrac{\partial \phi}{\partial p}, \dfrac{\partial \phi}{\partial q}, \dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}$ must vanish when $p = 0$, $q = 0$; and therefore ϕ is of the form

$$\begin{aligned}\phi = p^2 (\alpha + ax + a'y + Ap + A'q + \ldots) \\ + 2pq (\beta + bx + b'y + \ldots) \\ + q^2 (\gamma + cx + c'y + Cp + C'q + \ldots).\end{aligned}$$

We require the equations of the characteristics in the immediate vicinity of the point $x=0$, $y=0$, $z=0$ on the singular integral: in differential form, they are

$$\frac{dx}{2\alpha p + 2\beta q + \dots} = \frac{dy}{2\beta p + 2\gamma q + \dots} = \frac{dp}{p + \dots} = \frac{dq}{q + \dots},$$

integrals of which for the immediate vicinity of the origin are

$$\begin{aligned} p &= \kappa q + \dots, \\ x &= 2(\alpha\kappa + \beta) q + \dots, \\ y &= 2(\beta\kappa + \gamma) q + \dots, \end{aligned}$$

κ being an arbitrary constant. Also, as

$$\begin{aligned} dz &= p\,dx + q\,dy \\ &= 2(\alpha\kappa^2 + 2\beta\kappa + \gamma) q\,dq + \dots, \end{aligned}$$

we have

$$z = (\alpha\kappa^2 + 2\beta\kappa + \gamma) q^2 + \dots.$$

Hence the equations of the characteristic in the immediate vicinity of the origin are

$$\left.\begin{aligned} (\alpha\kappa + \beta) y &= (\beta\kappa + \gamma) x \\ 4(\alpha\gamma - \beta^2) z &= \gamma x^2 - 2\beta xy + \alpha y^2 \end{aligned}\right\},$$

where κ is an arbitrary quantity: and the solution is satisfactory unless $\alpha\gamma - \beta^2$ vanishes. Now the curve touches $z=0$ at the origin, and the tangent to the curve lies in this plane along the line

$$y = \frac{\beta\kappa + \gamma}{\alpha\kappa + \beta} x;$$

when κ is arbitrary, the quantity $\dfrac{\beta\kappa + \gamma}{\alpha\kappa + \beta}$ also is arbitrary unless $\alpha\gamma - \beta^2$ vanishes: and therefore a characteristic can be drawn touching every line in the plane. We thus have the former result:—

When an equation has a singular integral, then through any point on it there passes an infinitude of characteristics unless

$$\frac{\partial^2 f}{\partial p^2}\frac{\partial^2 f}{\partial q^2} - \left(\frac{\partial^2 f}{\partial p \partial q}\right)^2$$

vanishes at the point; each characteristic touches the singular integral there and has the point for an ordinary point, and no two characteristics have the same tangent. Moreover, all these

characteristics lie on a surface which touches the singular integral and has the point of contact for an ordinary point. For, in order to obtain this integral surface from the equations of the characteristics, all that needs (§ 84) to be done is to make the initial values satisfy the equations

$$f=0, \quad dz=p\,dx+q\,dy.$$

In the present case, both of these equations are satisfied at the point by taking

$$\alpha\kappa^2+2\beta\kappa+\gamma=0.$$

Hence, in the immediate vicinity of the point of intersection of the surface with the singular integral, we have

$$\begin{aligned} x &= 2(\alpha\kappa+\beta)\,q+\ldots, \\ y &= 2(\beta\kappa+\gamma)\,q+\ldots, \\ z &= \lambda q^3+\ldots, \end{aligned}$$

on the surface; and therefore

$$\kappa x+y=0, \quad z=\mu x^3+\ldots;$$

the point of intersection is an ordinary point on the surface, and the two surfaces touch there.

If however $\alpha\gamma-\beta^2=0$, without α, β, γ separately vanishing, we may take

$$\beta=\alpha\theta, \quad \gamma=\alpha\theta^2.$$

We still have an infinitude of characteristics through the point, given by the equations

$$\begin{aligned} p &= \kappa q+\ldots, \\ x &= 2(\kappa+\theta)\,\alpha q+\ldots, \\ y &= 2\theta(\kappa+\theta)\,\alpha q+\ldots, \\ z &= \alpha(\kappa+\theta)^2 q^2+\ldots; \end{aligned}$$

they touch the plane $z=0$ at the origin, and the tangents are given by

$$y=\theta x,$$

that is, the infinitude of characteristics have a common tangent at the origin: and as

$$4\alpha z=x^2+\ldots,$$

the origin is an ordinary point for each of the characteristics.

This infinitude of characteristics still lies on a surface (which of course gives an integral of the differential equation). To

obtain its equation from the equations of the characteristics, we make the initial values satisfy the equations

$$f=0, \quad dz = p\,dx + q\,dy.$$

In the present case, both of these equations are satisfied at the point by taking

$$\kappa + \theta = 0.$$

With these initial values, we have, for the characteristics,

$$\begin{aligned} p &= \kappa q + \ldots, \\ \frac{dx}{dq} &= \{3A\kappa^2 + (2A' + 4B)\kappa + 2B' + C\}\, q + \ldots \\ &= \lambda q + \ldots, \\ \frac{dy}{dq} &= \{(A' + 2B)\kappa^2 + (4B' + 2C)\kappa + 3C'\}\, q + \ldots \\ &= \mu q + \ldots, \end{aligned}$$

and therefore

$$\begin{aligned} x &= \tfrac{1}{2}\lambda q^2 + \ldots, \\ y &= \tfrac{1}{2}\mu q^2 + \ldots, \\ z &= \tfrac{1}{3}(\kappa\lambda + \mu)\, q^3 + \ldots. \end{aligned}$$

Hence, at the point of intersection of the singular integral and the surface that is the locus of the characteristics, we have

$$\begin{aligned} y &= \rho x + \ldots, \\ z &= \sigma x^{\frac{3}{2}} + \ldots, \end{aligned}$$

along the surface: and therefore the point is a singularity on the surface that is the locus of the characteristics. The two surfaces touch at the point which, on the assumptions implicitly made that neither λ nor μ nor $\kappa\lambda + \mu$ vanishes, is a cusp on the locus containing the characteristics through the point.

Even if α, β, γ be zero, the same kind of result holds: for it is sufficient in that event to make $\kappa = 0$: we merely have different values of λ and μ. Hence we have the result:

When an equation has a singular integral, and when the relation

$$\frac{\partial^2 f}{\partial p^2}\frac{\partial^2 f}{\partial q^2} = \left(\frac{\partial^2 f}{\partial p \partial q}\right)^2$$

is satisfied at any point of it, an infinitude of characteristics passes through the point having a common tangent that also touches (or lies

in) the singular integral. All the characteristics lie on a surface which touches the singular integral at the point and has the point for a singularity.

Ex. 1. Prove that, when an equation $f(x, y, z, p, q)=0$ possesses a singular integral in the form

$$z=\phi(x, y),$$

such that $\frac{\partial f}{\partial z}$ does not vanish at all points on the surface represented by that integral, the general integral which touches the singular integral along a curve can be represented in the form

$$z=\phi(x, y)+\zeta^2,$$

where ζ is a regular function of x and y, and the complete integral can be represented in the form

$$z=\phi(x, y)+u^2+v^2,$$

where u and v are regular functions of x and y. (Darboux.)

Ex. 2. Shew that the characteristics of the equation

$$(pz-x)^2=q^2(x^2+z^2-1)$$

are plane curves, and that the locus of their cusps is

$$x^2+z^2=1. \qquad \text{(Goursat.)}$$

Ex. 3. Discuss the various integrals of the equation

$$\{p(x^2+z^2-1)+qxy\}^2=q^2(1-z^2)(x^2+y^2+z^2-1).$$

(Goursat.)

116. *Case V.* Here we have

$$f=0, \quad P=0, \quad Q=0, \quad X=0, \quad Y=0, \quad Z=0.$$

These are six equations involving five variables: hence they can coexist only if connected by certain relations. If so connected, they may be equivalent to five equations, or to four, or to three, or to two: on the assumption that $f=0$ is irreducible, they cannot all be satisfied in virtue of one equation alone. We shall assume that f is a polynomial function of its arguments.

When the six equations are equivalent to five, the equations determine a limited number of sets of values for the five variables. Taking any one of these, we have to consider the form of the characteristic at the point. As before, we take the point for origin, and the tangent plane of the complete integral is made the plane $z=0$: so that we have

$$x=0, \quad y=0, \quad z=0, \quad p=0, \quad q=0$$

at the point. It is clear that there can be no terms of the first order in f: so that the constant c, of § 110, is zero. In other

respects the analysis of §§ 111, 112 applies in the present sub-case: and the nature of the point on the characteristic is the same as for the corresponding alternatives in that discussion.

When the six equations are equivalent to four, determining y, z, p, q as functions of x, the discussion of § 111 will suffice for the present sub-case. When they are equivalent to two only, their significance is similar to that of the corresponding equations already (§ 109) discussed.

It remains therefore to discuss them when they are equivalent to three equations only, expressing z, p, q as functions of x and y: we have seen that it was not possible* definitely to declare that the relation between x, y, z is a singular integral, because of the vanishing of Z. The method of § 104 is not applicable: for the equation $f=0$ cannot be resolved with regard to z because $Z=0$: indeed, it cannot be resolved with regard to any of the variables so as to give a regular equation because X, Y, Z. P, Q all vanish. Suppose that

$$f=0, \quad P=0, \quad Q=0,$$

are three independent equations giving z, p, q as functions of x and y: and that these values make X, Y, Z all vanish. When the values are substituted, they make $f=0$, $P=0$, $Q=0$ satisfied identically: hence†

$$\frac{\partial P}{\partial x}+\frac{\partial P}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial P}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial x}=0,$$

$$\frac{\partial P}{\partial y}+\frac{\partial P}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial P}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}=0.$$

If then $\dfrac{\partial P}{\partial z}$ does not vanish when the values of z, p, q are substituted in it, it will be sufficient that the equations

$$\frac{\partial P}{\partial x}+\frac{\partial P}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial P}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial P}{\partial z}p=0,$$

$$\frac{\partial P}{\partial y}+\frac{\partial P}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial P}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial P}{\partial z}q=0,$$

shall be satisfied in order to secure that the relation between x, y, z is an integral of the partial equation. The equation $Q=0$

* Except, of course, by direct substitution in the partial differential equation.

† The equation $f=0$ gives no further information when thus treated, because P, Q, X, Y, Z all vanish for the values in question.

can be treated similarly, and will give a similar result if $\frac{\partial Q}{\partial z}$ does not vanish for the values in question.

But the obvious tests, which really are the equivalent of substituting in the differential equation, are that values of p, q, z should satisfy

$$p = \frac{\partial}{\partial x}(z), \qquad q = \frac{\partial}{\partial y}(z):$$

if these relations are satisfied, the relation between x, y, z is an integral.

The subject, in the case when $Z = 0$, admits of considerable expansion: but it will not be pursued further in this place. Darboux has given some discussion* of a limited class of equations; there seems plenty of opening for further investigation.

Ex. 1. Consider the equation

$$f = (px + qy - z)^2 - p^2 - q^2 + \frac{z^2}{x^2 + y^2 - 1} = 0,$$

which already (§ 78) has been discussed. The equations

$$f = 0, \quad X = 0, \quad Y = 0, \quad Z = 0, \quad P = 0, \quad Q = 0$$

are satisfied in virtue of the two relations

$$\frac{p}{x} = \frac{q}{y} = \frac{z}{x^2 + y^2 - 1};$$

and these lead to an integral

$$z^2 = a^2 (x^2 + y^2 - 1),$$

which is not a singular integral as it certainly is not the envelope of the complete integrals of the equation. But, though it satisfies

$$Z = 0,$$

it is an integral of the original equation.

Ex. 2. Consider the equation

$$64 (p^2 + q^2)^3 = 27 z^2.$$

The equations

$$f = 0, \quad X = 0, \quad Y = 0, \quad Z = 0, \quad P = 0, \quad Q = 0$$

are satisfied by a relation

$$z = 0$$

(with $p = 0$, $q = 0$), which is an integral: and the complete integral is

$$\{(x - a)^2 + (y - b)^2\}^3 = 27 z^4.$$

What is the relation between the two integrals?

* In § 35 of his memoir.

Ex. 3. Consider the equation

$$a^2\{(p-1)^2+(q-1)^2\}=z^2.$$

The six equations

$$f=0, \quad X=0, \quad Y=0, \quad Z=0, \quad P=0, \quad Q=0$$

are satisfied by

$$z=0, \quad p-1=0, \quad q-1=0:$$

but $z=0$ is clearly not an integral of the partial differential equation.

The same holds of any equation

$$f(p+a,\ q+b)=z^m,$$

where the constant m is greater than unity, a and b are constants which do not vanish, and f is a regular function of its arguments containing no terms of order lower than two in $p+a$ and $q+b$ combined. All the six equations are satisfied by

$$z=0, \quad p+a=0, \quad q+b=0:$$

but $z=0$ is not an integral of the equation.

Ex. 4. Discuss the relation of the locus $z=0$ to the complete integral and to the general integral of

$$a^2\{(p-1)^2+(q-1)^2\}=z^2.$$

Ex. 5. Shew that all the integrals of

$$pq=z^2,$$

which touch the integral $z=0$ along the axis of y, are given by

$$z=Ax^n\frac{e^{(n^2+4xy)^{\frac{1}{2}}}}{\{n+(n^2+4xy)^{\frac{1}{2}}\}^n},$$

where A and n are arbitrary. (Darboux.)

Ex. 6. Obtain the complete integral and the general integral of the equation in the preceding example. Is $z=0$ a singular integral?

Ex. 7. Integrate the equations:

(i) $p^2+q^2+2zpq=z^2$:

(ii) $p^3+q^3+z(p+q)=z^2$:

(iii) $pq(p+q)+z(p^2+q^2)=z^2$,

discussing, for each of them, the relations between the integral $z=0$ and the other integrals.

CHAPTER VIII.

THE METHOD OF CHARACTERISTICS IN ANY NUMBER OF INDEPENDENT VARIABLES.

THE present chapter gives an account of Cauchy's method of characteristics as applied to a single equation in n independent variables: and the account is made brief, because the process and the results are a generalisation of the process and the results for two independent variables, as expounded at considerable length in the two preceding chapters. Moreover, as the geometry of ordinary space has been amply used for illustration of the simpler case, it is not deemed necessary to enter at any length into illustrations of the more general case derived from the hypergeometry of $n+1$ dimensions.

Reference may be made to the works of Cauchy and of Darboux, quoted at the beginning of chapter VI. Many of the results towards the end of this chapter are believed to be new: and the subject admits of considerable development.

117. The method of characteristics can be employed when there are n independent variables $x_1, \ldots, x_n$. Adopting Cauchy's use of Ampère's practice, we change the independent variables so that they become $x_1, u_2, \ldots, u_n$: the new variables are functions of $x_2, \ldots, x_n$ (and, it may be, of x_1 also) which are independent of one another, and they will be chosen so as to simplify relations. Conversely, $x_2, \ldots, x_n, z, p_1, \ldots, p_n$ can be regarded as functions of $x_1, u_2, \ldots, u_n$; and, whatever the differential equation may be, we have

$$\frac{\partial z}{\partial x_1} = p_1 + \sum_{r=2}^{n} p_r \frac{\partial x_r}{\partial x_1},$$

$$\frac{\partial z}{\partial u_i} = \sum_{r=2}^{n} p_r \frac{\partial x_r}{\partial u_i}, \qquad (i = 2, \ldots, n).$$

Differentiating the former with regard to u_i and the latter with regard to x_1, and subtracting, we find

$$\frac{\partial p_1}{\partial u_i} = \sum_{r=2}^{n} \left(\frac{\partial p_r}{\partial x_1} \frac{\partial x_r}{\partial u_i} - \frac{\partial p_r}{\partial u_i} \frac{\partial x_r}{\partial x_1} \right),$$

holding for $i = 2, \ldots, n$.

When proper values of $x_2, \ldots, x_n, z, p_1, \ldots, p_n$, in terms of $x_1, u_2, \ldots, u_n$, are substituted in the differential equation, which may be taken in the form

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0,$$

it becomes an identity: hence, if

$$\frac{\partial f}{\partial z} = Z, \quad \frac{\partial f}{\partial x_i} = X_i, \quad \frac{\partial f}{\partial p_j} = P_j,$$

for i and $j = 1, \ldots, n$, we have

$$X_1 + Z\frac{\partial z}{\partial x_1} + \sum_{s=2}^{n} X_s \frac{\partial x_s}{\partial x_1} + \sum_{i=1}^{n} P_i \frac{\partial p_i}{\partial x_1} = 0,$$

$$Z\frac{\partial z}{\partial u_\mu} + \sum_{s=2}^{n} X_s \frac{\partial x_s}{\partial u_\mu} + \sum_{i=1}^{n} P_i \frac{\partial p_i}{\partial u_\mu} = 0,$$

the latter holding for $\mu = 2, \ldots, n$. Substituting in the latter for $\dfrac{\partial z}{\partial u_\mu}$ and $\dfrac{\partial p_1}{\partial u_\mu}$, and rearranging the terms, we find

$$\sum_{r=2}^{n} \left\{\left(X_r + Zp_r + P_1\frac{\partial p_r}{\partial x_1}\right)\frac{\partial x_r}{\partial u_\mu}\right\} + \sum_{r=2}^{n} \left\{\left(P_r - P_1\frac{\partial x_r}{\partial x_1}\right)\frac{\partial p_r}{\partial u_\mu}\right\} = 0.$$

Thus far, the new variables $u_2, \ldots, u_n$ are at our disposal: let them, if possible, be chosen so that

$$P_r - P_1\frac{\partial x_r}{\partial x_1} = 0,$$

for $r = 2, \ldots, n$, these $n-1$ equations being formally independent of one another. On the choice thus made, the foregoing equation becomes

$$\sum_{r=2}^{n} \left\{\left(X_r + Zp_r + P_1\frac{\partial p_r}{\partial x_1}\right)\frac{\partial x_r}{\partial u_\mu}\right\} = 0;$$

and it holds for $\mu = 2, \ldots, n$. There is thus a set of $n-1$ equations, homogeneous and linear in $n-1$ quantities; the determinant of their coefficients, being

$$J\left(\frac{x_2, \ldots, x_n}{u_2, \ldots, u_n}\right),$$

does not vanish, and therefore the quantities themselves vanish, that is,

$$X_r + Zp_r + P_1\frac{\partial p_r}{\partial x_1} = 0,$$

for $r=2, \ldots, n$. Substituting the values of $X_2, \ldots, X_n$ thus given, and also the value of $\frac{\partial z}{\partial x_1}$, in the equation

$$X_1 + Z\frac{\partial z}{\partial x_1} + \sum_{s=2}^{n} X_s \frac{\partial x_s}{\partial x_1} + \sum_{i=1}^{n} P_i \frac{\partial p_i}{\partial x_1} = 0,$$

and reducing, we have

$$X_1 + Zp_1 + P_1 \frac{\partial p_1}{\partial x_1} = 0.$$

Consequently, the equations

$$P_r - P_1 \frac{\partial x_r}{\partial x_1} = 0, \quad X_i + Zp_i + P_1 \frac{\partial p_i}{\partial x_1} = 0,$$

for $r = 2, \ldots, n$, and $i = 1, \ldots, n$, are satisfied: it will be noticed that they involve no derivatives with regard to $u_2, \ldots, u_n$.

Now this aggregate of $2n-1$ equations can be taken in the form

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dp_1}{-(X_1 + Zp_1)} = \ldots = \frac{dp_n}{-(X_n + Zp_n)},$$

which are a set of ordinary equations; so far as they are concerned, the arbitrary quantities that arise in the integration can be made functions of the variables $u_2, \ldots, u_n$, which do not occur explicitly in the set. If we equate each member of the above aggregate to

$$\frac{dz}{p_1P_1 + \ldots + p_nP_n},$$

the equation

$$\frac{\partial z}{\partial x_1} = p_1 + \sum_{r=2}^{n} p_r \frac{\partial x_r}{\partial x_1}$$

will be satisfied by the integrals of the set; no limitation will thereby be imposed upon the arbitrary functions of $u_2, \ldots, u_n$ that occur in the integrals. But the equations

$$\frac{\partial z}{\partial u_\mu} = \sum_{r=2}^{n} p_r \frac{\partial x_r}{\partial u_\mu}$$

have also to be satisfied; these will obviously impose limitations upon the arbitrary functions of $u_2, \ldots, u_n$.

118. Accordingly, we take the equations in the form

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{p_1P_1 + \ldots + p_nP_n}$$
$$= \frac{dp_1}{-(X_1 + Zp_1)} = \ldots = \frac{dp_n}{-(X_n + Zp_n)} = dt,$$

introducing a new variable t so as to secure the general symmetry in Darboux's presentation of Cauchy's method: t is the independent variable for the system of ordinary equations. Let

$$z=\zeta,\quad x_1=\xi_1,\ \ldots,\ x_n=\xi_n,\quad p_1=\pi_1,\ \ldots,\ p_n=\pi_n,$$

be a set of initial values assumed by the variables in the aggregate of ordinary equations, subject at this stage to the sole condition

$$f(\xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n)=0.$$

We shall suppose that not all the quantities $P_1, \ldots, P_n, X_1+Zp_1, \ldots, X_n+Zp_n$ vanish for these initial values and that each of these quantities is a regular function of its arguments in the vicinity of the initial values. From the theory of ordinary equations, it is known that the equations possess a unique set of integrals which are regular functions of t and are such that, when $t=0$, they acquire the assigned initial values respectively: let these integrals be

$$\begin{aligned}
x_1 &= x_1\,(t,\ \xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n &= x_n\,(t,\ \xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n),\\
z &= z\ \ (t,\ \xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n),\\
p_1 &= p_1\,(t,\ \xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
p_n &= p_n\,(t,\ \xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n).
\end{aligned}$$

When in the expressions thus obtained we make the quantities $\xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n$ functions of $u_2, \ldots, u_n$, at present arbitrary subject solely to the condition

$$f(\xi_1,\ \ldots,\ \xi_n,\ \zeta,\ \pi_1,\ \ldots,\ \pi_n)=0,$$

it is necessary that they should satisfy the relation

$$\frac{\partial z}{\partial t}=p_1\frac{\partial x_1}{\partial t}+\ldots+p_n\frac{\partial x_n}{\partial t},$$

(which the differential equations in the ordinary system shew to be satisfied), and the relations

$$\frac{\partial z}{\partial u_\mu}=p_1\frac{\partial x_1}{\partial u_\mu}+\ldots+p_n\frac{\partial x_n}{\partial u_\mu},$$

for $\mu=2, \ldots, n$, if the above integrals thus modified are to provide integrals of the original partial equation. Let

$$L_\mu=\frac{\partial z}{\partial u_\mu}-p_1\frac{\partial x_1}{\partial u_\mu}-\ldots-p_n\frac{\partial x_n}{\partial u_\mu};$$

then, as

$$\frac{\partial z}{\partial t} = p_1 \frac{\partial x_1}{\partial t} + \dots + p_n \frac{\partial x_n}{\partial t},$$

so that

$$\frac{\partial^2 z}{\partial t \partial u_\mu} = \sum_{r=1}^{n} \left(p_r \frac{\partial^2 x_r}{\partial t \partial u_\mu} + \frac{\partial p_r}{\partial u_\mu} \frac{\partial x_r}{\partial t} \right),$$

we have

$$\begin{aligned}\frac{\partial L_\mu}{\partial t} &= \frac{\partial^2 z}{\partial t \partial u_\mu} - \sum_{r=1}^{n} \left(p_r \frac{\partial^2 x_r}{\partial t \partial u_\mu} + \frac{\partial p_r}{\partial t} \frac{\partial x_r}{\partial u_\mu} \right) \\ &= \sum_{r=1}^{n} \left(\frac{\partial p_r}{\partial u_\mu} \frac{\partial x_r}{\partial t} - \frac{\partial x_r}{\partial u_\mu} \frac{\partial p_r}{\partial t} \right) \\ &= \sum_{r=1}^{n} \left\{ P_r \frac{\partial p_r}{\partial u_\mu} + (X_r + p_r Z) \frac{\partial x_r}{\partial u_\mu} \right\}.\end{aligned}$$

The quantities must satisfy the equation

$$f(x_1, \dots, x_n, z, p_1, \dots, p_n) = 0,$$

when their values are substituted: hence

$$Z \frac{\partial z}{\partial u_\mu} + \sum_{r=1}^{n} \left(X_r \frac{\partial x_r}{\partial u_\mu} + P_r \frac{\partial p_r}{\partial u_\mu} \right) = 0,$$

and therefore

$$\begin{aligned}\frac{\partial L_\mu}{\partial t} &= -Z \frac{\partial z}{\partial u_\mu} + \sum_{r=1}^{n} p_r Z \frac{\partial x_r}{\partial u_\mu} \\ &= -Z L_\mu,\end{aligned}$$

so that

$$L_\mu = \Lambda_\mu e^{-\int_0^t Z\,dt},$$

where Λ_μ is the value of L_μ, when $t = 0$. Now Z is a regular function of t in the vicinity of $t = 0$, so that $\int_0^t Z dt$ is finite. Consequently, in order to satisfy the relation

$$L_\mu = 0,$$

it is necessary and sufficient that the relation

$$\Lambda_\mu = 0$$

should be satisfied, that is,

$$\frac{\partial \zeta}{\partial u_\mu} = \pi_1 \frac{\partial \xi_1}{\partial u_\mu} + \dots + \pi_n \frac{\partial \xi_n}{\partial u_\mu};$$

and this must hold for $\mu = 2, \dots, n$. We thus have $n - 1$ further conditions imposed upon the quantities $\xi_1, \dots, \xi_n, \zeta, \pi_1, \dots, \pi_n$, regarded as functions of $u_2, \dots, u_n$.

Further, when these conditions are satisfied, and when the quantities are substituted in $f(x_1, \ldots, x_n, z, p_1, \ldots, p_n)=0$, the equation is satisfied identically. For we have

$$\begin{aligned}\frac{\partial f}{\partial t} &= Z\frac{\partial z}{\partial t} + \sum_{r=1}^{n} X_r \frac{\partial x_r}{\partial t} + \sum_{r=1}^{n} P_r \frac{\partial p_r}{\partial t}\\ &= 0,\end{aligned}$$

from the differential equations that led to the construction of the variables as functions of $t, u_2, \ldots, u_n$. Also, as $L_\mu = 0$, we have

$$\frac{\partial z}{\partial u_\mu} = p_1 \frac{\partial x_1}{\partial u_\mu} + \ldots + p_n \frac{\partial x_n}{\partial u_\mu};$$

and as now

$$\frac{\partial L_\mu}{\partial t} = 0,$$

and as always

$$\frac{\partial L_\mu}{\partial t} = \sum_{r=1}^{n} \left\{P_r \frac{\partial p_r}{\partial u_\mu} + (X_r + p_r Z)\frac{\partial x_r}{\partial u_\mu}\right\},$$

we now have

$$\sum_{r=1}^{n} \left(P_r \frac{\partial p_r}{\partial u_\mu} + X_r \frac{\partial x_r}{\partial u_\mu}\right) + Z\sum_{r=1}^{n} p_r \frac{\partial x_r}{\partial u_\mu} = 0,$$

that is,

$$Z\frac{\partial z}{\partial u_\mu} + \sum_{r=1}^{n} \left(P_r \frac{\partial p_r}{\partial u_\mu} + X_r \frac{\partial x_r}{\partial u_\mu}\right) = 0,$$

and therefore

$$\frac{\partial f}{\partial u_\mu} = 0.$$

Thus

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial u_2} = 0, \ \ldots, \ \frac{\partial f}{\partial u_n} = 0,$$

and therefore

$$\begin{aligned}f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) &= \text{constant}\\ &= f(\xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n)\\ &= 0.\end{aligned}$$

Consequently, the expressions obtained for $x_1, \ldots, x_n, z, p_1, \ldots, p_n$ satisfy the equation

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$$

identically when their values are substituted, provided only that the relations

$$\Lambda_2 = 0, \ \ldots, \ \Lambda_n = 0$$

are satisfied. Moreover, they are such that

$$\frac{\partial z}{\partial t} = \sum_{r=1}^{n} p_r \frac{\partial x_r}{\partial t},$$

$$\frac{\partial z}{\partial u_\mu} = \sum_{r=1}^{n} p_r \frac{\partial x_r}{\partial u_\mu};$$

and therefore, if $z, p_1, \ldots, p_n$ can be expressed in terms of $x_1, \ldots, x_n$ alone, on the elimination of $t, u_2, \ldots, u_n$, the n quantities p are the derivatives of z with regard to the n variables x, that is, the relation between $z, x_1, \ldots, x_n$ provides an integral of the original partial equation. It therefore is sufficient that the initial quantities $\xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n$ should satisfy the $n-1$ relations

$$\frac{\partial \zeta}{\partial u_\mu} = \sum_{r=1}^{n} \pi_r \frac{\partial \xi_r}{\partial u_\mu},$$

together with the condition

$$f(\xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n) = 0;$$

and then the integral will be given by the elimination of the parametric quantities among the $n+1$ equations

$$\begin{aligned} x_1 &= x_1(t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= x_n(t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ z &= z(t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n). \end{aligned}$$

119. The conditions as regards the initial values may be satisfied in various ways.

In the first of these ways, all the quantities ζ and ξ involve $u_2, \ldots, u_n$; the $n-1$ relations, together with

$$f(\xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n) = 0,$$

become n equations for the determination of $\pi_1, \ldots, \pi_n$. These n equations can be resolved for these magnitudes, unless

$$\begin{vmatrix} \Pi_1, & \ldots, & \Pi_n \\ \dfrac{\partial \xi_1}{\partial u_2}, & \ldots, & \dfrac{\partial \xi_n}{\partial u_2} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial \xi_1}{\partial u_n}, & \ldots, & \dfrac{\partial \xi_n}{\partial u_n} \end{vmatrix}$$

should vanish, where

$$\Pi_r = \frac{\partial f(\xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n)}{\partial \pi_r},$$

so that Π_r is the initial value of P_r. We shall assume that this quantity does not vanish, recognising that an exceptional case occurs when $\Pi_1, \ldots, \Pi_n$ all vanish.

As the quantities $\zeta, \xi_1, \ldots, \xi_n$ involve the $n-1$ variables $u_2, \ldots, u_n$, then on the elimination of $u_2, \ldots, u_n$, we have two relations which may be represented in the form

$$\phi(\xi_1, \ldots, \xi_n) = 0,$$
$$g(\xi_1, \ldots, \xi_n) = \zeta:$$

in other words, the initial conditions are such that, when a relation

$$\phi(x_1, \ldots, x_n) = 0$$

is satisfied, z is to acquire a value $g(x_1, \ldots, x_n)$. All the requirements are now satisfied, without any further restrictions: so that g can be taken quite arbitrarily, as also can ϕ.

The integral thus obtained is the *general integral.*

In the second of the ways, some (but not all) of the quantities $\zeta, \xi_1, \ldots, \xi_n$ involve the $n-1$ variables $u_2, \ldots, u_n$. Suppose that $\xi_{i+1}, \ldots, \xi_n$ do not involve any of the variables: then we clearly have

$$\xi_{i+1} = a_{i+1}, \ldots, \xi_n = a_n,$$

where the quantities $a_{i+1}, \ldots, a_n$ are constants. The relations are

$$\frac{\partial \zeta}{\partial u_\mu} = \sum_{r=1}^{i} \pi_r \frac{\partial \xi_r}{\partial u_\mu},$$

for $\mu = 2, \ldots, n$; these shew that some functional relation exists in a form

$$\zeta = g(\xi_1, \ldots, \xi_i),$$

and they determine the values of $\pi_1, \ldots, \pi_i$: and then $\pi_{i+1}, \ldots, \pi_n$ can be taken as arbitrary functions of $u_2, \ldots, u_n$, subject to the equation

$$f(\xi_1, \ldots, \xi_i, a_{i+1}, \ldots, a_n, \zeta, \pi_1, \ldots, \pi_n) = 0.$$

All the requirements now are satisfied, without further restrictions; so that the function g can be taken quite arbitrarily. The initial conditions are such that, when

$$x_{i+1} = a_{i+1}, \ldots, x_n = a_n,$$

z is to acquire a value $g(x_1, \ldots, x_i)$.

The integral is of the general type: clearly it is a specialised case of the general integral.

In the third of the ways, all the quantities ζ, $\xi_1, \ldots, \xi_n$ may be independent of the variables: they then are constants. Among the $n+2$ equations

$$\begin{aligned} x_1 &= x_1(t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= x_n(t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ z &= z\ (t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ 0 &= f\ (\ \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \end{aligned}$$

we eliminate the $n+1$ quantities $t, \pi_1, \ldots, \pi_n$. The eliminant is an integral.

As there are $n+1$ constants, which are values of the variables, one of them may be looked upon as an initial value, and the rest of them may be regarded as arbitrary. The integral so obtained is a *complete integral*.

It is clear that the integral, provided by the second of the ways, is intermediate in character between the complete integral and the general integral. If in its expression, $i = n-1$, it effectively is the general integral; if, on the other hand, $i = 0$, it effectively is the complete integral.

As regards the possibility of giving (or even expecting) an explicit form, by the elimination of $t, u_2, \ldots, u_n$ among the $n+1$ equations

$$\begin{aligned} x_1 &= x_1\ (t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= x_n(t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \\ z &= z\ \ (t, \xi_1, \ldots, \xi_n, \zeta, \pi_1, \ldots, \pi_n), \end{aligned}$$

we may resolve the first n of the equations with regard to $t, u_2, \ldots, u_n$, and substitute the values so obtained in the last: the resulting form will be the integral. This process is theoretically possible, unless the quantity

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial t}, \ldots, \dfrac{\partial x_n}{\partial t} \\ \dfrac{\partial x_1}{\partial u_2}, \ldots, \dfrac{\partial x_n}{\partial u_2} \\ \ldots\ldots\ldots \\ \dfrac{\partial x_1}{\partial u_n}, \ldots, \dfrac{\partial x_n}{\partial u_n} \end{vmatrix}$$

vanishes identically, that is, unless the quantity

$$\begin{vmatrix} P_1, & \dots, & P_n \\ \frac{\partial x_1}{\partial u_2}, & \dots, & \frac{\partial x_n}{\partial u_2} \\ \dots & \dots & \dots \\ \frac{\partial x_1}{\partial u_n}, & \dots, & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

vanishes identically. In assuming this, we make no new assumption: for if it vanishes identically, its value, when $t = 0$, is zero: and this has already been assumed not to be the fact.

120. Various assumptions have been made which, as in the case of only two independent variables, restrict the application of the theorem and the process.

Thus, it has been assumed that the quantities $P_1, \dots, P_n$, $X_1 + p_1 Z, \dots, X_n + p_n Z$ are regular functions of their arguments within the vicinity of the assigned initial values. If, therefore, any, or some, or all, of these quantities are characterised by deviations from regularity, whether by singularities or by algebraic irrationalities or by places of indeterminate values (to mention only the more familiar examples), then the theorems relating to a set of ordinary equations no longer apply of necessity: and the further inferences are then not necessarily valid.

Again, it has been recognised that, if $P_1, \dots, P_n$ vanish simultaneously for the initial values, the argument is not completely effective: as in the case of two independent variables, deviation from regularity can be caused thereby: and the equations require further consideration.

Again, it has been assumed that $P_1, \dots, P_n$, $X_1 + p_1 Z, \dots$, $X_n + p_n Z$ do not simultaneously vanish for values of the variables connected by the relation $f = 0$; but instances are known in which this assumption is not justified, the equations

$$f = 0, \quad P_1 = 0, \dots, P_n = 0, \quad X_1 + p_1 Z = 0, \dots, X_n + p_n Z = 0,$$

being consistent with one another. In such an event, there are two alternatives. Either the quantities $p_1, \dots, p_n$ can be eliminated, and the eliminant is a relation between $z, x_1, \dots, x_n$: this relation then provides the *singular integral* of the equation. Or, though the elimination is not possible, the equations are satisfied ($f = 0$

being irreducible) by proper values for at least two of the quantities p in terms of the remainder: let there be m such equations, where $m \leqslant n$; then these equations form a complete system, and they have an integral involving $n-m+1$ constants, which integral is a specialised case of the complete integral.

Moreover, there is no guarantee at any stage that every possible integral of the equation can be derived by the processes adopted: and it has, in fact, been found to be the case that a partial equation can be satisfied by an integral of the type called *special*, not falling within any of the indicated classes.

121. As in the case of two independent variables, so in the case of n independent variables, one exceptional instance requires consideration. It may happen that, though no one of the quantities $P_1, \ldots, P_n$ vanishes, still the relation

$$p_1 P_1 + \ldots + p_n P_n = 0$$

might be satisfied: it would, for example, be satisfied if f were homogeneous in the derivatives p. One of the integrals of the ordinary equations then would be

$$\begin{aligned} z &= \text{quantity independent of } t \\ &= z_0, \end{aligned}$$

a relation which would be useless for purposes of elimination if the complete integral were being sought.

In such an event, we adopt (as in the case of two variables in the corresponding event) a Legendrian transformation of the type

$$z' = z - p_1 x_1,$$

or of some similar type. For the particular transformation, the associated variables are

$$\begin{gathered} x_2' = x_2, \ \ldots, \ x_n' = x_n, \quad x_1' = -p_1, \\ p_2' = p_2, \ \ldots, \ p_n' = p_n, \quad p_1' = x_1: \end{gathered}$$

the quantity $p_1' P_1' + \ldots + p_n' P_n'$ in the transformed system is then obtained from

$$x_1 (X_1 + p_1 Z) + p_2 P_2 + \ldots + p_n P_n,$$

that is, from

$$x_1 (X_1 + p_1 Z) - p_1 P_1,$$

by making the above substitutions. This quantity does not vanish, and so the process can be applied to the modified system; the integral of the original equation can be deduced as before.

This general method of integration has been illustrated for the case of two independent variables: it is unnecessary to illustrate it in detail for the general case.

Ex. Integrate, by Cauchy's method, the equations

(i) $p_1 p_2 \dots p_n = x_1 x_2 \dots x_n$;

(ii) $(p_1 - z)(p_2 - z)\dots(p_n - z) = p_1 p_2 \dots p_n$;

(iii) $p_1 x_1 + \dots + p_n x_n = p_1 p_2 \dots p_n$;

obtaining in each case an integral z which acquires an assigned value $\phi(x_2, \dots, x_n)$, when $x_1 = a_1$.

122. The ordinary equations subsidiary to the integration of the partial equation can also be obtained as follows. In space of $n+1$ dimensions, the integral represents a hypersurface, which can be regarded as the envelope of its tangent planes. The equation of any tangent plane is

$$\zeta - z = p_1(\xi_1 - x_1) + \dots + p_n(\xi_n - x_n);$$

when the envelope of this plane is formed, subject to the law

$$f(x_1, \dots, x_n, z, p_1, \dots, p_n) = 0,$$

we have

$$0 = (\xi_1 - x_1)\,\delta p_1 + \dots + (\xi_n - x_n)\,\delta p_n,$$
$$0 = P_1 \delta p_1 + \dots + P_n \delta p_n,$$

and therefore

$$\frac{\xi_1 - x_1}{P_1} = \dots = \frac{\xi_n - x_n}{P_n},$$

that is, in the vicinity of the point

$$\frac{dx_1}{P_1} = \dots = \frac{dx_n}{P_n},$$
$$= dt,$$

say, giving equations for a direction through the point. Moreover, as this direction belongs to an integral which satisfies the equation, the equation $f = 0$ will be satisfied identically when the proper values of $z, p_1, \dots, p_n$ are inserted: so that

$$X_r + Zp_r + P_1 \frac{\partial p_1}{\partial x_r} + \dots + P_n \frac{\partial p_n}{\partial x_r} = 0,$$

for $r = 1, \dots, n$. Thus

$$(X_r + Zp_r)\,dt + \frac{\partial p_1}{\partial x_r} dx_1 + \dots + \frac{\partial p_n}{\partial x_r} dx_n = 0.$$

Now

$$\frac{\partial p_n}{\partial x_r} = \frac{\partial^2 z}{\partial x_r \partial x_n} = \frac{\partial p_r}{\partial x_n};$$

and therefore

$$\frac{\partial p_1}{\partial x_r} dx_1 + \ldots + \frac{\partial p_n}{\partial x_r} dx_n = \frac{\partial p_r}{\partial x_1} dx_1 + \ldots + \frac{\partial p_r}{\partial x_n} dx_n$$
$$= dp_r,$$

so that

$$(X_r + Zp_r)\, dt + dp_r = 0,$$

for $r = 1, \ldots, n$. Also

$$dz = p_1 dx_1 + \ldots + p_n dx_n;$$

hence, gathering together the various equations, we have

$$\frac{dx_1}{P_1} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{p_1 P_1 + \ldots + p_n P_n} = \frac{-dp_1}{X_1 + p_1 Z} = \ldots = \frac{-dp_n}{X_n + p_n Z}$$
$$= dt,$$

which are the equations in question.

Next, consider the various integrals. There is a complete integral, which may be taken in the form

$$\phi\,(x_1, \ldots, x_n, z, a_1, \ldots, a_n) = 0.$$

The least restricted general integral is obtained by eliminating the n constants among the $n+1$ equations

$$\phi = 0, \quad a_1 = \theta\,(a_2, \ldots, a_n),$$
$$\frac{d\phi}{da_r} = \frac{\partial \phi}{\partial a_r} + \frac{\partial \phi}{\partial a_1}\frac{\partial \theta}{\partial a_r} = 0,$$

for $r = 2, \ldots, n$; it will be a single equation, and it represents the envelope of that family of complete surfaces selected by the relation

$$a_1 = \theta\,(a_2, \ldots, a_n).$$

In the uneliminated form, the equations represent a locus of one dimension, which is the intersection of n consecutive surfaces obtained by varying the $n-1$ independent parameters in

$$\phi\,(x_1, \ldots, x_n, z, \theta, a_2, \ldots, a_n) = 0.$$

On the analogy of ordinary space, such a curve is called a *characteristic*: clearly the general integral is a locus of characteristics. As a characteristic is a locus of one dimension, it can be represented by a set of ordinary equations, which are easily found as follows to be the preceding set.

The differential equation is obtained by the elimination of $a_1, \ldots, a_n$ among the $n+1$ equations

$$\left.\begin{aligned} \phi = \phi(x_1, \ldots, x_n, z, a_1, \ldots, a_n) = 0 \\ \phi_r = \frac{\partial \phi}{\partial x_r} + p_r \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\},$$

for $r = 1, \ldots, n$; and it is the sole equation resulting from that elimination. The only independent relations, that connect differential elements $dx_1, \ldots, dx_n, dz, dp_1, \ldots, dp_n$, are

$$d\phi = 0, \quad d\phi_1 = 0, \ldots, d\phi_n = 0;$$

and $df = 0$ is a relation connecting those elements: hence quantities $\mu, \mu_1, \ldots, \mu_n$, free from differential relations, must exist such that the relation

$$df = \mu d\phi + \mu_1 d\phi_1 + \ldots + \mu_n d\phi_n$$

is satisfied: and therefore

$$f = \mu\phi + \mu_1\phi_1 + \ldots + \mu_n\phi_n,$$

because $f, \phi, \phi_1, \ldots, \phi_n$ vanish together.

Because no one of the constants $a_1, \ldots, a_n$ appears in f, we have

$$0 = \mu \frac{\partial \phi}{\partial a_r} + \mu_1 \frac{\partial \phi_1}{\partial a_r} + \ldots + \mu_n \frac{\partial \phi_n}{\partial a_r},$$

for $r = 1, \ldots, n$. Again, p_s occurs in ϕ_s only and in no one of the other quantities ϕ; hence

$$\frac{\partial f}{\partial p_s} = \mu_s \frac{\partial \phi}{\partial z},$$

for $s = 1, \ldots, n$. For the integral in question, we have

$$\frac{\partial f}{\partial x_i} = \mu \frac{\partial \phi}{\partial x_i} + \sum_{j=1}^{n} \mu_j \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} + p_j \frac{\partial^2 \phi}{\partial x_i \partial z} \right),$$

$$\frac{\partial f}{\partial z} = \mu \frac{\partial \phi}{\partial z} + \sum_{j=1}^{n} \mu_j \left(\frac{\partial^2 \phi}{\partial x_j \partial z} + p_j \frac{\partial^2 \phi}{\partial z^2} \right),$$

and therefore

$$\frac{\partial f}{\partial x_i} + p_i \frac{\partial f}{\partial z} = \sum_{j=1}^{n} \mu_j \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} + p_j \frac{\partial^2 \phi}{\partial x_i \partial z} + p_i \frac{\partial^2 \phi}{\partial x_j \partial z} + p_i p_j \frac{\partial^2 \phi}{\partial z^2} \right),$$

for $i = 1, \ldots, n$.

The integral equations of the characteristic are

$$\phi = 0, \quad a_1 = \theta(a_2, \ldots, a_n),$$

$$\psi_\kappa = \frac{\partial\phi}{\partial a_\kappa} + \frac{\partial\phi}{\partial a_1}\frac{\partial a_1}{\partial a_\kappa} = 0,$$

for $\kappa = 2, \ldots, n$. From the preceding equations, we have

$$\mu\left(\frac{\partial\phi}{\partial a_r} + \frac{\partial a_1}{\partial a_r}\frac{\partial\phi}{\partial a_1}\right) + \mu_1\left(\frac{\partial\phi_1}{\partial a_r} + \frac{\partial a_1}{\partial a_r}\frac{\partial\phi_1}{\partial a_1}\right) + \ldots + \mu_n\left(\frac{\partial\phi_n}{\partial a_r} + \frac{\partial a_1}{\partial a_r}\frac{\partial\phi_n}{\partial a_1}\right) = 0;$$

therefore, along a characteristic,

$$\mu_1\left(\frac{\partial\phi_1}{\partial a_r} + \frac{\partial a_1}{\partial a_r}\frac{\partial\phi_1}{\partial a_1}\right) + \ldots + \mu_n\left(\frac{\partial\phi_n}{\partial a_r} + \frac{\partial a_1}{\partial a_r}\frac{\partial\phi_n}{\partial a_1}\right) = 0,$$

that is,

$$\mu_1\left(\frac{\partial\psi_r}{\partial x_1} + p_1\frac{\partial\psi_r}{\partial z}\right) + \ldots + \mu_n\left(\frac{\partial\psi_r}{\partial x_n} + p_n\frac{\partial\psi_r}{\partial z}\right) = 0;$$

and this holds for $r = 2, \ldots, n$. Now $\psi_r = 0$ holds permanently along the characteristic, so that

$$\frac{\partial\psi_r}{\partial x_1}dx_1 + \ldots + \frac{\partial\psi_r}{\partial x_n}dx_n + \frac{\partial\psi_r}{\partial z}dz = 0;$$

and

$$dz = p_1 dx_1 + \ldots + p_n dx_n$$

for every curve on $\phi = 0$: thus

$$\left(\frac{\partial\psi_r}{\partial x_1} + p_1\frac{\partial\psi_r}{\partial z}\right)dx_1 + \ldots + \left(\frac{\partial\psi_r}{\partial x_n} + p_n\frac{\partial\psi_r}{\partial z}\right)dx_n = 0;$$

and this holds for $r = 2, \ldots, n$. The $n-1$ equations for the ratios of $\mu_1, \ldots, \mu_n$ are exactly the same as the $n-1$ equations for the ratios of $dx_1, \ldots, dx_n$; hence

$$\frac{dx_1}{\mu_1} = \ldots = \frac{dx_n}{\mu_n} = u,$$

say, where $dx_1, \ldots, dx_n$ are elements of the characteristic. Consequently,

$$u\left(\frac{\partial f}{\partial x_i} + p_i\frac{\partial f}{\partial z}\right) = \sum_{j=1}^{n}\left(\frac{\partial^2\phi}{\partial x_i\partial x_j} + p_j\frac{\partial^2\phi}{\partial x_i\partial z} + p_i\frac{\partial^2\phi}{\partial x_j\partial z} + p_i p_j\frac{\partial^2\phi}{\partial z^2}\right)dx_j$$

$$= d\left(\frac{\partial\phi}{\partial x_i} + p_i\frac{\partial\phi}{\partial z}\right) - \frac{\partial\phi}{\partial z}dp_i$$

$$= -\frac{\partial\phi}{\partial z}dp_i,$$

because the relation

$$\frac{\partial \phi}{\partial x_i} + p_i \frac{\partial \phi}{\partial z} = 0$$

holds permanently in connection with the equation. Also

$$\begin{aligned} u \frac{\partial f}{\partial p_s} &= u\mu_s \frac{\partial \phi}{\partial z} \\ &= \frac{\partial \phi}{\partial z} dx_s . \end{aligned}$$

Hence, gathering together the various equations that are satisfied along the characteristic, we have

$$\begin{aligned} \frac{dx_1}{\dfrac{\partial f}{\partial p_1}} = \dots = \frac{dx_n}{\dfrac{\partial f}{\partial p_n}} = \frac{dz}{p_1 \dfrac{\partial f}{\partial p_1} + \dots + p_n \dfrac{\partial f}{\partial p_n}} &= \frac{-dp_1}{\dfrac{\partial f}{\partial x_1} + p_1 \dfrac{\partial f}{\partial z}} = \dots \\ &= \frac{-dp_n}{\dfrac{\partial f}{\partial x_n} + p_n \dfrac{\partial f}{\partial z}} = \frac{u}{\dfrac{\partial \phi}{\partial z}} . \end{aligned}$$

These are the former subsidiary equations, which accordingly are the differential equations of the characteristics.

Ex. 1. Prove that the envelope of the amplitudes

$$\tfrac{1}{4} z^2 = (a - x_1)(a - x_2)(a - x_3),$$

where a is a variable parameter, is a general integral of the differential equation

$$z = p_1 x_1 + p_2 x_2 + p_3 x_3 + p_1 p_2 p_3 ;$$

and find the relation among the arbitrary constants in the complete integral which leads to this general integral.

Ex. 2. An amplitude of one dimension is given by the equations

$$x_1 + bc = x_2 + ca = x_3 + ab,$$
$$z + 2abc = 0, \quad a + b + c = 0 :$$

find the general form of the partial differential equation of the first order for which this amplitude can be a characteristic, and verify that the equation in the preceding example is a particular case.

Contact of the Integrals.

123. The complete integral is an amplitude of n dimensions, represented by

$$\phi(x_1, \dots, x_n, z, a_1, \dots, a_n) = 0.$$

The general integral is an amplitude also of n dimensions, obtained as the result of eliminating $a_2, \ldots, a_n$ between the equations

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0,$$

on taking $a_1 = \theta(a_2, \ldots, a_n)$: the n equations represent an amplitude of one dimension, being the characteristic: and various loci, of dimensions of all orders between unity and n, are given by the elimination of the various sets of constants that can be selected from $a_2, \ldots, a_n$. And there are various classes of general integrals: that general integral, which is represented by means of the foregoing equations, is the most comprehensive of them all.

The singular integral (when it exists) is an amplitude also of n dimensions, obtained as the result of eliminating $a_1, \ldots, a_n$ between the equations

$$\phi = 0, \quad \frac{\partial\phi}{\partial a_1} = 0, \quad \frac{\partial\phi}{\partial a_2} = 0, \ldots, \frac{\partial\phi}{\partial a_n} = 0.$$

The values of $p_1, \ldots, p_n$ at any position on the complete integral are given by

$$\frac{\partial\phi}{\partial x_r} + p_r \frac{\partial\phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$.

The values of $p_1, \ldots, p_n$ at any position on the general integral are given by

$$\frac{\partial\phi}{\partial x_r} + p_r \frac{\partial\phi}{\partial z} + \sum_{s=2}^{n} \frac{d\phi}{da_s}\frac{\partial a_s}{\partial x_r} = 0,$$

that is, by

$$\frac{\partial\phi}{\partial x_r} + p_r \frac{\partial\phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$.

The values of $p_1, \ldots, p_n$ at any position on the singular integral are given by

$$\frac{\partial\phi}{\partial x_r} + p_r \frac{\partial\phi}{\partial z} + \sum_{t=1}^{n} \frac{\partial\phi}{\partial a_t}\frac{\partial a_t}{\partial x_r} = 0,$$

that is, by

$$\frac{\partial\phi}{\partial x_r} + p_r \frac{\partial\phi}{\partial z} = 0,$$

for $r = 1, \ldots, n$.

Hence, at any position common to any two of these three integrals, the values of $p_1, \ldots, p_n$ are the same. Regarding $z, x_1, \ldots,$

$x_n, p_1, \ldots, p_n$ as defining an element of an integral of the differential equation, we can express this last result in the form that, *at any position common to any two of the three amplitudes represented by the complete integral, a general integral, and the singular integral, the two amplitudes have a common element.*

Moreover, the equations

$$\phi = 0, \quad \frac{\partial \phi}{\partial a_1} = 0, \ldots, \frac{\partial \phi}{\partial a_n} = 0$$

usually determine one set of values, or a limited number of sets of values, of $z, x_1, \ldots, x_n$, in terms of $a_1, \ldots, a_n$: that is, the number of positions common to the complete integral and the singular integral is limited and, as has just been proved, the two amplitudes have a common element at each common position. Also, a set of values of $a_1, \ldots, a_n$ determines a position on the singular integral.

Again, a relation

$$a_1 = \theta(a_2, \ldots, a_n)$$

determines an amplitude of one dimension within the singular integral; and at every position on this amplitude within the singular integral, the equations

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0$$

are satisfied. These are the equations of the general integral, which accordingly contains the amplitude. Hence the general integral and the singular integral have an amplitude of one dimension in common and, as has been proved, the two integrals have a common element at every position on this amplitude.

The complete integral and a general integral have a characteristic in common, being an amplitude of one dimension. Moreover, the equations of the characteristic determine the relations between z, x, p uniquely from initial values, except when those initial values belong to singularities: hence *if two complete integrals have an element in common at any position, they have in common all the elements along the characteristic through the position.*

Among the special loci to be considered is the amplitude which is the envelope of all the characteristics on any general integral. Taking two consecutive characteristics, we have

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0,$$

along one of them; along the other, we have

$$\phi + \frac{d\phi}{da_2}\,da_2 + \ldots + \frac{d\phi}{da_n}\,da_n = 0,$$

$$\frac{d^2\phi}{da_r\,da_2}\,da_2 + \ldots + \frac{d^2\phi}{da_r\,da_n}\,da_n = 0,$$

the latter holding for $r = 2, \ldots, n$. At positions common to both, the first of the second set of equations is satisfied by means of the first set of equations; eliminating the ratios $da_2 : da_3 : \ldots : da_n$ from the rest, we have

$$\begin{vmatrix} \dfrac{d^2\phi}{da_2^2}, & \ldots, & \dfrac{d^2\phi}{da_2 da_n} \\ \cdots & \cdots & \cdots \\ \dfrac{d^2\phi}{da_2 da_n}, & \ldots, & \dfrac{d^2\phi}{da_n^2} \end{vmatrix} = 0,$$

say $H(\phi) = 0$. Thus the required envelope is given by the elimination of $a_2, \ldots, a_n$ among the equations

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0, \quad H(\phi) = 0;$$

it is an amplitude of $n-1$ dimensions. The $n+1$ equations usually determine a set of values, or a limited number of sets of values, of $z, x_1, \ldots, x_n$ in terms of $a_2, \ldots, a_n$; hence the number of positions common to a complete integral and the envelope of the characteristics on a general integral is limited: and, of course, they are isolated positions on the amplitude of one dimension, along which the complete integral and the general integral have elements in common.

Again, consider a locus intermediate in dimensions between those of a characteristic and a general integral: such an one is obtained, for example, by the elimination of $a_2, \ldots, a_r$ between the equations

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0.$$

The result of the elimination will consist of $n-(r-1)$ equations; and therefore it will represent an amplitude of r dimensions in the hyperspace under consideration. Its equations involve the $n-r$ constants $a_{r+1}, \ldots, a_n$. To find its envelope, we take a consecutive amplitude: at any position on the first, we have

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0:$$

at any position on the second, we have

$$\phi + \frac{d\phi}{da_{r+1}} da_{r+1} + \ldots + \frac{d\phi}{da_n} da_n = 0,$$

$$\frac{d^2\phi}{da_s da_{r+1}} da_{r+1} + \ldots + \frac{d^2\phi}{da_s da_n} da_n = 0,$$

for $s = 2, \ldots, n$. At a position on the envelope, the first of the latter set is satisfied by the earlier equations: eliminating da_{r+1}, $\ldots$, da_n, we have

$$\left\| \begin{array}{ccc} \dfrac{d^2\phi}{da_2 da_{r+1}}, & \ldots, & \dfrac{d^2\phi}{da_n da_{r+1}} \\ \cdots & \cdots & \cdots \\ \dfrac{d^2\phi}{da_2 da_n}, & \ldots, & \dfrac{d^2\phi}{da_n^2} \end{array} \right\| = 0.$$

These equations, which are equivalent to r independent equations in general, together with

$$\phi = 0, \quad \frac{d\phi}{da_2} = 0, \ldots, \frac{d\phi}{da_n} = 0,$$

give the envelope: it is an amplitude of $n - r + 1$ dimensions.

124. It is not proposed to make the complete generalisation of all the properties in ordinary space associated with partial differential equations in two independent variables: only one other property will be generalised here. We shall consider the order of contact in which a common element is possessed by different integrals.

Assuming that the singular integral exists, we know that it is given by the single equation which results from the elimination of $a_1, \ldots, a_n$ among the equations

$$\phi(x_1, \ldots, x_n, z, a_1, \ldots, a_n) = 0,$$

$$\frac{\partial\phi}{\partial a_1} = 0, \ldots, \frac{\partial\phi}{\partial a_n} = 0.$$

Let this single equation be supposed resolved with regard to z, with the result

$$z = \psi(x_1, \ldots, x_n);$$

and introduce a new dependent variable ζ, defined by the relation

$$\zeta = z - \psi(x_1, \ldots, x_n).$$

The complete integral is now

$$\phi(x_1, \ldots, x_n, \zeta + \psi, a_1, \ldots, a_n) = 0;$$

the derivatives with regard to $a_1, \ldots, a_n$ are unaffected. Hence, when we make the elimination as before and resolve this new eliminant with regard to ζ, the same resolution as before leads to

$$\zeta = 0.$$

We therefore may take the singular integral in the form

$$z = 0.$$

It is true that this equation has arisen out of the resolution of another equation, and that therefore it may not (and generally will not) represent the singular integral in the whole of its extent: but the immediate purpose is the discussion of the closeness of possession of an element, common to the singular integral and to any other integral at a common position, and therefore only the immediate vicinity of any position on $z=0$ need be considered. Let any position on $z=0$ be taken: when chosen, it is made the origin, so that we are considering the immediate vicinity of

$$z=0, \quad x_1=0, \ \ldots, \ x_n=0.$$

Moreover, at that position (and at any other) on the part of the singular integral under consideration, we have

$$p_1=0, \ \ldots, \ p_n=0,$$

because z is steadily zero.

Let

$$\phi(x_1, \ldots, x_n, z, a_1, \ldots, a_n)=0$$

be an integral: the discrimination, as to whether it is general or complete, will depend upon the other equations (if any) that are associated with it. As $z=0$ is the singular integral, then at every position common to $\phi=0$ and the singular integral, we have

$$\frac{\partial \phi}{\partial a_1}=0, \ \ldots, \ \frac{\partial \phi}{\partial a_n}=0;$$

and we know that, at any such position, the two integrals have a common element so that the values of $p_1, \ldots, p_n$ for $\phi=0$, as given by

$$\frac{\partial \phi}{\partial x_r}+p_r\frac{\partial \phi}{\partial z}=0,$$

for $r=1, \ldots, n$, must be the same as those for $z=0$. The latter vanish: hence, at a common element,

$$\frac{\partial \phi}{\partial x_1}=0, \ \ldots, \ \frac{\partial \phi}{\partial x_n}=0.$$

In general, $\frac{\partial \phi}{\partial z}$ will not vanish there: the position then would be a singularity on $\phi = 0$, and circumstances would require to be very special in order that a singularity of the amplitude $\phi = 0$ should lie upon its envelope.

Assuming the origin to be the common element in question, we thus have

$$\frac{\partial \phi}{\partial a_1} = 0, \ \ldots, \ \frac{\partial \phi}{\partial a_n} = 0, \quad \frac{\partial \phi}{\partial x_1} = 0, \ \ldots, \ \frac{\partial \phi}{\partial x_n} = 0,$$

at the position $0, \ldots, 0$ on the integral $\phi = 0$.

In order to simplify the consideration of the small variations along two integrals in the immediate vicinity of a common element, Darboux proceeds as follows. As in the case of two independent variables, so, generally, in the case of n independent variables, the discussion centres round an aggregate of terms of the second order of the type

$$\Sigma \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j + \frac{\partial^2 \phi}{\partial x_i \partial a_\mu} dx_i da_\mu + \frac{\partial^2 \phi}{\partial a_\mu \partial a_r} da_\mu da_r \right),$$

where the values $z = 0$, $x_1 = 0$, $\ldots$, $x_n = 0$ and the corresponding values of $a_1, \ldots, a_n$ are to be substituted in the coefficients of the bilinear terms. Let a homogeneous linear change be effected upon the variables x, and another upon the constants a: these do not affect the position of the common element and, among other things, they can be used to render the position of the axes of $x_1, \ldots, x_n$ more precise. The number of constants at our disposal in two such transformations is $2n^2$; let them be chosen so as, if possible, to make all the quantities

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j}, \quad \frac{\partial^2 \phi}{\partial x_i \partial a_j}, \quad \frac{\partial^2 \phi}{\partial a_i \partial a_j},$$

for all values of i and j from $1, \ldots, n$ that are distinct from one another, vanish at the common element. In general, these conditions amount to $2n(n-1)$ relations among the constants; for these, the $2n^2$ constants more than suffice. Hence, in addition to the equations

$$\frac{\partial \phi}{\partial a_1} = 0, \ \ldots, \ \frac{\partial \phi}{\partial a_n} = 0, \quad \frac{\partial \phi}{\partial x_1} = 0, \ \ldots, \ \frac{\partial \phi}{\partial x_n} = 0,$$

satisfied at a common element, we may also suppose that the equations

$$\frac{\partial^2\phi}{\partial a_i \partial a_j}=0,\quad \frac{\partial^2\phi}{\partial a_i \partial x_j}=0,\quad \frac{\partial^2\phi}{\partial x_i \partial x_j}=0,$$

for i and $j=1, 2, \ldots, n$ but unequal to one another, also are satisfied.

Consider, first, small variations along the singular integral in the vicinity of the common element: these can be represented by $dx_1, \ldots, dx_n$. There is no variation of z, for z is steadily zero: the small variations give rise to variations of $a_1, \ldots, a_n$, which may be represented by $\delta a_1, \ldots, \delta a_n$. All these variations are to be subject to the equations

$$\phi=0,\quad \frac{\partial\phi}{\partial a_r}=0,\quad \frac{\partial\phi}{\partial x_r}=0,$$

for $r=1, \ldots, n$: hence, taking account of all the quantities that vanish, we have

$$\frac{\partial^2\phi}{\partial a_r^2}\delta a_r+\frac{\partial^2\phi}{\partial a_r \partial x_r}dx_r=0,$$

$$\frac{\partial^2\phi}{\partial x_r \partial a_r}\delta a_r+\frac{\partial^2\phi}{\partial x_r^2}dx_r=0,$$

for $r=1, \ldots, n$. Consequently, other n equations

$$\frac{\partial^2\phi}{\partial a_r^2}\frac{\partial^2\phi}{\partial x_r^2}=\left(\frac{\partial^2\phi}{\partial a_r \partial x_r}\right)^2$$

are satisfied at the common element: these replace n of the equations containing differential elements; and the other n of those equations give the variations $\delta a_1, \ldots, \delta a_n$, belonging to the singular integral and determined by means of $dx_1, \ldots, dx_n$. Hence, at the common element, we may take

$$\frac{\partial^2\phi}{\partial x_r^2}=\mu_r\frac{\partial^2\phi}{\partial x_r \partial a_r}=\mu_r^2\frac{\partial^2\phi}{\partial a_r^2}.$$

Now consider a general integral possessing the element at the origin: and suppose that it is of class m, that is, such that m relations are postulated among the n quantities $a_1, \ldots, a_n$. We require variations along the general integral, determined by $dx_1, \ldots, dx_n$: these may be denoted by $da_1, \ldots, da_n$: and then the m postulated relations may be taken in the differential form

$$da_1=c_{1,m+1}\,da_{m+1}+\ldots+c_{1,n}\,da_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$da_m=c_{m,m+1}da_{m+1}+\ldots+c_{m,n}da_n.$$

The equation of the general integral is given by

$$\phi = 0,$$

$$\phi_r = \frac{\partial \phi}{\partial a_1} c_{1, m+r} + \frac{\partial \phi}{\partial a_2} c_{2, m+r} + \ldots + \frac{\partial \phi}{\partial a_m} c_{m, m+r} + \frac{\partial \phi}{\partial a_{m+r}} = 0,$$

for $r = 1, \ldots, n - m$; it results from the elimination of $a_1, \ldots, a_n$ among these $n - m + 1$ equations and the m postulated relations.

The order of contact of the common element at the point (being the generalisation of the order of contact in ordinary space) depends upon the magnitude of dz, belonging to the general integral and expressed in terms of $dx_1, \ldots, dx_n$. To find an expression for dz, we expand ϕ along the general integral in the vicinity of the origin, we insert the initial values in the coefficients, and then we retain only the most important terms. The result can be expressed in the form

$$\begin{aligned} -2 \frac{\partial \phi}{\partial z} dz &= \sum_{r=1}^{n} \left(\frac{\partial^2 \phi}{\partial x_r^2} dx_r^2 + 2 \frac{\partial^2 \phi}{\partial x_r \partial a_r} dx_r da_r + \frac{\partial^2 \phi}{\partial a_r^2} da_r^2 \right) \\ &= \sum_{r=1}^{n} \frac{\partial^2 \phi}{\partial a_r^2} (da_r + \mu_r dx_r)^2, \end{aligned}$$

on using the former relations. The quantities $da_1, \ldots, da_n$ are given, by means of the m differential relations and by variations of

$$\phi_1 = 0, \ \ldots, \ \phi_{n-m} = 0,$$

in terms of $dx_1, \ldots, dx_n$: these $n - m$ variations are

$$\begin{aligned} \left(\frac{\partial^2 \phi}{\partial a_1^2} da_1 + \frac{\partial^2 \phi}{\partial a_1 \partial x_1} dx_1 \right) c_{1, m+r} + \ldots + \left(\frac{\partial^2 \phi}{\partial a_m^2} da_m + \frac{\partial^2 \phi}{\partial a_m \partial x_m} dx_m \right) c_{m, m+r} \\ + \frac{\partial^2 \phi}{\partial a^2_{m+r}} da_{m+r} + \frac{\partial^2 \phi}{\partial a_{m+r} \partial x_{m+r}} dx_{m+r} = 0, \end{aligned}$$

for $r = 1, \ldots, n - m$, account having been taken of the values of the coefficients of the differential elements at the origin. Using the former relations, we have

$$\begin{aligned} \frac{\partial^2 \phi}{\partial a_1^2} (da_1 + \mu_1 dx_1) c_{1, m+r} + \ldots + \frac{\partial^2 \phi}{\partial a_m^2} (da_m + \mu_m dx_m) c_{m, m+r} \\ + \frac{\partial^2 \phi}{\partial a^2_{m+r}} (da_{m+r} + \mu_{m+r} dx_{m+r}) = 0, \end{aligned}$$

for $r = 1, \ldots, n - m$. Also, the former differential relations can be written in the form

$$\begin{aligned} da_s + \mu_s dx_s - \sum_{t=1}^{n-m} c_{s, m+t} (da_{m+t} + \mu_{m+t} dx_{m+t}) \\ = \mu_s dx_s - \sum_{t=1}^{n-m} (c_{s, m+t} \mu_{m+t} dx_{m+t}), \end{aligned}$$

for $s=1, \ldots, m$; hence all the quantities $da_r + \mu_r dx_r$ (for $r=1, \ldots, n$) can be expressed linearly in terms of the m quantities

$$\mu_s dx_s - \sum_{t=1}^{n-m} (c_{s,m+t}\, \mu_{m+t}\, dx_{m+t}).$$

Now the quantity dz is given by

$$-2\frac{\partial\phi}{\partial z} dz = \sum_{r=1}^{n} \frac{\partial^2\phi}{\partial a_r^2}(da_r + \mu_r dx_r)^2,$$

which, after substitution, comes to be a bilinear function of the foregoing m quantities. This bilinear function does not vanish for all values of $dx_1, \ldots, dx_n$, except under special conditions; and therefore *the contact of an element, common to a general integral of any class and the singular integral, is usually of the first order.*

While the bilinear function does not vanish for all values of $dx_1, \ldots, dx_n$ except under special conditions, there are certain ratios of the values of the differential elements (which may be called hyperdirections through the origin) for which the function does vanish. Let

$$\theta_\kappa = \frac{\mu_\kappa dx_\kappa - \sum_{t=1}^{n-m} c_{\kappa,m+t}\, \mu_{m+t}\, dx_{m+t}}{\mu_1 dx_1 - \sum_{t=1}^{n-m} c_{1,m+t}\, \mu_{m+t}\, dx_{m+t}},$$

for $\kappa = 2, \ldots, m$; then we have

$$-2\frac{\partial\phi}{\partial z} dz = \left(\mu_1 dx_1 - \sum_{t=1}^{n-m} c_{1,m+t}\, \mu_{m+t} dx_{m+t}\right)^2 Q(\theta_2, \ldots, \theta_m),$$

where Q is a quadratic function of its arguments. It is clear that dz will be of the third order whenever

$$Q(\theta_2, \ldots, \theta_m) = 0,$$

an equation which, in general, gives two values of θ_2 in terms of $\theta_3, \ldots, \theta_m$. Hence we may say that, *when variations $dx_2, \ldots, dx_n$ are taken arbitrarily, there are generally two values of dx_1 which can be associated with them so as to make dz belong to a general integral of the third order of small quantities.*

Other results are given in the examples which follow: in particular, exceptions to the last result are indicated.

125. The preceding analysis is not merely a generalisation of that adopted, in § 105, for the case when $n=2$: it is therefore worth

while setting it out briefly for that case, in order to allow of comparison with the discussion there given.

We take $z=0$ as the singular solution: any position on it is chosen and made the origin, so that, for an element there, we have

$$z=0, \quad x=0, \quad y=0, \quad p=0, \quad q=0.$$

Hence for any general integral $\phi(x, y, z, a, b)=0$, possessing that element, we have

$$\frac{\partial\phi}{\partial a}=0, \quad \frac{\partial\phi}{\partial b}=0, \quad \frac{\partial\phi}{\partial x}=0, \quad \frac{\partial\phi}{\partial y}=0.$$

Moreover, we may assume that the relations

$$\frac{\partial^2\phi}{\partial a\partial b}=0, \quad \frac{\partial^2\phi}{\partial a\partial y}=0, \quad \frac{\partial^2\phi}{\partial b\partial x}=0, \quad \frac{\partial^2\phi}{\partial x\partial y}=0,$$

are satisfied at the point: if they are not satisfied in the form in which the equation arises, they can be made to be so by making linear transformations of the variables x and y, and linear transformations of the constants a and b.

The critical condition for contact of order closer than the usual contact was found (§ 105) to be

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}=\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2;$$

and it was indicated (§ 105, Ex. 1) that this equation implies, and is implied by, the equation

$$\frac{\partial^2\phi}{\partial a^2}\frac{\partial^2\phi}{\partial b^2}=\left(\frac{\partial^2\phi}{\partial a\partial b}\right)^2.$$

Thus, for the transformations adopted, these conditions will be

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}=0, \quad \frac{\partial^2\phi}{\partial a^2}\frac{\partial^2\phi}{\partial b^2}=0,$$

respectively.

Let small variations along the singular integral in the immediate vicinity of the element at the origin be denoted by dx, dy, δa, δb: these must satisfy

$$\phi=0, \quad \frac{\partial\phi}{\partial a}=0, \quad \frac{\partial\phi}{\partial b}=0, \quad \frac{\partial\phi}{\partial x}=0, \quad \frac{\partial\phi}{\partial y}=0.$$

Hence, taking account of the various vanishing quantities, we have

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial a^2}\,\delta a+\frac{\partial^2\phi}{\partial a\,\partial x}\,dx=0\\ \frac{\partial^2\phi}{\partial a\,\partial x}\,\delta a+\frac{\partial^2\phi}{\partial x^2}\,dx=0\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{\partial^2\phi}{\partial b^2}\,\delta b+\frac{\partial^2\phi}{\partial b\,\partial y}\,dy=0\\ \frac{\partial^2\phi}{\partial b\,\partial y}\,\delta b+\frac{\partial^2\phi}{\partial y^2}\,dy=0\end{aligned}\right\}.$$

Consequently, the two relations

$$\frac{\partial^2\phi}{\partial a^2}\,\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial^2\phi}{\partial a\,\partial x}\right)^2,\qquad \frac{\partial^2\phi}{\partial b^2}\,\frac{\partial^2\phi}{\partial y^2}=\left(\frac{\partial^2\phi}{\partial b\,\partial y}\right)^2,$$

are satisfied at the common element: in virtue of these two, the four equations reduce to two only, which then determine δa and δb in terms of dx and dy. These two relations may be expressed by the equations

$$\frac{\partial^2\phi}{\partial x^2}=\rho\,\frac{\partial^2\phi}{\partial a\,\partial x}=\rho^2\,\frac{\partial^2\phi}{\partial a^2},$$

$$\frac{\partial^2\phi}{\partial y^2}=\sigma\,\frac{\partial^2\phi}{\partial b\,\partial y}=\sigma^2\,\frac{\partial^2\phi}{\partial b^2}.$$

Now consider the general integral: let the single relation between a and b, when expressed in differential form, be

$$db=c\,da,$$

where da and db are variations along the general integral; the equations of this integral are

$$\phi=0,\quad \frac{\partial\phi}{\partial a}+c\,\frac{\partial\phi}{\partial b}=0.$$

At the element, $\dfrac{\partial\phi}{\partial a}$ and $\dfrac{\partial\phi}{\partial b}$ vanish: hence da and db are such that

$$\frac{\partial^2\phi}{\partial a^2}\,da+\frac{\partial^2\phi}{\partial a\,\partial x}\,dx+\left(\frac{\partial^2\phi}{\partial b^2}\,db+\frac{\partial^2\phi}{\partial b\,\partial y}\,dy\right)c=0,$$

taking account of vanishing quantities. Thus, by the preceding relations, we have

$$\frac{\partial^2\phi}{\partial a^2}\,(da+\rho\,dx)+c\,\frac{\partial^2\phi}{\partial b^2}\,(db+\sigma\,dy)=0.$$

Also

$$c\,(da+\rho\,dx)-(db+\sigma\,dy)=c\rho\,dx-\mu\,dy,$$

so that

$$\left.\begin{aligned}\left(\frac{\partial^2\phi}{\partial a^2}+c^2\,\frac{\partial^2\phi}{\partial b^2}\right)(da+\rho\,dx)&=c\,\frac{\partial^2\phi}{\partial b^2}\,(c\rho\,dx-\mu\,dy)\\ \left(\frac{\partial^2\phi}{\partial a^2}+c^2\,\frac{\partial^2\phi}{\partial b^2}\right)(db+\sigma\,dy)&=-\frac{\partial^2\phi}{\partial a^2}\,(c\rho\,dx-\mu\,dy)\end{aligned}\right\},$$

which give the necessary values of da and db. Also, variation of the equation $\phi=0$ leads, when account is taken of vanishing quantities, to the relation

$$
\begin{aligned}
-2\frac{\partial\phi}{\partial z}dz &= \frac{\partial^2\phi}{\partial x^2}dx^2 + 2\frac{\partial^2\phi}{\partial a\partial x}da\,dx + \frac{\partial^2\phi}{\partial a^2}da^2 \\
&\qquad + \frac{\partial^2\phi}{\partial y^2}dy^2 + 2\frac{\partial^2\phi}{\partial b\partial y}db\,dy + \frac{\partial^2\phi}{\partial b^2}db^2 \\
&= \frac{\partial^2\phi}{\partial a^2}(da+\rho dx)^2 + \frac{\partial^2\phi}{\partial b^2}(db+\sigma dy)^2 \\
&= \frac{\dfrac{\partial^2\phi}{\partial a^2}\dfrac{\partial^2\phi}{\partial b^2}}{\dfrac{\partial^2\phi}{\partial a^2}+c^2\dfrac{\partial^2\phi}{\partial b^2}}(c\rho dx-\mu dy)^2.
\end{aligned}
$$

Hence dz is usually of the second order in dx and dy: and the contact is usually of the first order.

There is a single direction along which dz is of the third order: it is given by

$$c\rho = \mu\frac{dy}{dx},$$

and the point is then a contact-point of the branches of the intersection of the surfaces in question, having this direction for its tangent*.

If, however, the relation

$$\frac{\partial^2\phi}{\partial a^2}\frac{\partial^2\phi}{\partial b^2}=0$$

is satisfied at the point, dz is of the third order for all variations of x and y through the point: the general integral and the singular integral then have contact of the second order. (This relation agrees with the result to be expected from the earlier case.) Moreover, when this relation is satisfied at the point, it clearly is satisfied independently of any functional relation between a and b: it therefore is satisfied for all such relations and consequently, *if the singular integral has contact of the second order with any general integral, it has contact of the second order with every general integral.* This is the former result (§ 105).

* This establishes the statement made in § 104.

When the contact between the two integrals is only of the first order, the single direction in which the contact is of the second order is given by

$$c\rho = \mu \frac{dy}{dx}.$$

The quantities μ and ρ belong to the singular integral at the point: hence this direction usually changes from one general integral to another.

126. We now proceed to give some examples of the general theory which has just been expounded.

Ex. 1. When we are dealing with what is called the general integral, being the integral for which there is only a single postulated relation among the parameters, the formulæ become simple and lead easily to further results.

Let the postulated relation be

$$da_1 = b_2 da_2 + \ldots + b_n da_n;$$

then the equations of the general integral are

$$\phi = 0, \quad \frac{\partial\phi}{\partial a_2} + b_2 \frac{\partial\phi}{\partial a_1} = 0, \quad \ldots, \quad \frac{\partial\phi}{\partial a_n} + b_n \frac{\partial\phi}{\partial a_1} = 0.$$

Assuming all the properties in the text, we have the expression for dz in the form

$$-2 \frac{\partial\phi}{\partial z} dz = \sum_{r=1}^{n} \frac{\partial^2\phi}{\partial a_r^2} (da_r + \mu_r dx_r)^2.$$

The quantities $da_1, \ldots, da_n$, determined for the general integral in terms of $dx_1, \ldots, dx_n$, are given by the above relation and by

$$\frac{\partial^2\phi}{\partial a_r^2} (da_r + \mu_r dx_r) + b_r \frac{\partial^2\phi}{\partial a_1^2} (da_1 + \mu_1 dx_1) = 0,$$

for $r = 2, \ldots, n$. We take the relation in the form

$$da_1 + \mu_1 dx_1 - \sum_{s=2}^{n} b_s (da_s + \mu_s dx_s) = \mu_1 dx_1 - \sum_{s=2}^{n} b_s \mu_s dx_s:$$

and we have

$$da_1 + \mu_1 dx_1 = \frac{1}{\Delta} \left(\mu_1 dx_1 - \sum_{s=2}^{n} b_s \mu_s dx_s \right),$$

$$da_r + \mu_r dx_r = - \frac{b_r}{\dfrac{\partial^2\phi}{\partial a_r^2}} \frac{\partial^2\phi}{\partial a_1^2} \frac{1}{\Delta} \left(\mu_1 dx_1 - \sum_{s=2}^{n} b_s \mu_s dx_s \right),$$

for $r = 2, \ldots, n$, where

$$\Delta = 1 + \frac{\partial^2\phi}{\partial a_1^2} \left\{ \frac{b_2^2}{\dfrac{\partial^2\phi}{\partial a_2^2}} + \ldots + \frac{b_n^2}{\dfrac{\partial^2\phi}{\partial a_n^2}} \right\}.$$

Substituting these values, we have the value of dz in the form

$$-2\frac{\partial\phi}{\partial z}dz=\frac{1}{\Delta}\frac{\partial^2\phi}{\partial a_1^2}\left(\mu_1 dx_1-\sum_{s=2}^{n} b_s\mu_s dx_s\right)^2.$$

Hence, for the general integral (or the integral of the first class, only one relation among the parameters being postulated), an element common with the singular integral has contact of only the first order. This is the former result.

For this general integral, we may take arbitrary variations $dx_2, \ldots, dx_n$: and then there is one variation dx_1, given by

$$\mu_1 dx_1=\sum_{s=2}^{n} b_s\mu_s dx_s,$$

for which dz is of the third order of small quantities; accordingly, for those variations, the element has contact of the second order. The property is special to this general integral: for the general integrals of other classes, where the number of postulated relations is greater than one, there are two variations dx_1 for which the corresponding dz is of the third order.

Ex. 2. The relations defining a general integral of the second class can be taken in the form

$$\left.\begin{aligned}da_1&=b_3da_3+\ldots+b_nda_n\\ da_2&=c_3da_3+\ldots+c_nda_n\end{aligned}\right\}.$$

An element at the origin is possessed in common by this integral and the singular integral: and the various quantities at this position have the same values as in the investigation in the text. Let

$$\phi_{ss}=\frac{\partial^2\phi}{\partial a_s^2},$$

for $s=1, \ldots, n$; also let

$$\Delta=1+\phi_{11}\sum_{r=3}^{n}\frac{b_r^2}{\phi_{rr}}+\phi_{22}\sum_{r=3}^{n}\frac{c_r^2}{\phi_{rr}}+\phi_{11}\phi_{22}\left\{\sum_{r=3}^{n}\frac{b_r^2}{\phi_{rr}}\sum_{r=3}^{n}\frac{c_r^2}{\phi_{rr}}-\left(\sum_{r=3}^{n}\frac{b_rc_r}{\phi_{rr}}\right)^2\right\};$$

and let

$$du=\mu_1dx_1-\mu_3b_3dx_3-\ldots-\mu_nb_ndx_n,$$
$$dv=\mu_2dx_2-\mu_3c_3dx_3-\ldots-\mu_nc_ndx_n.$$

Obtain the value of dz, which measures the order of possession of the element common to this general integral and the singular integral, in the form

$$\begin{aligned}-2\Delta\frac{\partial\phi}{\partial z}dz\\ =\left(\phi_{11}+\phi_{11}\phi_{22}\sum_{r=3}^{n}\frac{c_r^2}{\phi_{rr}}\right)du^2+\left(\phi_{22}+\phi_{11}\phi_{22}\sum_{r=3}^{n}\frac{b_r^2}{\phi_{rr}}\right)dv^2\\ -2\phi_{11}\phi_{22}\sum_{r=3}^{n}\frac{b_rc_r}{\phi_{rr}}du\,dv.\end{aligned}$$

Hence shew, in general,

(i) that the order of possession of the common element is usually the first:

(ii) that usually there are two distinct sets of variations, having $dx_2, \ldots, dx_n$ arbitrary, for which the element is possessed to the second order:

(iii) that, if either $\dfrac{\partial^2\phi}{\partial a_1^2}$ or $\dfrac{\partial^2\phi}{\partial a_2^2}$ should vanish, there is only one set of variations, arbitrary in $dx_2, \ldots, dx_n$, for which the element is possessed to the second order:

(iv) that, if the element is possessed to the second order for all variations, it is sufficient that

$$\frac{\partial^2\phi}{\partial a_1^2}=0, \quad \frac{\partial^2\phi}{\partial a_2^2}=0.$$

Are the two conditions in the last result necessary as well as sufficient?

Shew also that, if

$$\frac{\partial^2\phi}{\partial a_3^2}=0,$$

while none of the other second derivatives vanish, then

$$-2\Delta' \frac{\partial\phi}{\partial z} dz = \phi_{11}\phi_{22}(c_3 du - b_3 dv)^2,$$

where

$$\Delta' = b_3^2\phi_{11} + c_3^2\phi_{22} + \phi_{11}\phi_{22}\sum_{r=4}^{n}\frac{(b_3c_r - b_rc_3)^2}{\phi_{rr}};$$

and discuss this form.

Ex. 3. Adopting the notation of § 125, and assuming that all the quantities which occur have the values there given, prove that in order to have an element common to a general integral, defined by three relations among the quantities $da_1, \ldots, da_n$, and the singular integral, possessed to the second order, it is necessary and sufficient that three of the quantities

$$\frac{\partial^2\phi}{\partial a_1^2}, \quad \frac{\partial^2\phi}{\partial a_2^2}, \quad \ldots, \quad \frac{\partial^2\phi}{\partial a_n^2}$$

should vanish.

In case fewer than three of these quantities should vanish, what are the sets of variations for which the element is possessed to the second order?

Ex. 4. Shew that, if one general integral of any class has an element common with the singular integral possessed to the second order, then every general integral of that class has the same property.

127. The discussion of the order, in which an element common to a complete integral and the singular integral is possessed, is simpler. Taking the element on the singular integral as before (§ 124), we require variations along the complete integral: these

are given by $dx_1, \ldots, dx_n, dz$ alone, because $a_1, \ldots, a_n$ are constants for this integral. The earlier analysis shews at once that the most important terms in these variations are given by

$$-2\frac{\partial\phi}{\partial z}dz = \frac{\partial^2\phi}{\partial x_1^2}dx_1^2 + \ldots + \frac{\partial^2\phi}{\partial x_n^2}dx_n^2,$$

assuming that the quantities belonging to the singular integral at the point are the same as before. It is clear that, except under very special conditions, *an element common to a complete integral and to the singular integral is usually possessed only to the first order.*

Ex. Discuss the order of possession of an element, common to a complete integral and to a general integral, for which $n-1$ relations are postulated among the parameters.

CHAPTER IX.

LIE'S METHODS APPLIED TO EQUATIONS OF THE FIRST ORDER.

MANY of Lie's investigations are concerned with the integration of partial differential equations of the first order: broadly speaking, he has devised two general methods of proceeding which have considerable features in common. It may be added that they arise as illustrations of processes with wider issues and of analysis having a more extended significance.

One of the methods depends upon the use of tangential transformations (or contact transformations, as they are more frequently called). So far as concerns the properties of these transformations and (as an incident in their application to Pfaff's problem) their application to the integration of a single partial differential equation of the first order, an exposition has already been given* in Part I of the present work; it will be sufficient therefore to give, in this place, merely a statement of the results.

The other of the methods due to Lie depends upon the theory of groups of functions as developed, in part, through the theory of contact transformations. It is applied to a system of simultaneous equations in the first instance and naturally it can be applied to the simplest case when there is only a single equation.

Some references are given in the sections in Part I that have already been mentioned. Of these, the most important are Lie's memoir in the 8th volume (1875) of the *Mathematische Annalen*, as regards the fundamental properties of contact transformations and their simpler applications to the integration of partial differential equations, and the second volume (1890) of his *Theorie der Transformationsgruppen*, which contains many properties, developments, and applications of the transformations in question.

Reference may also be made advantageously to the exposition given by Goursat in chapters XI and XII of his treatise already (p. 55) quoted.

* See chapters IX, X; for the application, see specially §§ 136, 142.

Contact Transformations.

128. The distinctive idea of contact transformations is derived from geometrical considerations applied to hyperspace. An element of surface at any position in a space of $n+1$ dimensions is determined by means of $z, x_1, \ldots, x_n$, the coordinates of the position, and of $p_1, \ldots, p_n$, the coordinates of the orientation, of the element; thus, if $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ be regarded as $2n+1$ independent magnitudes in general, the element of surface will be given by an equation

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0.$$

Transformations of the variables are conceived by means of relations between $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ and new variables $z', x_1', \ldots, x_n', p_1', \ldots, p_n'$; if these relations are such that

$$dz' - \sum_{i=1}^{n} p_i' dx_i' = \rho \left(dz - \sum_{i=1}^{n} p_i dx_i\right),$$

where ρ is a non-vanishing quantity independent of differential elements, the transformation is said to be a *contact transformation*; it obviously transforms two elements of surface, that touch one another, into two other elements of surface that also touch one another. Accordingly, Lie's definition * of the most general contact transformation is:—

Let $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$ be $2n+1$ independent functions of $2n+1$ independent variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, such that the relation

$$dZ - \sum_{i=1}^{n} P_i dX_i = \rho \left(dz - \sum_{i=1}^{n} p_i dx_i\right)$$

is satisfied identically, when ρ is a non-vanishing quantity independent of differential elements: then the equations

$$z' = Z, \quad x_i' = X_i, \quad p_i' = P_i,$$

for $i = 1, \ldots, n$, define a contact transformation.

The most general contact transformation will be given when the most general functions $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$, satisfying the preceding relations, are known. These are given† by the theorem:—

* *Math. Ann.*, t. VIII (1875), p. 220.

† The analysis establishing the theorem is set out in §§ 134, 135 of Part I of this work. A difference in sign should be noted: it is due to the fact that the quantity $[P, Q]$ is used in this volume in its usual sense, and has the reverse sign of $[P, Q]$ as given in Part I.

When the $n+1$ quantities $Z, X_1, \ldots, X_n$ are obtained as $n+1$ functionally independent integrals of the equations

$$[Z, X_i]=0, \quad [X_i, X_j]=0,$$

for i and $j=1, \ldots, n$; when the quantities $P_1, \ldots, P_n$ are determined, either from the n equations

$$\frac{\partial Z}{\partial x_i}+p_i\frac{\partial Z}{\partial z}=\sum_{r=1}^{n} P_r\left(\frac{\partial X_r}{\partial x_i}+p_i\frac{\partial X_r}{\partial z}\right),$$

for $i=1, \ldots, n$, or from the n equations

$$\frac{\partial Z}{\partial p_i}=\sum_{r=1}^{n} P_r\frac{\partial X_r}{\partial p_i},$$

for $i=1, \ldots, n$, the two sets of n equations being equivalent to one another; and when ρ denotes the non-vanishing quantity

$$\frac{\partial Z}{\partial z}-\sum_{r=1}^{n} P_r\frac{\partial X_r}{\partial z}:$$

then the relation

$$dZ-\sum_{i=1}^{n} P_i dX_i=\rho\left(dz-\sum_{i=1}^{n} p_i dx_i\right)$$

is satisfied identically. The conditions are both necessary and sufficient to secure the property: and other equations satisfied by $Z, X_1, \ldots, X_n, P_1, \ldots, P_n, \rho$, are

$$[Z, P_i]=\rho P_i, \quad [P_i, X_i]=-\rho,$$
$$[P_i, X_j]=0 \quad , \quad [P_i, P_j]= \quad 0,$$

for $i=1, \ldots, n$, and values of j unequal to i.

This is Mayer's form* of Lie's theorem relating to the determination of the most general contact transformation. As regards the conditions, it is to be noted that ρ is known as soon as $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$ have been obtained, and $P_1, \ldots, P_n$ are known as soon as $Z, X_1, \ldots, X_n$ have been obtained.

These quantities are subject to the equations

$$[Z, X_i]=0, \quad [X_i, X_j]=0;$$

and as (§ 53) we have

$$[[Z, X_i], X_j]+[[X_i, X_j], Z]-[[Z, X_j], X_i]$$
$$=-\frac{\partial Z}{\partial z}[X_i, X_j]+\frac{\partial X_i}{\partial z}[Z, X_j]-\frac{\partial X_j}{\partial z}[Z, X_i],$$

for any functions whatever of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, the equations are consistent with one another and coexist. But the conditions

* *Math. Ann.*, t. VIII (1875), p. 309.

for the existence of the contact transformation do not require any equation to be satisfied by any one of the quantities $Z, X_1, \ldots, X_n$ alone; and the preceding relation for coexistence holds for all functions which satisfy the equations. Hence there is an arbitrary element in the equations that define the contact transformation: thus we could choose any one of the quantities $Z, X_1, \ldots, X_n$ arbitrarily: or any other arbitrary relation could be chosen that is not inconsistent with the aggregate of conditions and equations.

129. There is one most important form of contact transformation, viz. that in which the difference between the old variables and the new variables is small. Such transformations are called *infinitesimal*: they can be represented by

$$Z = z + \epsilon\zeta, \quad X_i = x_i + \epsilon\xi_i, \quad P_i = p_i + \epsilon\pi_i, \qquad (i = 1, \ldots, n),$$

where $\zeta, \xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n$ are functions of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, and ϵ is a quantity so small that its square and higher powers may be neglected. Moreover, when we have the identical transformation, then $\rho = 1$; hence, for an infinitesimal transformation, we have

$$\rho = 1 + \epsilon\sigma,$$

where σ will be a quantity to be determined. Then the equations

$$[Z, X_i] = 0, \quad [X_i, X_j] = 0, \quad [P_i, X_j] = 0, \quad [P_i, P_j] = 0,$$

for unequal values of i and j, respectively give

$$-\frac{\partial\zeta}{\partial p_i} + \sum_{r=1}^{n} p_r \frac{\partial\xi_i}{\partial p_r} = 0,$$

$$-\frac{\partial\xi_i}{\partial p_j} + \frac{\partial\xi_j}{\partial p_i} = 0,$$

$$-\left(\frac{\partial\xi_j}{\partial x_i} + p_i\frac{\partial\xi_j}{\partial z}\right) - \frac{\partial\pi_i}{\partial p_j} = 0,$$

$$\frac{\partial\pi_j}{\partial x_i} + p_i\frac{\partial\pi_j}{\partial z} - \left(\frac{\partial\pi_i}{\partial x_j} + p_j\frac{\partial\pi_i}{\partial z}\right) = 0.$$

The last three equations, taken for all values of i and j, shew that a function U of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ exists, such that

$$\xi_i = \frac{\partial U}{\partial p_i},$$

$$\pi_i = -\frac{\partial U}{\partial x_i} - p_i\frac{\partial U}{\partial z},$$

for all values of $i=1, \ldots, n$. The equation

$$[P_i, X_i] = -\rho = -1 - \epsilon\sigma$$

then gives

$$\sigma = \frac{\partial \pi_i}{\partial p_i} - \left(\frac{\partial \xi_i}{\partial x_i} + p_i \frac{\partial \xi_i}{\partial z}\right)$$

$$= -\frac{\partial U}{\partial z};$$

and the equations

$$[Z, P_i] = \rho P_i$$

then give

$$\frac{\partial \zeta}{\partial x_i} + p_i \frac{\partial \zeta}{\partial z} = -p_i \frac{\partial U}{\partial z} + \pi_i - \sum_{r=1}^{n} p_r \frac{\partial \pi_i}{\partial p_r}.$$

When these are taken in conjunction with the equations

$$\frac{\partial \zeta}{\partial p_i} = \sum_{r=1}^{n} p_r \frac{\partial \xi_i}{\partial p_r},$$

we find the value of ζ to be

$$\zeta = \sum_{r=1}^{n} \left(p_r \frac{\partial U}{\partial p_r}\right) - U;$$

consequently *the most general infinitesimal transformation is given by the equations*

$$Z = z + \epsilon\zeta, \quad X_i = x_i + \epsilon\xi_i, \quad P_i = p_i + \epsilon\pi_i, \quad \rho = 1 + \epsilon\sigma,$$

where

$$\zeta = \sum_{r=1}^{n} \left(p_r \frac{\partial U}{\partial p_r}\right) - U, \quad \xi_i = \frac{\partial U}{\partial p_i}, \quad -\pi_i = \frac{\partial U}{\partial x_i} + p_i \frac{\partial U}{\partial z}, \quad \sigma = -\frac{\partial U}{\partial z},$$

for $i=1, \ldots, n$, and U denotes any arbitrary function of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$.

130. Sometimes it is necessary to know the contact transformations for which $X_1, \ldots, X_n, P_1, \ldots, P_n$ are *explicitly independent* of z, so that Z is the only quantity that involves z. The results* are as follows:—

The quantity ρ is constant and may be made unity; then Z is given by

$$Z = Az + \Pi,$$

where Π is a function of $x_1, \ldots, x_n, p_1, \ldots, p_n$ that does not involve z, and A is a constant. The quantities $X_1, \ldots, X_n$ are functionally independent integrals of the equations

$$(X_i, X_j) = 0,$$

* See Part I of this work, § 137. The same remark as to difference of sign from the results in Part I applies here as in the foot-note on p. 315 of this volume.

for i and $j = 1, \ldots, n$, and Π is an integral of the equations

$$(\Pi, X_i) = -A \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r},$$

which is functionally independent of $X_1, \ldots, X_n$. The quantities $P_1, \ldots, P_n$ are then given by any n independent equations of the set

$$\frac{\partial \Pi}{\partial x_i} - \sum_{r=1}^{n} P_r \frac{\partial X_r}{\partial x_i} = -Ap_i, \quad \frac{\partial \Pi}{\partial p_i} - \sum_{r=1}^{n} P_r \frac{\partial X_r}{\partial p_i} = 0,$$

for $i = 1, \ldots, n$. The functions thus determined satisfy the relation

$$d\Pi - \sum_{i=1}^{n} P_i dX_i = -A \sum_{i=1}^{n} p_i dx_i$$

identically: and other equations satisfied are

$$(X_i, P_i) = A, \quad (\Pi, P_i) = A\left(P_i - \sum_{r=1}^{n} p_r \frac{\partial P_i}{\partial p_r}\right),$$

$$(P_i, X_j) = 0, \quad (P_i, P_j) = 0,$$

for all values of i and j unequal to one another. The constant A is usually taken to be unity.

The determination of Π can be effected by a quadrature, when $X_1, \ldots, X_n$ are known. To see this, let the variables be changed to $y_1, \ldots, y_n, v_1, \ldots, v_n$, where

$$y_1 = X_1, \ldots, y_n = X_n,$$

and $v_1, \ldots, v_n$ are n functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$, chosen so that the new variables make up an aggregate of $2n$ independent functions. Now, because

$$(X_i, X_j) = 0,$$

the quantity (Π, X_i) vanishes when Π is made equal to any of the quantities $X_1, \ldots, X_n$; and therefore, when the new variables are taken, no one of the derivatives

$$\frac{\partial \Pi}{\partial y_1}, \ldots, \frac{\partial \Pi}{\partial y_n}$$

occurs in (Π, X_i). Hence the n equations

$$(\Pi, X_i) = -A \sum_{r=1}^{n} p_r \frac{\partial X_i}{\partial p_r}$$

can be resolved with respect to

$$\frac{\partial \Pi}{\partial v_1}, \ldots, \frac{\partial \Pi}{\partial v_n},$$

giving these quantities in terms of the variables: a quadrature then determines Π.

Conversely, *if we regard the contact transformation as changing the variables from* Z, $X_1, \ldots, X_n$, $P_1, \ldots, P_n$, *the results are similar.* We then have

$$z = aZ + \Pi,$$

where Π is a function of $X_1, \ldots, X_n$, $P_1, \ldots, P_n$, and a is a constant; the differential relation is

$$d\Pi - \sum_{i=1}^{n} p_i dx_i = -a \sum_{i=1}^{n} P_i dX_i;$$

the quantities $x_1, \ldots, x_n$ are functionally independent integrals of the equations

$$(x_i, x_j) = 0,$$

for i and $j = 1, \ldots, n$, where the independent variables are $X_1, \ldots, X_n$, $P_1, \ldots, P_n$; and Π is an integral of the equations

$$(\Pi, x_i) = -a \sum_{r=1}^{n} P_r \frac{\partial x_i}{\partial P_r},$$

which is functionally independent of $x_1, \ldots, x_n$. The quantities $p_1, \ldots, p_n$ are given by any n independent equations of the set

$$\frac{\partial \Pi}{\partial X_i} - \sum_{r=1}^{n} p_r \frac{\partial x_r}{\partial X_i} = -aP_i, \quad \frac{\partial \Pi}{\partial P_i} - \sum_{r=1}^{n} p_r \frac{\partial x_r}{\partial P_i} = 0,$$

for $i = 1, \ldots, n$; and other equations satisfied are

$$(x_i, p_i) = a, \quad (\Pi, p_i) = a\left(p_i - \sum_{r=1}^{n} P_r \frac{\partial p_i}{\partial P_r}\right),$$

for $i = 1, \ldots, n$, as well as

$$(p_i, x_j) = 0, \quad (p_i, p_j) = 0,$$

for unequal values of i, j from the series $1, \ldots, n$.

And $aA = 1$: so that, as A is usually unity, so also is a.

As regards these results, it is to be noted that ρ has become a constant which has justifiably been made unity. The quantities $P_1, \ldots, P_n$ are known as soon as Π, $X_1, \ldots, X_n$ are known: and the quantity Π is to be constructed after $X_1, \ldots, X_n$ are known. These quantities are subject to the equations

$$(X_i, X_j) = 0:$$

so that, as (§ 52) the relation

$$((X_i, X_j)\, X_k) + ((X_j, X_k)\, X_i) + ((X_k, X_i)\, X_j) = 0$$

is satisfied for any functions whatever of $x_1, \ldots, x_n, p_1, \ldots, p_n$, the equations are consistent with one another and coexist. The conditions for the existence of the contact transformation do not impose any equation involving only a single one of the quantities $X_1, \ldots, X_n$; and the preceding conditions for coexistence of the equations are satisfied identically, whatever be the functions $X_1, \ldots, X_n$. Hence the equations defining the contact transformation under consideration contain an arbitrary element: thus any one of the quantities $X_1, \ldots, X_n$ can be assigned arbitrarily, or any other arbitrary relation can be chosen that is not inconsistent with the aggregate of conditions and equations.

COROLLARY 1. There is one special case of this contact transformation, usually called the *infinitesimal* transformation of this type.

It is characterised by the properties

$$Z = z + \epsilon\zeta, \quad X_i = x_i + \epsilon\xi_i, \quad P_i = p_i + \epsilon\pi_i, \qquad (i = 1, \ldots, n),$$

where ϵ is a small quantity of such a magnitude that squares and higher powers may be neglected. The critical equations impose limitations upon the forms of ζ, ξ_i, π_i, for $i = 1, \ldots, n$; thus the equations

$$(X_i, X_j) = 0, \quad (X_i, P_j) = 0, \quad (P_i, P_j) = 0, \quad (X_i, P_i) = 1,$$

give the conditions

$$\frac{\partial \xi_i}{\partial p_j} - \frac{\partial \xi_j}{\partial p_i} = 0,$$

$$\frac{\partial \pi_j}{\partial p_i} + \frac{\partial \xi_i}{\partial x_j} = 0,$$

$$\frac{\partial \pi_i}{\partial x_j} - \frac{\partial \pi_j}{\partial x_i} = 0,$$

$$\frac{\partial \pi_i}{\partial x_i} + \frac{\partial \xi_i}{\partial x_i} = 0,$$

respectively. Hence there is some function U of $x_1, \ldots, x_n, p_1, \ldots, p_n$ such that

$$\xi_i = \frac{\partial U}{\partial p_i}, \quad \pi_i = -\frac{\partial U}{\partial x_i}, \qquad (i = 1, \ldots, n);$$

and any function U of these variables will enable all the conditions to be satisfied. Also writing

$$\Pi = \epsilon\zeta,$$

we find the equations for ζ to be

$$\frac{\partial \zeta}{\partial x_i} = -\frac{\partial U}{\partial x_i} + \sum_{r=1}^{n} p_r \frac{\partial^2 U}{\partial p_r \partial x_i},$$

$$\frac{\partial \zeta}{\partial p_i} = \sum_{r=1}^{n} p_r \frac{\partial^2 U}{\partial p_i \partial p_r},$$

so that

$$\zeta = -U + \sum_{r=1}^{n} p_r \frac{\partial U}{\partial p_r}.$$

Hence an infinitesimal contact transformation is given by the equations

$$X_i - x_i = \epsilon \frac{\partial U}{\partial p_i},$$

$$P_i - p_i = -\epsilon \frac{\partial U}{\partial x_i},$$

$$Z - z = \epsilon \left\{ \sum_{r=1}^{n} \left(p_r \frac{\partial U}{\partial p_r} \right) - U \right\}$$

The equations

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

$$dZ = P_1 dX_1 + \ldots + P_n dX_n,$$

are simultaneously satisfied, when either is satisfied: it is easy to verify this property from the foregoing values. Hence an infinitesimal contact transformation, *in which the changes of* x_1, ..., x_n, p_1, ..., p_n *do not involve* z, *is given by the equations*

$$\delta x_i = \epsilon \frac{\partial U}{\partial p_i}, \quad \delta p_i = -\epsilon \frac{\partial U}{\partial x_i}, \qquad (i = 1, \ldots, n),$$

$$\delta z = \epsilon \left\{ \sum_{r=1}^{n} \left(p_r \frac{\partial U}{\partial p_r} \right) - U \right\},$$

where U *is any function of the variables* $x_1, \ldots, x_n, p_1, \ldots, p_n$.

COROLLARY 2. There is another special case of this type of contact transformation, in which X_1, ..., X_n *are homogeneous functions of zero dimension in the variables* p_1, ..., p_n: *the quantity* Z *is given by*

$$Z = z + c,$$

where c *is a constant; and* P_1, ..., P_n *are then homogeneous, of one dimension in the variables* p_1, ..., p_n, *given by*

$$\sum_{r=1}^{n} P_r \frac{\partial X_r}{\partial x_i} = p_i,$$

for $i = 1, \ldots, n$, *the quantities* $X_1, \ldots, X_n$ *still satisfying the equations*

$$(X_i, X_j) = 0,$$

for i *and* $j = 1, \ldots, n$; *and the differential relation*

$$\sum_{i=1}^{n} P_i dX_i = \sum_{i=1}^{n} p_i dx_i$$

is then satisfied identically. Other equations satisfied are

$$(X_i, P_i) = 1, \quad (P_i, X_j) = 0, \quad (P_i, P_j) = 0,$$

for all values of i *and* j *unequal to one another.*

Such transformations* are called *homogeneous*. As before, any one of the quantities $X_1, \ldots, X_n$ may be chosen arbitrarily: or some other arbitrary relation may be assigned that is not inconsistent with the other relations and equations.

COROLLARY 3. The corresponding *infinitesimal* homogeneous contact transformation has already† been given. It affects only the values of $x_1, \ldots, x_n, p_1, \ldots, p_n$: and *it can be taken in the form*

$$\left.\begin{aligned} X_i &= x_i + \epsilon \frac{\partial H}{\partial p_i} \\ P_i &= p_i - \epsilon \frac{\partial H}{\partial x_i} \end{aligned}\right\}, \qquad (i = 1, \ldots, n),$$

where H *is a function of* $x_1, \ldots, x_n, p_1, \ldots, p_n$, *which is homogeneous of one dimension in* $p_1, \ldots, p_n$ *and is otherwise arbitrary.*

If we write

$$X_i - x_i = dx_i, \quad P_i - p_i = dp_i, \quad \epsilon = dt,$$

these equations take the form

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i},$$

for $i = 1, \ldots, n$. These equations, exactly in this form, will occur later as a canonical system of equations in theoretical dynamics.

131. One remark, indicating a relation between these infinitesimal transformations and the integration of a partial differential equation

$$f(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

* *L.c.*, § 139. The remark, in the foot-note, p. 315, as to change of sign applies here also.

† See vol. I of this work, § 140.

may at once be made*. If it be required to find the infinitesimal contact transformations

$$\delta z = \epsilon\zeta, \quad \delta x_i = \epsilon\xi_i, \quad \delta p_i = \epsilon\pi_i, \quad (i = 1, \ldots, n),$$

which transform f into itself, we clearly must have

$$\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \xi_i + \frac{\partial f}{\partial p_i} \pi_i \right) = 0.$$

Denoting by U any arbitrary function of $x_1, \ldots, x_n, p_1, \ldots, p_n$ (thus omitting z from its arguments), we know that the quantities ξ and π can be taken

$$\xi_i = \frac{\partial U}{\partial p_i}, \qquad \pi_i = -\frac{\partial U}{\partial x_i};$$

thus the appropriate infinitesimal contact transformations will arise through a function U such that

$$(f, U) = 0.$$

Consequently, the determination of all such transformations is equivalent to the integration of the original equation

$$f = 0.$$

Applications of Contact Transformations to the Integration of an Equation or Equations.

132. The application of the properties of finite contact transformations to the integration of a single partial differential equation is immediate.

First, suppose that the dependent variable z occurs explicitly in the equation so that the given equation may be written in the form

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0.$$

We then take

$$Z = f(x_1, \ldots, x_n, z, p_1, \ldots, p_n),$$

and, applying the preceding results, we assume that we have $2n + 1$ independent functions of the $2n + 1$ variables $z, p_1, \ldots, p_n, x_1, \ldots, x_n$, being $Z, P_1, \ldots, P_n, X_1, \ldots, X_n$, such that the relation

$$Z - \sum_{i=1}^{n} P_i dX_i = \rho \left(dz - \sum_{i=1}^{n} p_i dx_i \right)$$

* Lie, *Math. Ann.*, t. VIII (1875), p. 240.

is satisfied identically for a non-vanishing quantity ρ that does not involve the differential elements. All the required quantities $P_1, \ldots, P_n$ are known, if $X_1, \ldots, X_n$ are known, by equations characteristic of a contact transformation; and these quantities $X_1, \ldots, X_n$ are such that the equations

$$[Z, X_i] = 0, \quad [X_i, X_j] = 0,$$

for $i = 1, \ldots, n$, and $j = 1, \ldots, n$ but unequal to i, are satisfied. It has been seen that, even when Z is arbitrarily assigned, these equations are consistent and coexist. Accordingly, assuming the general results of the theory of contact transformations, we may assume that quantities $X_1, \ldots, X_n$ are determined by these equations and that the quantities $P_1, \ldots, P_n$ have subsequently been obtained.

Now what is desired is an integral of the equation

$$Z = f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0,$$

so that, as $p_1, \ldots, p_n$ are derivatives of z with regard to $x_1, \ldots, x_n$ respectively, we have

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0;$$

hence the quantities, defining the contact transformation, must satisfy the relation

$$dZ - P_1 dX_1 - \ldots - P_n dX_n = 0,$$

or, since Z is a permanent zero, we must have

$$P_1 dX_1 + \ldots + P_n dX_n = 0.$$

This relation is the only relation, except $Z = 0$, that needs to be satisfied in order to secure the existence of the relation

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0;$$

and, subject to the specified exception, $P_1, \ldots, P_n, X_1, \ldots, X_n$ are independent functions of the variables involved. We therefore require an integral equivalent of the relation

$$P_1 dX_1 + \ldots + P_n dX_n = 0,$$

where $P_1, \ldots, P_n, X_1, \ldots, X_n, Z$ are $2n + 1$ independent functions of the quantities $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, the relation being satisfied identically.

Such a relation possesses* three types of integral equivalents.

(i) It is satisfied identically if

$$X_1 = a_1, \ldots, X_n = a_n,$$

where $a_1, \ldots, a_n$ are n arbitrary constants: these n equations give n differential relations

$$dX_1 = 0, \ldots, dX_n = 0,$$

in virtue of which the differential relation obviously is satisfied.

(ii) It is satisfied identically, if μ equations of the type

$$X_i = g_i(X_{\mu+1}, \ldots, X_n),$$

for $i = 1, \ldots, \mu$ (where $\mu < n$), are postulated, provided the equations

$$\sum_{i=1}^{\mu} P_i \frac{\partial g_i}{\partial X_j} + P_j = 0,$$

for $j = \mu + 1, \ldots, n$, also are satisfied: for these equations give μ differential relations

$$dX_i = \sum_{j=1}^{n-\mu} \frac{\partial g_i}{\partial X_j} dX_j,$$

for $i = 1, \ldots, \mu$ which, in connection with the other $n - \mu$ relations, obviously satisfy the required differential relation.

(iii) It is satisfied identically if

$$P_1 = 0, \ldots, P_n = 0;$$

these relations, however, do not possess (and do not necessarily imply) any differential character.

We consider the significance of these three types of integral equivalent in succession.

In the first place, we have

$$X_1 = a_1, \ldots, X_n = a_n,$$

concurrently with the equation $Z = 0$; and these relations are sufficient to secure the differential relation

$$dz = p_1 dx_1 + \ldots + p_n dx_n.$$

As $Z, X_1, \ldots, X_n$ are independent functions of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, it is possible to eliminate $p_1, \ldots, p_n$ among the n equations

$$Z = 0, \quad X_1 = a_1, \ldots, X_n = a_n,$$

the eliminant being an equation involving $z, x_1, \ldots, x_n, a_1, \ldots, a_n$. The differential relation shews that the value of z thus provided is an integral of $Z = 0$: manifestly, it is the *complete integral.*

* See Part I of this work, § 142, *foot-note.*

In the second place, we have μ relations

$$X_i = g_i(X_{\mu+1}, \dots, X_n),$$

for $i = 1, \dots, \mu$, and $n - \mu$ relations

$$\sum_{i=1}^{\mu} P_i \frac{\partial g_i}{\partial X_j} + P_j = 0,$$

for $j = \mu + 1, \dots, n$, concurrently with the equation $Z = 0$: and these equations are sufficient to secure the differential relation

$$dz = p_1 dx_1 + \dots + p_n dx_n.$$

As $Z, X_1, \dots, X_n, P_1, \dots, P_n$ are independent functions of their arguments, the n relations are independent of one another and of $Z = 0$: thus it is possible to eliminate $p_1, \dots, p_n$ among the n relations and $Z = 0$, and the eliminant is an equation involving $z, x_1, \dots, x_n$ and the functional forms. The differential relation shews that the value of z thus provided is an integral of $Z = 0$: manifestly, it is a *general integral*. Clearly, there will be a number of classes of such integrals, a class being determined by the number of functional relations between the quantities $X_1, \dots, X_n$ initially postulated: as before, the most comprehensive general integral occurs when only one such relation of the most unrestricted type is postulated.

The equations are expressed* in another form by Lie, as follows. Let H denote a function of $P_1, \dots, P_\mu, X_{\mu+1}, \dots, X_n$, which is homogeneous and linear in $P_1, \dots, P_\mu$ and otherwise is quite arbitrary; then the equations are given by

$$X_i = \frac{\partial H}{\partial P_i}, \quad P_j = -\frac{\partial H}{\partial X_j},$$

for $i = 1, \dots, \mu$; $j = \mu + 1, \dots, n$.

For, since H is homogeneous and linear in $P_1, \dots, P_\mu$, we have

$$\begin{aligned} H &= P_1 \frac{\partial H}{\partial P_1} + \dots + P_\mu \frac{\partial H}{\partial P_\mu} \\ &= P_1 X_1 + \dots + P_\mu X_\mu; \end{aligned}$$

hence

$$\begin{aligned} \sum_{i=1}^{\mu} P_i dX_i + \sum_{i=1}^{\mu} X_i dP_i &= dH \\ &= \sum_{i=1}^{\mu} \frac{\partial H}{\partial P_i} dP_i + \sum_{j=\mu+1}^{n} \frac{\partial H}{\partial X_j} dX_j \\ &= \sum_{i=1}^{\mu} X_i dP_i - \sum_{j=\mu+1}^{n} P_j dX_j, \end{aligned}$$

and therefore

$$\sum_{r=1}^{n} P_r dX_r = 0,$$

which is the equation to be satisfied.

* *Math. Ann.*, t. IX (1876), p. 250.

In the third place, we have n equations

$$P_1 = 0, \ldots, P_n = 0,$$

concurrently with $Z = 0$: but as pointed out, they do not definitely secure the differential relation

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0,$$

for they have no differential character. If, however, they do secure it, we then have

$$dZ = \rho\,(dz - p_1 dx_1 - \ldots - p_n dx_n):$$

consequently

$$\frac{\partial Z}{\partial z} = \rho, \quad \frac{\partial Z}{\partial x_i} = -\rho p_i, \quad \frac{\partial Z}{\partial p_i} = 0,$$

that is,

$$\frac{\partial Z}{\partial p_i} = 0, \quad \frac{\partial Z}{\partial x_i} + p_i \frac{\partial Z}{\partial z} = 0,$$

for $i = 1, \ldots, n$. Thus $2n$ equations must be satisfied, in addition to

$$Z = 0, \quad P_1 = 0, \ldots, P_n = 0;$$

and as ρ is not to vanish, $\dfrac{\partial Z}{\partial z}$ is not zero. Assuming that all the equations coexist, and that it is possible to eliminate $p_1, \ldots, p_n$ so as to leave an equation expressing z in terms of $x_1, \ldots, x_n$, the value of z thus provided is an integral: it is the *singular integral*.

Ex. 1. Let

$$Z = p_1 x_1 + p_2 x_2 + p_3 x_3 - z = 0.$$

The quantities X_1, X_2, X_3 are subject to the equation

$$[Z, X_1] = 0, \quad [Z, X_2] = 0, \quad [Z, X_3] = 0,$$
$$[X_2, X_3] = 0, \quad [X_1, X_3] = 0, \quad [X_1, X_2] = 0;$$

and it is easy to verify that these are satisfied by

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = p_3.$$

The quantities P_1, P_2, P_3 are then given by the equations

$$\frac{\partial Z}{\partial p_i} = \sum_{r=1}^{3} P_r \frac{\partial X_r}{\partial p_i},$$

for $i = 1, 2, 3$: evidently

$$P_1 = x_1, \quad P_2 = x_2, \quad P_3 = x_3.$$

It is clear that the quantities Z, X_1, X_2, X_3, P_1, P_2, P_3 are independent of one another: also

$$dZ - P_1 dX_1 - P_2 dX_2 - P_3 dX_3 = -(dz - p_1 dx_1 - p_2 dx_2 - p_3 dx_3):$$

the value of ρ is -1.

The complete integral is given by

$$p_1 = X_1 = a_1, \quad p_2 = X_2 = a_2, \quad p_3 = X_3 = a_3, \quad Z = 0,$$

where a_1, a_2, a_3 are arbitrary constants: it is

$$z = a_1 x_1 + a_2 x_2 + a_3 x_3.$$

One class of general integrals is given by

$$p_1 = g(p_2, p_3),$$

together with

$$x_1 \frac{\partial g}{\partial p_2} + x_2 = 0, \quad x_1 \frac{\partial g}{\partial p_3} + x_3 = 0, \quad Z = 0.$$

Clearly p_2, p_3, and therefore also p_1, are functions of $\frac{x_2}{x_1}$ and $\frac{x_3}{x_1}$; and when g is as unrestricted as possible, they will be arbitrary functions of these quantities. Thus

$$\begin{aligned} z &= p_1 x_1 + p_2 x_2 + p_3 x_3 \\ &= x_1 \left(p_1 + \frac{x_2}{x_1} p_2 + \frac{x_3}{x_1} p_3 \right) \\ &= x_1 G\left(\frac{x_2}{x_1}, \frac{x_3}{x_1} \right), \end{aligned}$$

where G is an arbitrary function of its arguments: the integral represents the first class of general integrals.

Another class of general integrals is given by

$$p_1 = g_1(p_3), \quad p_2 = g_2(p_3),$$

together with

$$x_1 \frac{\partial g_1}{\partial p_3} + x_2 \frac{\partial g_2}{\partial p_3} + x_3 = 0, \quad Z = 0.$$

Manifestly, it is given by the elimination of p_3 between the equations

$$\left. \begin{aligned} z &= x_1 g_1(p_3) + x_2 g_2(p_3) + x_3 p_3 \\ 0 &= x_1 \frac{\partial g_1}{\partial p_3} + x_2 \frac{\partial g_2}{\partial p_3} + x_3 \end{aligned} \right\};$$

it represents the second class of general integrals.

The equations

$$x_1 = P_1 = 0, \quad x_2 = P_2 = 0, \quad x_3 = P_3 = 0,$$

with $Z = 0$, clearly do not provide an integral: there is no singular integral.

Ex. 2. Integrate in the same way the equations:—

(i) $z = p_1 x_1 + p_2 x_2 + p_3 x_3 + p_1 p_2 p_3$;

(ii) $z = p_1 x_2 + p_2 x_3 + p_3 x_1 + p_1 p_2 p_3$;

(iii) $z^2 = (x_1 - a p_1)^2 + (x_2 - a p_2)^2 + (x_3 - a p_3)^2$.

133. Next, suppose that the dependent variable z does not occur explicitly in the differential equation to be solved, which therefore is of the form

$$f(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0.$$

We then take

$$X_n = f(x_1, \ldots, x_n, p_1, \ldots, p_n),$$

and, applying the results of § 130, we assume that we have functions $\Pi, X_1, \ldots, X_{n-1}, P_1, \ldots, P_n$ of $x_1, \ldots, x_n, p_1, \ldots, p_n$, such that the relation

$$d\Pi - \sum_{i=1}^{n} P_i dX_i = - \sum_{i=1}^{n} p_i dx_i$$

is satisfied identically. The quantities $P_1, \ldots, P_n$ are known by means of equations characteristic of the contact transformation, when once $\Pi, X_1, \ldots, X_n$ are known: and Π is determined by a number of equations, as soon as $X_1, \ldots, X_n$ are known. These quantities are subject to the equations

$$(X_i, X_j) = 0,$$

for i and $j = 1, \ldots, n$, the values of i and j being unequal. As has been seen, any one of these quantities can be arbitrarily assumed: accordingly, we assign $f(x_1, \ldots, x_n, p_1, \ldots, p_n)$ as the value of X_n. And then, adopting the general results of the theory of contact transformations, we may suppose that the quantities $X_1, \ldots, X_{n-1}$, $\Pi, P_1, \ldots, P_n$ are known.

What is desired is an integral of the equation

$$X_n = f(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0.$$

For that purpose, $p_1, \ldots, p_n$ are derivatives of z with regard to $x_1, \ldots, x_n$ respectively, so that

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0;$$

hence the quantities, defining the contact transformation in the present circumstances, must satisfy the relation

$$d\Pi - \sum_{i=1}^{n} P_i dX_i = - dz,$$

or, since X_n is a permanent zero, we must have

$$d(z + \Pi) - P_1 dX_1 - \ldots - P_{n-1} dX_{n-1} = 0.$$

This is the only relation, other than $X_n = 0$, which needs to be satisfied in order to secure the existence of the relation

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0;$$

we therefore require an integral equivalent of the differential relation

$$d(z + \Pi) - P_1 dX_1 - \ldots - P_{n-1} dX_{n-1} = 0,$$

so that it may be satisfied identically. As in the former case, there are varieties of integral equivalents: but, as will be seen, the singular integral does not occur.

(i) The differential relation is satisfied identically if

$$z + \Pi = c, \quad X_1 = a_1, \ldots, X_{n-1} = a_{n-1},$$

where $c, a_1, \ldots, a_{n-1}$ are arbitrary constants. These equations are to be taken concurrently with $X_n = 0$; and the quantities Π, $X_1, \ldots, X_{n-1}, X_n$ are functionally independent of one another. Hence we may consider the equations

$$X_1 = a_1, \ldots, X_{n-1} = a_{n-1}, \quad X_n = 0$$

resolved* with regard to $p_1, \ldots, p_n$, and the deduced values substituted in Π: the result is of the form

$$z + h(x_1, \ldots, x_n, a_1, \ldots, a_{n-1}) = c.$$

The integral clearly is the *complete integral.*

(ii) The differential relation is satisfied identically if a number of relations

$$z + \Pi = g\,(X_{\mu+1}, \ldots, X_{n-1}),$$
$$X_i = g_i(X_{\mu+1}, \ldots, X_{n-1}),$$

for $i = 1, \ldots, \mu$, hold, provided the further relations

$$\frac{\partial g}{\partial X_j} - \sum_{i=1}^{\mu} P_i \frac{\partial g_i}{\partial X_j} - P_j = 0,$$

for $j = \mu + 1, \ldots, n - 1$, also hold. Eliminating $p_1, \ldots, p_n$ among these n relations and $X_n = 0$, we have an integral: it is a *general integral.* Obviously there will be a number of classes of general integrals, a class being determined by the number of functional relations postulated. The most comprehensive general integral is given by the equations

$$z + \Pi = F(X_1, \ldots, X_{n-1}),$$
$$\frac{\partial F}{\partial X_r} = P_r, \quad X_n = 0,$$

for $r = 1, \ldots, n - 1$, the function F being completely arbitrary.

(iii) The differential relation is satisfied identically if

$$P_1 = 0, \ldots, P_{n-1} = 0, \quad z + \Pi = a,$$

* Exceptions might arise, when resolution with regard to $p_1, \ldots, p_n$ is either inconvenient or impossible: we should then proceed as in §§ 58, 59.

where a is a constant that may be zero. Eliminating $p_1, \ldots, p_n$ among these n relations and $X_n = 0$, we have an integral which is an exceedingly special case of the foregoing general integral obtained by taking a as the expression of $F(X_1, \ldots, X_{n-1})$. The relation

$$dz + d\Pi = 0$$

gives

$$d\Pi = -\sum_{i=1}^{n} p_i dx_i,$$

so that

$$\frac{\partial \Pi}{\partial p_i} = 0, \quad \frac{\partial \Pi}{\partial x_i} = -p_i,$$

for $i = 1, \ldots, n$, which are in agreement with the equations characteristic of the contact transformation in this case.

And these exhaust the modes of satisfying the differential relation: thus a singular integral does not arise.

It appears, from the results of § 132 and from the results just obtained, that the construction of the various integrals of a partial differential equation

$$U = 0$$

can immediately be effected, if a contact transformation of which U is an element is known; hence this method of proceeding requires either a knowledge, or the determination, of such a contact transformation. If it has to be determined, then a number of simultaneous partial equations have to be solved.

Ex. As an illustration, consider the equation

$$p_1^2 + p_2^2 + p_3^2 = 1.$$

We take

$$X_3 = p_1^2 + p_2^2 + p_3^2 - 1;$$

a contact transformation of which X_3 is an element, is given by

$$X_1 = p_1, \quad X_2 = p_2,$$

$$\Pi = -p_1 x_1 - p_2 x_2 - p_3 x_3,$$

$$P_1 = -x_1 + \frac{p_1}{p_3} x_3, \quad P_2 = -x_2 + \frac{p_2}{p_3} x_3, \quad P_3 = -\frac{x_3}{2p_3}.$$

The complete integral of the equation is given by the elimination of p_1, p_2, p_3 between

$$z + \Pi = c, \quad X_1 = a_1, \quad X_2 = a_2, \quad X_3 = 0:$$

that is, the complete integral is given by

$$z - a_1 x_1 - a_2 x_2 - a_3 x_3 = c,$$

where

$$a_3^2 = 1 - a_1^2 - a_2^2.$$

There are two classes of general integrals. A general integral of the first class is given by

$$z+\Pi=f(X_1, X_2), \quad X_3=0,$$
$$P_1=\frac{\partial f}{\partial X_1}, \quad P_2=\frac{\partial f}{\partial X_2},$$

where f is an arbitrary function: that is, it is given by the elimination of p_1, p_2, p_3 among the equations

$$\left.\begin{aligned} p_1^2+p_2^2+p_3^2&=1\\ z-p_1x_1-p_2x_2-p_3x_3&=f(p_1, p_2)\\ -x_1+\frac{p_1}{p_3}x_3&=\frac{\partial f}{\partial p_1}\\ -x_2+\frac{p_2}{p_3}x_3&=\frac{\partial f}{\partial p_2} \end{aligned}\right\}.$$

A general integral of the second class is given by

$$z+\Pi=f(X_2), \quad X_1=g(X_2), \quad X_3=0,$$
$$\frac{df}{dX_2}-P_1\frac{dg}{dX_2}-P_2=0,$$

that is, by the elimination of p_1, p_2, p_3 among the equations

$$\left.\begin{aligned} p_1^2+p_2^2+p_3^2&=1\\ z-p_1x_1-p_2x_2-p_3x_3&=f(p_2)\\ p_1&=g(p_2)\\ \frac{df}{dp_2}+\left(x_1-\frac{p_1}{p_3}x_3\right)\frac{dg}{dp_2}+x_2-\frac{p_2}{p_3}x_3&=0 \end{aligned}\right\}.$$

There is no singular integral.

134. In the same manner, the properties of contact transformations can be applied to obtain the integrals of a system of equations. We may assume that the system is complete, that is, that all the relations

$$[X_i, X_j]=0, \quad (X_i, X_j)=0,$$

according as the dependent variable does or does not occur, are satisfied, either identically or in virtue of the equations of the system: and we may also assume that the number of such equations is less than $n+1$ or less than n, in the two cases respectively.

In the first place, suppose that the dependent variable does occur: and let the complete system be

$$Z=0, \quad X_1=0, \ldots, X_m=0,$$

where $m<n$, the relations

$$[Z, X_i]=0, \quad [X_i, X_j]=0,$$

being satisfied. If these relations are satisfied identically*, the quantities $Z, X_1, \ldots, X_m$ can be constituents of a contact transformation $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$: for the quantities $Z, X_1, \ldots, X_n$ are determined by equations

$$[Z, X_r] = 0, \quad [X_r, X_s] = 0,$$

for r and $s = 1, \ldots, n$: and the quantities $P_1, \ldots, P_n$ are then given by linear algebraic equations. This contact transformation leads to the differential relation

$$dZ - P_1 dX_1 - \ldots - P_n dX_n = \rho (dz - p_1 dx_1 - \ldots - p_n dx_n),$$

where ρ is a non-vanishing quantity.

When the quantity z is an integral of the given system, we must have

$$dz = p_1 dx_1 + \ldots + p_n dx_n:$$

also

$$dZ = 0, \quad dX_1 = 0, \ldots, dX_m = 0,$$

because $Z = 0, X_1 = 0, \ldots, X_m = 0$ permanently: hence the above relation becomes

$$P_{m+1} dX_{m+1} + \ldots + P_n dX_n = 0,$$

and when this relation is satisfied, we have

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0,$$

so that an integral of the equation is provided. As before, the relation can be satisfied in three kinds of ways.

(i) The relation will be satisfied if

$$X_{m+1} = a_{m+1}, \ldots, X_n = a_n,$$

where $a_{m+1}, \ldots, a_n$ are constants which may be taken arbitrarily. The quantities $Z, X_1, \ldots, X_n$ are independent of one another: hence, eliminating $p_1, \ldots, p_n$ between these $n - m$ equations and the $m + 1$ equations of the complete system, we have a relation between $z, x_1, \ldots, x_n$, which involves $n - m$ arbitrary constants and provides an integral of the system. The integral thus provided is clearly the *complete integral*.

(ii) The relation will be satisfied if the quantities $X_{m+1}, \ldots,$

* The alternative, when these relations are satisfied only in virtue of the members of the complete system, will be a matter for subsequent consideration.

X_n are connected by q equations (where q is less than $n-m$) which may be taken in the form

$$g_1(X_{m+1}, \ldots, X_n) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$g_q(X_{m+1}, \ldots, X_n) = 0;$$

but in addition, the equations

$$P_{m+r} = \lambda_1 \frac{\partial g_1}{\partial X_{m+r}} + \ldots + \lambda_q \frac{\partial g_q}{\partial X_{m+r}},$$

for $r = 1, \ldots, n-m$, must be satisfied. We now have the $m+1$ equations of the original system, the q relations of condition among the variables $X_{m+1}, \ldots, X_n$, and the deduced $n-m$ equations connecting the quantities $P_{m+1}, \ldots, P_n$ with the variables $X_{m+1}, \ldots, X_n$: that is, there are $n+1+q$ equations in all. Eliminating $p_1, \ldots, p_n, \lambda_1, \ldots, \lambda_q$ among this aggregate, we obtain a single equation as the eliminant: the form of the equation is affected by the form of the q postulated relations of condition. The value of z thus provided is an integral: obviously it is a *general integral.*

Manifestly there are $n-m-1$ classes of general integrals, determined by the assignment of $1, 2, \ldots, n-m-1$ relations of condition among the variables $X_{m+1}, \ldots, X_n$. The most comprehensive class is that determined by the assignment of only a single relation: if this relation be taken in the form

$$X_{m+1} = h(X_{m+2}, \ldots, X_n),$$

then

$$P_{m+r} + \frac{\partial h}{\partial X_{m+r}} P_{m+1} = 0,$$

for $r = 2, \ldots, n-m-2$: and, as before, we obtain the integral by elimination.

(iii) The relation will be satisfied if

$$P_{m+1} = 0, \ \ldots, \ P_n = 0.$$

We then have

$$dZ - P_1 dX_1 - \ldots - P_m dX_m = \rho(dz - p_1 dx_1 - \ldots - p_n dx_n),$$

so that

$$\frac{\partial Z}{\partial z} - P_1 \frac{\partial X_1}{\partial z} - \ldots - P_m \frac{\partial X_m}{\partial z} = \rho,$$
$$\frac{\partial Z}{\partial x_i} - P_1 \frac{\partial X_1}{\partial x_i} - \ldots - P_m \frac{\partial X_m}{\partial x_i} = -\rho p_i,$$
$$\frac{\partial Z}{\partial p_j} - P_1 \frac{\partial X_1}{\partial p_j} - \ldots - P_m \frac{\partial X_m}{\partial p_j} = 0,$$

for i and $j=1, \ldots, n$. Eliminating $P_1, \ldots, P_m, \rho$ among these $2n+1$ equations, we have $2n-m$ conditions to be satisfied among the derivatives of $Z, X_1, \ldots, X_m$, which can be expressed in the form

$$\left\| \begin{array}{cccccc} \frac{dZ}{dx_1}, & \ldots, & \frac{dZ}{dx_n}, & \frac{\partial Z}{\partial p_1}, & \ldots, & \frac{\partial Z}{\partial p_n} \\ \frac{dX_1}{dx_1}, & \ldots, & \frac{dX_1}{dx_n}, & \frac{\partial X_1}{\partial p_1}, & \ldots, & \frac{\partial X_1}{\partial p_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dX_m}{dx_1}, & \ldots, & \frac{dX_m}{dx_n}, & \frac{\partial X_m}{\partial p_1}, & \ldots, & \frac{\partial X_m}{\partial p_n} \end{array} \right\| = 0.$$

When these are satisfied, either identically or in virtue of

$$Z=0, \quad X_1=0, \ldots, X_m=0, \quad P_{m+1}=0, \ldots, P_n=0,$$

then the elimination of $p_1, \ldots, p_n$ among these n equations leads to an equation connecting $z, x_1, \ldots, x_n$. This value of z is an integral of the equation: clearly it is the *singular* integral.

Ex. Obtain integrals of the equations

$$\left.\begin{array}{r} Z=z-p_1x_1-p_2x_2-p_3x_3-a=0 \\ X_1=p_1^2+p_2^2+p_3^2-c=0 \end{array}\right\},$$

which satisfy the condition

$$[Z, X_1]=0$$

of coexistence: the quantities a and c are given constants.

The quantities Z and X_1 are constituents of a contact transformation, represented in full by

$$Z, \quad X_1, \quad X_2=p_1, \quad X_3=p_2,$$

$$P_1=\tfrac{1}{2}\frac{x_3}{p_3}, \quad P_2=x_1-x_3\frac{p_1}{p_3}, \quad P_3=x_2-x_3\frac{p_2}{p_3}.$$

The complete integral is represented by

$$X_2=a_2, \quad X_3=a_3,$$

where a_2 and a_3 are arbitrary constants: combining these with

$$Z=0, \quad X_1=0,$$

we find the complete integral given by

$$z=a_1x_1+a_2x_2+a_3x_3+a,$$

where a_1 and a_2 are arbitrary, and a_3 is given by

$$a_1^2+a_2^2+a_3^2=1:$$

or, expressed solely in terms of independent constants, the integral is

$$(z-a_1x_1-a_2x_2-a)^2=x_3^2(1-a_1^2-a_2^2).$$

To find the general integral, we take

$$X_2 = g(X_3),$$
$$P_2 g'(X_3) + P_3 = 0;$$

that is, the general integral is given by the elimination of p_1, p_2, p_3 among the equations

$$\left.\begin{aligned} z - p_1x_1 - p_2x_2 - p_3x_3 - a &= 0 \\ p_1^2 + p_2^2 + p_3^2 - c &= 0 \\ p_1 &= g(p_2) \\ \left(x_1 - x_3\frac{p_1}{p_3}\right) g'(p_2) + x_2 - x_3\frac{p_2}{p_3} &= 0 \end{aligned}\right\}.$$

The singular integral, if it exists, is given by

$$Z = 0, \quad X_1 = 0, \quad P_2 = 0, \quad P_3 = 0.$$

The further necessary conditions are satisfied. The elimination of p_1, p_2, p_3 among these four equations leads to a relation

$$(z - a)^2 = c(x_1^2 + x_2^2 + x_3^2),$$

which is therefore the singular integral.

135. Next, suppose that the dependent variable does not occur explicitly and that the complete system is

$$X_1 = 0, \ \ldots, \ X_m = 0,$$

where the relations

$$(X_i, X_j) = 0,$$

for i and $j = 1, \ldots, m$, are satisfied, the integer m being less than n. As these relations are satisfied, the quantities $X_1, \ldots, X_m$ are constituents of a contact transformation such that

$$Z = z + \Pi,$$

and quantities $\Pi, X_{m+1}, \ldots, X_n, P_1, \ldots, P_n$, being functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$, exist such that the differential relation

$$d\Pi - \sum_{i=1}^{n} P_i dX_i = -\sum_{i=1}^{n} p_i dx_i$$

is satisfied identically.

What is desired is an integral of the system

$$X_1 = 0, \ \ldots, \ X_m = 0,$$

so that we must have

$$dX_1 = 0, \ \ldots, \ dX_m = 0;$$

also, as the value of z is to be an integral of this system, we have

$$dz = p_1 dx_1 + \ldots + p_n dx_n.$$

Thus the differential relation can be taken in the form

$$d(z+\Pi) - P_{m+1}dX_{m+1} - \ldots - P_n dX_n = 0.$$

When this relation is satisfied concurrently with the m equations of the system, then the relation

$$dz = p_1 dx_1 + \ldots + p_n dx_n$$

also is satisfied: so that the value of z, expressed in terms of $x_1, \ldots, x_n$, is an integral.

As in the case of a single equation, there are two distinct kinds of integral equivalents of the differential relation, leading to a complete integral and to the general integrals respectively: there is no singular integral.

(i) The differential relation is satisfied identically if

$$z + \Pi = c, \quad X_{m+1} = a_{m+1}, \ \ldots, \ X_n = a_n,$$

where $c, a_{m+1}, \ldots, a_n$ are arbitrary constants. These equations exist concurrently with the equations

$$X_1 = 0, \ \ldots, \ X_m = 0;$$

and the quantities $\Pi, X_1, \ldots, X_m, X_{m+1}, \ldots, X_n$ are functionally independent of one another. We may therefore resolve the set of equations

$$X_1 = 0, \ \ldots, \ X_m = 0, \quad X_{m+1} = a_{m+1}, \ \ldots, \ X_n = a_n$$

with regard to the variables* $p_1, \ldots, p_n$: when we substitute their values in

$$z + \Pi = c,$$

we have an integral of the system of equations in a form

$$z + H(x_1, \ldots, x_n, a_{m+1}, \ldots, a_n) = c,$$

involving $n - m + 1$ variables. The integral is clearly the *complete integral.*

(ii) The differential relation is satisfied identically if a number of relations

$$z + \Pi = g \ (X_{\mu+1}, \ \ldots, \ X_n),$$
$$X_{m+r} = g_r (X_{\mu+1}, \ \ldots, \ X_n),$$

* Exceptions may arise when the resolution with regard to another set of n of the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ is more convenient: we then proceed as in §§ 58, 59.

for $r = 1, \ldots, \mu - m$, hold provided the further relations

$$\frac{\partial g}{\partial X_j} - \sum_{r=1}^{\mu-m} P_{m+r} \frac{\partial g_r}{\partial X_j} - P_j = 0,$$

for $j = \mu + 1, \ldots, n$, also hold. Eliminating $p_1, \ldots, p_n$ among these $n - m + 1$ relations and the equations of the original system, we have an integral of that system. It is a *general integral.*

Obviously, there will be a number of classes of general integrals, a class being determined by the number of functional relations postulated. The most comprehensive general integral is given by the elimination of $p_1, \ldots, p_n$ among the equations

$$z + \Pi = h(X_{m+1}, \ldots, X_n),$$
$$\frac{\partial h}{\partial X_r} = P_r, \quad X_1 = 0, \ldots, X_m = 0,$$

for $r = m + 1, \ldots, n$.

(iii) The differential relation is satisfied identically if

$$P_{m+1} = 0, \ \ldots, \ P_n = 0, \quad z + \Pi = a,$$

where a is a constant that may be zero. Eliminating $p_1, \ldots, p_n$ among these $n - m + 1$ relations and the m equations of the complete system, we obtain an integral which is an exceedingly special case of the foregoing comprehensive general integral, the case arising by taking a as the expression of the function $h(X_{m+1}, \ldots, X_n)$. The relation

$$dz + d\Pi = 0$$

gives

$$d\Pi = -\sum_{i=1}^{n} p_i dx_i,$$

so that

$$\frac{\partial \Pi}{\partial p_i} = 0, \quad \frac{\partial \Pi}{\partial x_i} = -p_i,$$

for $i = 1, \ldots, n$; these equations are characteristic of the contact transformation for the present case.

The modes of satisfying the differential relation are exhausted: hence a singular integral does not arise for the complete system in the supposed circumstance that the complete system does not explicitly involve the dependent variable.

Ex. 1. Integrate the simultaneous equations

$$\left.\begin{array}{l} F=x_2p_1+x_1p_2+ap_3(p_1-p_2)-c=0 \\ G=(p_1+p_2)(x_1+x_2)-b=0 \end{array}\right\},$$

where a, b, and c are constants.

It is easy to verify that the condition

$$(F, G)=0$$

is satisfied identically, and that, if

$$H=p_3,$$

then the equations

$$(F, H)=0, \quad (G, H)=0,$$

also are satisfied identically. Hence, taking

$$\begin{aligned} X_1&=x_2p_1+x_1p_2+ap_3(p_1-p_2)-c, \\ X_2&=(p_1+p_2)(x_1+x_2)-b, \\ X_3&=p_3, \end{aligned}$$

the quantities X_1, X_2, X_3 can be constituents of a contact transformation.

To complete the contact transformation, the quantities Π, P_1, P_2, P_3 are required. Of these, Π satisfies the three equations

$$\begin{aligned} (\Pi, X_3)&=-\sum_{r=1}^{3} p_r\frac{\partial X_3}{\partial p_r}=-p_3, \\ (\Pi, X_2)&=-\sum_{r=1}^{3} p_r\frac{\partial X_2}{\partial p_r}=-(p_1+p_2)(x_1+x_2), \\ (\Pi, X_1)&=-\sum_{r=1}^{3} p_r\frac{\partial X_1}{\partial p_r}=-p_1(x_2+ap_3)-p_2(x_1-ap_3)-ap_3(p_1-p_2) \\ &=-p_1x_2-p_2x_1-2ap_3(p_1-p_2). \end{aligned}$$

From the first of these, we have

$$\Pi+p_3x_3=u,$$

where u is any function of x_1, x_2, p_1, p_2, p_3. From the second of them, we have

$$u+\tfrac{1}{2}(X_2+b)\log(x_1+x_2)=v,$$

where v is any function of

$$X_2+b, \quad x_2-x_1, \quad p_2-p_1, \quad p_3,$$

or say of X_2+b, α, β, p_3, where

$$\alpha=x_2-x_1, \quad \beta=p_2-p_1.$$

From the third of the equations, we have

$$\frac{2v}{\beta(\alpha+2ap_3)+b}+\log(\alpha+2ap_3)=w,$$

where w is any function of X_2+b, $\beta(\alpha+2ap_3)+b$, p_3. Now for the integration of our equations, $X_1=0$, $X_2=0$; and

$$\begin{aligned} \beta(\alpha+2ap_3)&=X_2-2X_1+b-2c \\ &=b-2c, \end{aligned}$$

so that effectively w is a function of p_3: consequently,

$$\Pi + p_3 x_3 + \tfrac{1}{2}(X_2 + b)\log(x_1 + x_2)$$
$$= \tfrac{1}{2}(X_2 - 2X_1 + 2b - 2c)\{-\log(x_2 - x_1 + 2ap_3) + \phi(p_3)\}.$$

The complete integral is given by

$$z + \Pi = A, \quad X_1 = 0, \quad X_2 = 0, \quad X_3 = C,$$

where A and C are arbitrary constants; consequently, it is

$$A = z - Cx_3 - \tfrac{1}{2}b\log(x_1 + x_2) - (b - c)\{-\log(x_2 - x_1 + 2aC) + \phi(C)\},$$

or as $(b-c)\phi(C)$ can be absorbed into A, this complete integral is

$$z - Cx_3 - \tfrac{1}{2}b\log(x_1 + x_2) - (b - c)\log(x_2 - x_1 + 2aC) = A.$$

The general integral is given by

$$z + \Pi = g(X_3), \quad X_1 = 0, \quad X_2 = 0,$$

together with the relation

$$\frac{dg}{dX_3} = P_3.$$

Also, the quantities P_1, P_2, P_3 are given by any three of the six equations

$$\frac{\partial \Pi}{\partial x_i} - \sum_{r=1}^{3} P_r \frac{\partial X_r}{\partial x_i} = -p_i, \qquad \frac{\partial \Pi}{\partial p_i} - \sum_{r=1}^{3} P_r \frac{\partial X_r}{\partial p_i} = 0$$

which are independent of one another: in particular,

$$P_3 = \frac{\partial \Pi}{\partial p_3} + \frac{a(p_1 - p_2)}{x_2 - x_1 + 2ap_3}\left(\frac{\partial \Pi}{\partial p_1} - \frac{\partial \Pi}{\partial p_2}\right),$$

which, on substitution, gives

$$P_3 = -x_3 + (b - c)\left\{\phi'(p_3) - \frac{2a}{x_2 - x_1 + 2ap_3}\right\}.$$

Hence the general integral is given by the elimination of p_3 between the equations

$$z - p_3 x_3 - \tfrac{1}{2}b\log(x_1 + x_2) - (b - c)\{\log(x_2 - x_1 + 2ap_3) - \phi(p_3)\} = g(p_3),$$
$$-x_3 + (b - c)\left\{\phi'(p_3) - \frac{2a}{x_2 - x_1 + 2ap_3}\right\} = g'(p_3),$$

or, writing

$$g(p_3) - (b - c)\phi(p_3) = h(p_3),$$

so that $h(p_3)$ is a new arbitrary function, we have the general integral given by the elimination of p_3 between

$$\left.\begin{aligned} z - p_3 x_3 - \tfrac{1}{2}b\log(x_1 + x_2) - (b - c)\log(x_2 - x_1 + 2ap_3) &= h(p_3) \\ -x_3 - \frac{2a(b - c)}{x_2 - x_1 + 2ap_3} &= h'(p_3) \end{aligned}\right\}.$$

And there is no singular integral.

Note. The example is worked out in order to illustrate the relation between the constituents of the contact transformation and the construction of the integral.

If the Jacobian method (Chap. IV) were adopted for the integration of

$$F = 0, \quad G = 0,$$

we should find a third equation to associate with these: proceeding in the usual way, we could take either

$$p_3 = a_3,$$

or

$$a(p_1 - p_2) - x_3 = c_3,$$

where a_3 and c_3 are arbitrary constants. We then resolve the equations

$$F=0, \quad G=0,$$

with one of the equations

$$p_3 = a_3, \quad a(p_1 - p_2) - x_3 = c_3,$$

for p_1, p_2, p_3; we substitute in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3,$$

and effect a quadrature. The resulting equation gives the complete integral: and the general integral would be deduced by the customary process.

Writing

$$p_3 = H, \quad a(p_1 - p_2) - x_3 = K,$$

we have

$$(F, G) = 0, \quad (F, H) = 0, \quad (G, H) = 0,$$

all satisfied identically, so that F, G, H can be the constituents X_1, X_2, X_3 of a contact transformation. Also

$$(F, K) = 0, \quad (G, K) = 0,$$

satisfied identically, so that F, G, K can be the constituents X_1, X_2, X_3 of another contact transformation. The integration of

$$F=0, \quad G=0,$$

by means of the latter contact transformation is left as an exercise.

But

$$(H, K) = 1;$$

hence F, H, K cannot be the constituents X_1, X_2, X_3 of a contact transformation, nor can G, H, K be those constituents.

Ex. 2. Integrate by this method the equations

$$\left.\begin{aligned} x_2 p_1 + x_1 p_2 + x_3 p_3 - c = 0 \\ a(p_1 - p_2) - x_3 = 0 \end{aligned}\right\},$$

where a and c are constants.

Ex. 3. Similarly integrate the equations

$$\left.\begin{aligned} x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4 = a \\ x_3 p_1 + x_4 p_2 - x_1 p_3 - x_2 p_4 = c \end{aligned}\right\},$$

where a and c are constants.

136. It thus appears that, when a single differential equation is given in a form

$$U = 0,$$

the quantity U can always be made a constituent of a contact transformation, the explicit expression of which adopts one or other

of two forms according as U does, or does not, explicitly involve the dependent variable: and the various kinds of integrals* can be deduced when the full expression of the transformation is known.

Again, when a complete system of equations is given in a form

$$X_1 = 0, \ldots, X_m = 0,$$

(where m is less than the number of independent variables), so that the relations

$$[X_i, X_j] = 0, \text{ or } (X_i, X_j) = 0,$$

(according as the equations in the system do, or do not, involve the dependent variable explicitly) are satisfied, it has been proved that, if the relations are satisfied identically, the theory of contact transformations can be applied† immediately by making $X_1, \ldots, X_m$ constituents of the transformation in the mode indicated. The various kinds of integrals‡ could be deduced when the full expression of the transformation is known.

Consequently, in order that the method of contact transformations may be made effective for the practical integration of a single equation or of a complete system of equations, it is necessary to devise a practical process for the completion of a contact transformation when one constituent can be assumed or when several constituents can be assumed. These constituents have been supposed, in every case thus far considered, to belong to the X-type and not to the P-type in a relation

$$dZ - \sum_{i=1}^{n} P_i dX_i = \rho\left(dz - \sum_{i=1}^{n} p_i dx_i\right);$$

alternative suppositions have not been considered.

* No indication of the occurrence of *special* integrals has been given: these, however, usually depend on the details of form of particular equations, and such details have been ignored. They would, of course, have to be taken into account if a complete theory were being based upon contact transformations and their properties, and upon these alone: but the present chapter has no such purpose. What is intended, is the exposition of the more important general methods of integration, in a sufficient amount of detail to make their scope clear: among such methods, contact transformations must find a place.

† No implication has been made that the theory cannot be applied unless the relations are satisfied identically: the condition was requisite merely for the immediate use of the propositions quoted in §§ 128—130.

‡ With the same exception of special integrals as occurs sometimes when a single equation is propounded for integration.

Moreover, it is quite possible (examples, indeed, have occurred freely in illustration of preceding discussions) that a system $X_1 = 0, \ldots, X_m = 0$ should be complete, even though the relations

$$[X_i, X_j] = 0, \text{ or } (X_i, X_j) = 0$$

should not be satisfied identically; all that is necessary, is that the relations should be satisfied simultaneously with the equations of the system without the intervention of any other equation. The question thus is raised as to the use (if any) that may be made of the property, when the relations of coexistence are satisfied only in virtue of the equations of the system; and the answer to the question is to be found in the properties of groups of functions.

Groups of Functions.

137. Accordingly, we proceed to the consideration of the simpler properties of *groups of functions**: they are based upon the properties of contact transformations, and their development is sufficiently distinct to cause their application to be regarded as a distinct method for the integration of systems of equations.

Also, partly for the sake of some simplicity in the formulæ, and partly because there is no intention of developing the full theory, it will be assumed that we are dealing with the limited contact transformations of § 130, so that we may take

$$z' = z + \Pi, \quad p_i' = P_i, \quad x_i' = X_i,$$

where $i = 1, \ldots, n$, and $\Pi, P_1, \ldots, P_n, X_1, \ldots, X_n$ are functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$, as the type of transformation to be discussed.

The general notion of a group is of exceedingly comprehensive range: for the immediate purpose, it may be described by saying that the aggregate of a number of entities is called a group when, if those entities are compounded in all possible ways which conform to assigned laws, the results of the composition can be expressed in terms of those entities by forms which are subject to other assigned laws.

* This theory is, of course, only a part of Lie's comprehensive theory of transformation-groups. A full exposition is given in his treatise *Theorie der Transformationsgruppen*, vol. II, pp. 178 *et seq.*: see also *Math. Ann.*, t. VIII (1875), pp. 248 *et seq.*, *ib.*, t. XI (1877), pp. 465 *et seq.*

The arguments of the functions are $2n$ in number, taken to be $x_1, \ldots, x_n, p_1, \ldots, p_n$ in two complementary sets, and to be independent of one another, so far as the properties of the sets of functions are concerned. When two functions u and v of the group are known, the assigned law of composition is that they shall be combined in the form

$$\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial p_i}-\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial x_i}\right);$$

in accordance with the notation already used, this combination is represented by

$$(u, v).$$

Then r functions $u_1, \ldots, u_r$ of the $2n$ variables are, for our purpose, said to be a *group* when the following conditions are satisfied:—

(i) the functions $u_1, \ldots, u_r$ are algebraically independent of one another:

(ii) every combination (u_i, u_j), for i and $j = 1, \ldots, r$, is expressible by means of the r functions $u_1, \ldots, u_r$, the expressions not requiring the introduction of other functions of the variables.

When the conditions are satisfied, the group is said to be of *order* r: and every function of the variables which, when combined with the functions $u_1, \ldots, u_r$ in turn, satisfies the condition of expression in terms of the members $u_1, \ldots, u_r$ of the group, is said to *belong to* the group.

Further, it will appear possible that, in a group of order r, there may be a set of r' functions which, taken by themselves, satisfy the conditions for a group: naturally, in this case, $r' < r$. The set of r' functions is then called a *sub-group* of the group of order r.

When all the combinations (u_i, u_j), for i and $j = 1, \ldots, r$, vanish, the group is called a *system in involution*.

A function v of the $2n$ variables, such that $(v, u_i) = 0$ for $i = 1, \ldots, n$, is said to be *in involution with the group* $u_1, \ldots, u_r$.

Some simple properties of groups, that are practically obvious, may at once be noted.

(i) Every function of the members of a group belongs to the group.

(ii) If r functions $v_1, \ldots, v_r$ of the members of $u_1, \ldots, u_r$ of a group of order r be algebraically independent of one another, they constitute a group. This group, composed of $v_1, \ldots, v_r$, is regarded merely as another form of the original group: thus a group can have an unlimited number of forms.

When any form of a group is in involution, every form of the group is also in involution.

(iii) When r functions $u_1, \ldots, u_r$ are connected by s, and by not more than s, relations, and when all the combinations (u_i, u_j) are expressible in terms of $u_1, \ldots, u_r$, the functions belong to a group of order $r-s$.

(iv) When two different groups have members in common with one another, the aggregate of those common members constitutes a group; for if the aggregate be $t_1, \ldots, t_m$, the combinations (t_i, t_j), for i and $j = 1, \ldots, m$, are expressible in terms of members of the first group and also in terms of members of the second group, that is, they are expressible in terms of the members common to the two groups.

(v) A group of functions, involving $2n$ variables in two complementary sets, is of order not greater than $2n$; for if the number of members were greater than $2n$, they could not be algebraically independent.

We shall see later that the order of a group in involution cannot be greater than n.

(vi) A contact transformation changes a group of order r into another group of order r.

Let the variables introduced by the transformation be $x_1', \ldots, x_n', p_1', \ldots, p_n'$, so that the equations characteristic of the transformation are

$$(x_i', x_j') = 0, \quad (p_i', p_j') = 0, \quad (p_i', x_j') = 0, \quad (p_i', x_i') = -1,$$

where i and $j = 1, \ldots, n$, and are unequal to one another, the independent variables being $x_1, \ldots, x_n, p_1, \ldots, p_n$: also

$$(x_i, x_j) = 0, \quad (p_i, p_j) = 0, \quad (p_i, x_j) = 0, \quad (p_i, x_i) = -1,$$

the independent variables being $x_1', \ldots, x_n', p_1', \ldots, p_n'$.

Let $u_1, \ldots, u_r$ be the group to be transformed, and let v_i be the value of u_i (for $i = 1, \ldots, r$) after the transformation has been

effected; as $u_1, \dots, u_r$ are algebraically independent of one another, so also are $v_1, \dots, v_r$. Then, since

$$\frac{\partial v_i}{\partial x_s'} = \sum_{\mu=1}^{n} \left(\frac{\partial u_i}{\partial x_\mu} \frac{\partial x_\mu}{\partial x_s'} + \frac{\partial u_i}{\partial p_\mu} \frac{\partial p_\mu}{\partial x_s'} \right),$$

$$\frac{\partial v_i}{\partial p_s'} = \sum_{\mu=1}^{n} \left(\frac{\partial u_i}{\partial x_\mu} \frac{\partial x_\mu}{\partial p_s'} + \frac{\partial u_i}{\partial p_\mu} \frac{\partial p_\mu}{\partial p_s'} \right),$$

we have

$$\begin{aligned}
(v_i, v_j) &= \sum_{s=1}^{n} \left(\frac{\partial v_i}{\partial x_s'} \frac{\partial v_j}{\partial p_s'} - \frac{\partial v_i}{\partial p_s'} \frac{\partial v_j}{\partial x_s'} \right) \\
&= \sum_{\mu=1}^{n} \left\{ \left(\frac{\partial u_i}{\partial x_\mu} \frac{\partial u_j}{\partial p_\mu} - \frac{\partial u_i}{\partial p_\mu} \frac{\partial u_j}{\partial x_\mu} \right) \sum_{s=1}^{n} \left(\frac{\partial x_\mu}{\partial x_s'} \frac{\partial p_\mu}{\partial p_s'} - \frac{\partial x_\mu}{\partial p_s'} \frac{\partial p_\mu}{\partial x_s'} \right) \right\} \\
&= \sum_{\mu=1}^{n} \left(\frac{\partial u_i}{\partial x_\mu} \frac{\partial u_j}{\partial p_\mu} - \frac{\partial u_i}{\partial p_\mu} \frac{\partial u_j}{\partial x_\mu} \right) (x_\mu, p_\mu) \\
&= \sum_{\mu=1}^{n} \left(\frac{\partial u_i}{\partial x_\mu} \frac{\partial u_j}{\partial p_\mu} - \frac{\partial u_i}{\partial p_\mu} \frac{\partial u_j}{\partial x_\mu} \right) \\
&= (u_i, u_j).
\end{aligned}$$

But (u_i, u_j) is expressible in terms of $u_1, \dots, u_r$ alone, so that

$$\begin{aligned}
(u_i, u_j) &= \theta_{ij} (u_1, \dots, u_r) \\
&= \theta_{ij} (v_1, \dots, v_r);
\end{aligned}$$

hence

$$(v_i, v_j) = \theta_{ij} (v_1, \dots, v_r),$$

and therefore the functions $v_1, \dots, v_r$ form a group of order r.

A question is thus suggested, as follows. Given two groups of order r, each involving $2n$ variables in complementary sets: what are the contact transformations, if any, which transform one of the groups into the other? An answer to the question will be obtained through the determination of a canonical form to represent a group.

138. Passing now to properties of groups that are less obvious than those just given, we have the following theorem, due* to Lie:—

Let $u_1, \dots, u_r$ be a group of order r in $2n$ variables, composed of two complementary sets; then the r linear partial differential equations

$$(u_1, f) = 0, \dots, (u_r, f) = 0$$

are a complete system.

* It appears to have been the earliest result in Lie's researches, and was first published in 1872: see *Forh. af Christ.*, (1872), pp. 133—135, *ib.* (1873), pp. 237—262.

In the first place, the r equations are linearly independent of one another: for otherwise a set of determinants involving the derivatives of $u_1, \ldots, u_r$ would vanish and so would imply identical relations among the quantities $u_1, \ldots, u_r$, contrary to the hypothesis that the r functions constitute a group of order r.

When, for convenience, we write

$$(u_i, f) = A_i f,$$

for $i = 1, \ldots, r$, the Poisson-Jacobi identity

$$((u_i, u_j) f) + ((u_j, f) u_i) + ((f, u_i) u_j) = 0$$

becomes

$$((u_i, u_j) f) - A_i (A_j f) + A_j (A_i f) = 0,$$

that is,

$$\begin{aligned} A_i (A_j f) - A_j (A_i f) &= ((u_i, u_j) f) \\ &= \frac{\partial \theta_{ij}}{\partial u_1} A_1 f + \ldots + \frac{\partial \theta_{ij}}{\partial u_r} A_r f, \end{aligned}$$

where, as before,

$$(u_i, u_j) = \theta_{ij}(u_1, \ldots, u_r).$$

Hence all the equations

$$A_i (A_j f) - A_j (A_i f) = 0,$$

for i and $j = 1, \ldots, r$, are satisfied in virtue of the r equations

$$A_1 f = 0, \ldots, A_r f = 0,$$

which therefore (§ 37) are a complete system. The proposition is thus established.

As the system of r equations in $2n$ variables is complete, it possesses $2n - r$ distinct integrals: let these be $v_1, \ldots, v_{2n-r}$. It is known (§ 41) that every other integral of the complete system is expressible in terms of these $2n - r$ functionally distinct integrals; and by the Poisson-Jacobi theorem, (v_i, v_j) is an integral of the system: hence

$$(v_i, v_j) = \phi_{ij}(v_1, \ldots, v_{2n-r}),$$

and therefore the $2n - r$ functions $v_1, \ldots, v_{2n-r}$ form a group.

Applying Lie's theorem to the group $v_1, \ldots, v_{2n-r}$, we see that the system of $2n - r$ equations

$$(v_1, g) = 0, \ldots, (v_{2n-r}, g) = 0$$

is complete, so that it possesses r fundamentally distinct integrals. Evidently the members of the group $u_1, \ldots, u_r$ can be taken as these integrals; and all the equations

$$(u_i, v_j) = 0$$

are satisfied identically, owing to the property (§ 41) of integrals of homogeneous linear partial differential equations of the first order. We thus have the further theorem :—

A group of functions of order r in $2n$ variables determines another group of functions of order $2n-r$ in those variables, and conversely: every function of either group is in involution with the other group.

The two groups thus associated are called, sometimes *reciprocal* to one another, sometimes *polars* of one another.

COROLLARY I. *When a contact transformation is effected upon two reciprocal groups, the transformed groups are reciprocal to one another.* Let $u_1, \ldots, u_r$, and $v_1, \ldots, v_{2n-r}$, be the reciprocal groups, transformed into $u_1', \ldots, u_r'$, and $v_1', \ldots, v'_{2n-r}$, by the contact transformation: then, as before (§ 137),

$$\begin{aligned}(u_i', v_j') &= (u_i, v_j)\\ &= 0,\end{aligned}$$

for all values of i and j. This result is the analytical expression of the property stated.

COROLLARY II. *The order of a system in involution cannot be greater than n,* the total number of variables being $2n$ in two complementary sets. Let $u_1, \ldots, u_r$ be a system in involution, so that the equations

$$(u_i, u_j) = 0$$

are satisfied for all values of i and j. The equations

$$(u_1, f) = 0, \ \ldots, \ (u_r, f) = 0$$

are a complete Jacobian system, possessing $2n-r$ functionally independent integrals. Owing to the equations which express the involution, it is clear that $u_1, \ldots, u_r$ are integrals of the complete Jacobian system, and they are independent of one another: hence

$$r \not> 2n - r,$$

so that r cannot be greater than n.

COROLLARY III. The converse of Lie's theorem also is true: that is, if

$$(u_1, f) = 0, \ \ldots, \ (u_r, f) = 0,$$

are a complete system, and if $u_1, \ldots, u_r$ be functionally distinct from one another, then $u_1, \ldots, u_r$ form a group. For, as before, the equation

$$((u_i, u_j)\, f) = 0$$

is satisfied in virtue of the system. Now (u_i, u_j), being some function of the variables, can be expressed in terms of $u_1, \ldots, u_r$, and of $2n-r$ of the variables, say $x_{r+1}, \ldots, x_n, p_1, \ldots, p_n$: hence, if

$$(u_i, u_j) = \theta_{ij}(u_1, \ldots, u_r, x_{r+1}, \ldots, x_n, p_1, \ldots, p_n),$$

we have

$$((u_i, u_j) f) = \sum_{s=1}^{r} \frac{\partial \theta_{ij}}{\partial u_s}(u_s, f) + \sum_{\mu=r+1}^{n} \frac{\partial \theta_{ij}}{\partial x_\mu}(x_\mu, f) + \sum_{\sigma=1}^{n} \frac{\partial \theta_{ij}}{\partial p_\sigma}(p_\sigma, f),$$

and therefore

$$0 = \sum_{\mu=r+1}^{n} \frac{\partial \theta_{ij}}{\partial x_\mu}(x_\mu, f) + \sum_{\sigma=1}^{n} \frac{\partial \theta_{ij}}{\partial p_\sigma}(p_\sigma, f),$$

that is,

$$0 = \sum_{\mu=r+1}^{n} \frac{\partial \theta_{ij}}{\partial x_\mu}\frac{\partial f}{\partial p_\mu} - \sum_{\sigma=1}^{n} \frac{\partial \theta_{ij}}{\partial p_\sigma}\frac{\partial f}{\partial x_\sigma}.$$

Now the given system is complete, so that f satisfies no other equation: hence the last equation must be evanescent, and therefore

$$\frac{\partial \theta_{ij}}{\partial x_\mu} = 0, \quad \frac{\partial \theta_{ij}}{\partial p_\sigma} = 0,$$

for $\mu = r+1, \ldots, n$, and $\sigma = 1, \ldots, n$. Thus

$$(u_i, u_j) = \theta_{ij}(u_1, \ldots, u_r),$$

and $u_1, \ldots, u_r$ are supposed to be functionally distinct from one another: hence they form a group.

Indicial Functions of a Group.

139. A function, which belongs to a group $u_1, \ldots, u_r$ and is in involution with all the functions of the group, is called* *indicial.* Each indicial function f satisfies the equations

$$(u_1, f) = 0, \ \ldots, \ (u_r, f) = 0;$$

and therefore the number of indicial functions belonging to the group is the number of integrals of these equations, which are independent of one another and can be expressed in terms of $u_1, \ldots, u_r$. These equations are

$$A_1 f = (u_1, u_1)\frac{\partial f}{\partial u_1} + (u_1, u_2)\frac{\partial f}{\partial u_2} + \ldots + (u_1, u_r)\frac{\partial f}{\partial u_r} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_r f = (u_r, u_1)\frac{\partial f}{\partial u_1} + (u_r, u_2)\frac{\partial f}{\partial u_2} + \ldots + (u_r, u_r)\frac{\partial f}{\partial u_r} = 0;$$

* The German title is *ausgezeichnete Function*; the French title is *fonction distinguée.*

the coefficients of the derivatives of f are expressible in terms of $u_1, \ldots, u_r$, so that apparently there are r equations in r variables.

Now these equations are not linearly independent of one another: one obvious linear relation is

$$(A_1 f)\frac{\partial f}{\partial u_1} + \ldots + (A_r f)\frac{\partial f}{\partial u_r} = 0;$$

and other linear relations may exist. Accordingly, suppose that there are m such relations, so that the r equations reduce to $r - m$ linearly independent equations; the conditions, necessary and sufficient to secure this reduction, are that the determinant

$$\begin{vmatrix} (u_1, u_1), & (u_1, u_2), & \ldots, & (u_1, u_r) \\ (u_2, u_1), & (u_2, u_2), & \ldots, & (u_2, u_r) \\ \ldots & \ldots & \ldots & \ldots \\ (u_r, u_1), & (u_r, u_2), & \ldots, & (u_r, u_r) \end{vmatrix},$$

and its minors of all orders up to $m - 1$ inclusive, but not all the minors of order m, shall vanish.

We may assume that the linearly independent equations are the first $r - m$ of the set. As

$$A_i(A_j f) - A_j(A_i f) = \frac{\partial f_{ij}}{\partial u_1} A_1 f + \ldots + \frac{\partial f_{ij}}{\partial u_r} A_r f,$$

where $(u_i, u_j) = f_{ij}(u_1, \ldots, u_r)$, and as each of the quantities $A_{r-m+1}f, \ldots, A_r f$ is linearly expressible in terms of $A_1 f, \ldots, A_{r-m} f$, it follows that the equations

$$A_i(A_j f) - A_j(A_i f) = 0,$$

for i and $j = 1, \ldots, r - m$, are satisfied in virtue of the set of $r - m$ retained equations; hence these $r - m$ equations are a complete system. Any integral is an indicial function, and the complete system of $r - m$ equations in r variables possesses m independent integrals; hence, *under the conditions associated with the preceding determinant, the group possesses m indicial functions.*

Note 1. The number of indicial functions of a group subjected to a contact transformation is unaffected by the transformation. Let the transformed group be $u_1', \ldots, u_r'$: then, as in § 137, we have

$$(u_i', u_j') = (u_i, u_j),$$

so that the critical determinant and its minors are unaltered, and therefore the number of indicial functions is unaffected.

Note 2. The number of indicial functions of a group is independent of the form of the group.

Let $U_1, \ldots, U_r$ be a form of the group $u_1, \ldots, u_r$, so that the Jacobian

$$J\left(\frac{U_1, \ldots, U_r}{u_1, \ldots, u_r}\right)$$

does not vanish, because the functions $U_1, \ldots, U_r$ are independent. Now

$$(U_i, f) = \sum_{s=1}^{r} \frac{\partial U_i}{\partial u_s}(u_s, f),$$

for $i = 1, \ldots, r$; hence every function, which is indicial for the form $u_1, \ldots, u_r$, is indicial for the form $U_1, \ldots, U_r$: and, as the determinant of the coefficients of $(u_1, f), \ldots, (u_r, f)$ is the non-vanishing Jacobian, every function indicial for the form $U_1, \ldots, U_r$ is indicial also for the form $u_1, \ldots, u_r$.

Note 3. The critical determinant is obviously skew: it vanishes if r be odd. Hence every group of odd order possesses at least one indicial function.

Note 4. When there are m indicial functions, they are the independent integrals of $r - m$ homogeneous linear equations in r variables. The actual expression of the indicial functions may therefore be assumed known, on constructing the integrals of those equations by any of the methods explained in Chapter III.

Ex. The functions

$$u_1 = p_1 p_2 - x_3 x_4, \quad u_2 = p_3 p_4 - x_1 x_2, \quad u_3 = p_1 x_1 + p_2 x_2 - p_3 x_3 - p_4 x_4,$$

form a group of order three, because

$$(u_1, u_2) = u_3, \quad (u_1, u_3) = -2u_1, \quad (u_2, u_3) = 2u_2.$$

The critical determinant is

$$\begin{vmatrix} 0, & u_3, & -2u_1 \\ -u_3, & 0, & 2u_2 \\ 2u_1, & -2u_2, & 0 \end{vmatrix};$$

it is easy to see that $m = 1$, so that the group possesses one indicial function.

This indicial function satisfies the equations

$$\left.\begin{array}{r} u_3 \dfrac{\partial f}{\partial u_2} - 2u_1 \dfrac{\partial f}{\partial u_3} = 0 \\ -u_3 \dfrac{\partial f}{\partial u_1} \qquad\qquad + 2u_2 \dfrac{\partial f}{\partial u_3} = 0 \\ 2u_1 \dfrac{\partial f}{\partial u_1} - 2u_2 \dfrac{\partial f}{\partial u_2} \qquad\qquad = 0 \end{array}\right\},$$

which are equivalent to two independent equations. These possess one integral; it is easily found to be

$$4u_1 u_2 + u_3^2,$$

which accordingly is the indicial function of the group.

It has been seen that a group $u_1, \ldots, u_r$ determines a reciprocal group $v_1, \ldots, v_{2n-r}$, the members of the latter being the $2n-r$ independent integrals of

$$(u_1, f) = 0, \ldots, (u_r, f) = 0;$$

and any integral of these equations is expressible in terms of $v_1, \ldots, v_{2n-r}$. Now the indicial functions of the original group are integrals of these equations; hence there are m relations between the members of the two groups of the form

$$U_i(u_1, \ldots, u_r) = \phi_i(v_1, \ldots, v_{2n-r}),$$

for $i = 1, \ldots, m$.

Conversely, a relation of this character implies the existence of an indicial function: because, as the equations

$$(u_1, f) = 0, \ldots, (u_r, f) = 0,$$

are satisfied for $f = v_1, \ldots, v_{2n-r}$, we have

$$(u_1, \phi_i) = 0, \ldots, (u_r, \phi_i) = 0,$$

that is,

$$(u_1, U_i) = 0, \ldots, (u_r, U_i) = 0,$$

shewing that U_i is an indicial function. Denote the value of this indicial function by w_i, so that w_i can be expressed in terms of $u_1, \ldots, u_r$ alone, and therefore w_i belongs to the original group: it can be expressed in terms of $v_1, \ldots, v_{2n-r}$ alone, and therefore w_i belongs to the reciprocal group. Owing to the reciprocity between two polar groups, we know that r independent integrals of the equations

$$(v_1, g) = 0, \ldots, (v_{2n-r}, g) = 0$$

are $u_1, \ldots, u_r$. As

$$w_i = U_i(u_1, \ldots, u_r),$$

it follows that

$$(v_1, U_i) = 0, \ldots, (v_{2n-r}, U_i) = 0,$$

and therefore

$$(v_1, w_i) = 0, \ldots, (v_{2n-r}, w_i) = 0.$$

Now w_i can be expressed in terms of $v_1, \ldots, v_{2n-r}$; hence w_i is an indicial function of the reciprocal group.

Gathering together these results, we can enunciate them as follows:—

Two reciprocal groups $u_1, \ldots, u_r$, *and* $v_1, \ldots, v_{2n-r}$ *have the same indicial functions* $w_1, \ldots, w_m$: *and the existence of each indicial function implies the existence of a relation between functions of the two groups of the type*

$$U_i(u_1, \ldots, u_r) = w_i = \phi_i(v_1, \ldots, v_{2n-r}),$$

for $i = 1, \ldots, m$.

COROLLARY. *If a function belongs to two reciprocal groups, it is an indicial function for each of them.*

Ex. In an earlier example (p. 352), it was seen that

$$u_1 = p_1p_2 - x_3x_4, \quad u_2 = p_3p_4 - x_1x_2, \quad u_3 = p_1x_1 + p_2x_2 - p_3x_3 - p_4x_4,$$

form a group of order three. The reciprocal group is of order five, and is composed of five independent integrals of the equations

$$(u_1, f) = 0, \quad (u_2, f) = 0, \quad (u_3, f) = 0.$$

Five such integrals can be taken in the form

$$\begin{aligned} v_1 &= p_1p_3 - x_2x_4, \\ v_2 &= p_2p_4 - x_1x_3, \\ v_3 &= p_1p_4 - x_2x_3, \\ v_4 &= p_2p_3 - x_1x_4, \\ v_5 &= p_1x_1 - p_2x_2 + p_3x_3 - p_4x_4, \end{aligned}$$

which accordingly constitute the reciprocal group.

It was seen that $u_3^2 + 4u_1u_2$ is the indicial function of the original group: it is therefore, by the preceding theorem, the indicial function of the reciprocal group, and one relation must subsist between the functions u_1, u_2, u_3 and the functions v_1, v_2, v_3, v_4, v_5. It is easy to verify that

$$u_3^2 + 4u_1u_2 = v_5^2 + 4v_1v_2,$$

the common value of these quantities being the indicial function for the two groups.

CANONICAL FORM OF A GROUP.

140. When a group is a system in involution, it is obvious, from the characteristic equations

$$(u_i, u_j) = 0,$$

that every member of the group is an indicial function.

Also, if a group of order r certainly possesses $r-1$ indicial functions, it is a system in involution; for if $u_1, \ldots, u_{r-1}$ are the indicial functions, and if v is the other member,

$$(v, u_1) = 0, \ldots, (v, u_{r-1}) = 0,$$

that is,

$$(u_1, v) = 0, \ldots, (u_{r-1}, v) = 0;$$

therefore v is an indicial function also, and the group is a system in involution.

Hence, if a group of order r is not a system in involution, it cannot possess more than $r-2$ indicial functions. We now proceed to obtain, for groups that are not systems in involution, a canonical form which shall obviously shew the number and the incidence of the indicial functions of the group*: the main result is contained in the theorem:—

The order of a group, that is not a system in involution, exceeds the number of its indicial functions by an even integer; and a group of order $2q+m$, which possesses m indicial functions, can be transformed into a group $X_1, \ldots, X_{q+m}, P_1, \ldots, P_q$, such that

$$(X_i, X_j) = 0, \quad (P_\mu, P_\kappa) = 0, \quad (X_i, P_\mu) = 0,$$
$$(P_\mu, X_\mu) = 1,$$

for which i and $j = 1, \ldots, q+m$; μ and $\kappa = 1, \ldots, q$; while i and μ are unequal to one another.

It will appear that the unity, as the value of (P_μ, X_μ), is merely a determinate constant: and it is clear that $X_{q+1}, \ldots, X_{q+m}$ are the m indicial functions of the transformed group. The form, thus selected for the group, is usually called the *canonical form*.

The proposition is established as follows. Let the group be of order r and let its members be $u_1, \ldots, u_r$; also, let m be the number of indicial functions so that, after the earlier explanations, $m \leqslant r-2$. Suppose that u_1 is a member of the group which is not an indicial function: then not all the quantities (u_1, u_2), $(u_1, u_3), \ldots, (u_1, u_r)$ vanish, and so the equation

$$(u_1, \theta) = (u_1, u_2)\frac{\partial \theta}{\partial u_2} + \ldots + (u_1, u_r)\frac{\partial \theta}{\partial u_r} = 1$$

* It is unnecessary to deal with a system in involution: every function is an indicial function: and every form of the group is a system in involution.

is possible and possesses integrals which, being functions of $u_2, \ldots, u_r$ or of some of them, belong to the group. Take one of these integrals, and let it be denoted by u_2; thus

$$(u_1, u_2) = 1.$$

Now consider the two equations

$$(u_1, v) = 0, \quad (u_2, v) = 0;$$

they are a complete system, because

$$(u_1 (u_2, v)) - (u_2 (u_1, v)) = ((u_1, u_2) v) = (1, v) = 0.$$

In full expression, they are

$$\left.\begin{aligned} \frac{\partial v}{\partial u_2} + (u_1, u_3) \frac{\partial v}{\partial u_3} + \ldots + (u_1, u_r) \frac{\partial v}{\partial u_r} &= 0 \\ -\frac{\partial v}{\partial u_1} \quad + (u_2, u_3) \frac{\partial v}{\partial u_3} + \ldots + (u_2, u_r) \frac{\partial v}{\partial u_r} &= 0 \end{aligned}\right\};$$

they determine v in terms of $u_1, \ldots, u_r$; and being a complete system of two equations in r variables, they possess $r-2$ functionally independent integrals. Let these be $v_1, \ldots, v_{r-2}$, which accordingly are independent of one another; also they are functions of $u_1, \ldots, u_r$.

Then $u_1, u_2, v_1, \ldots, v_{r-2}$ are a set of r independent functions: for otherwise, some relation

$$g(u_1, u_2, v_1, \ldots, v_{r-2}) = 0$$

would be satisfied identically, and then

$$\begin{aligned} 0 = (u_1, g) &= (u_1, u_2) \frac{\partial g}{\partial u_2} + (u_1, v_1) \frac{\partial g}{\partial v_1} + \ldots + (u_1, v_{r-2}) \frac{\partial g}{\partial v_{r-2}} \\ &= \frac{\partial g}{\partial u_2}, \\ 0 = (u_2, g) &= -\frac{\partial g}{\partial u_1}, \end{aligned}$$

so that g would not involve u_1 or u_2, and the relation would subsist between $v_1, \ldots, v_{r-2}$, which are known to be functionally independent. Hence our original group can be replaced by a group $u_1, u_2, v_1, \ldots, v_{r-2}$ such that

$$(u_1, u_2) = 1, \quad (u_1, v_i) = 0, \quad (u_2, v_i) = 0,$$

for $i = 1, \ldots, r-2$. We have seen (§ 139, Note 2) that the number of indicial functions is independent of the form of the group; and it is clear that neither u_1 nor u_2 is an indicial function.

Let $w_1, \ldots, w_{2n-r}$ be the polar of the original group, so that

$$(u_1, w_i) = 0, \quad (u_2, w_i) = 0,$$

for $i = 1, \ldots, 2n-r$; and as the $2n-r$ quantities form a group, all the quantities (w_i, w_j) are expressible in terms of $w_1, \ldots, w_{2n-r}$. Then $u_1, u_2, w_1, \ldots, w_{2n-r}$ form a group: for they are $2n-r+2$ independent functions and, as

$$(u_1, u_2) = 1, \quad (u_1, w_i) = 0, \quad (u_2, w_i) = 0,$$

all the combinations of members of the group are expressible in terms of those members. Now

$$(v_i, u_1) = 0, \quad (v_i, u_2) = 0, \quad (v_i, w_j) = 0,$$

for $i = 1, \ldots, r-2$, and $j = 1, \ldots, 2n-r$: the first two are the equations defining v_i, and the rest are satisfied because v_i belongs to the group that is reciprocal to $w_1, \ldots, w_{2n-r}$. Hence $v_1, \ldots, v_{r-2}$ is a group reciprocal to $u_1, u_2, w_1, \ldots, w_{2n-r}$: so that $v_1, \ldots, v_{r-2}$ is a sub-group of the group $u_1, u_2, v_1, \ldots, v_{r-2}$. Moreover, the indicial functions of our group are functions of $u_1, u_2, v_1, \ldots, v_{r-2}$: denoting any one of them by $\theta(u_1, u_2, v_1, \ldots, v_{r-2})$, we know that the equations

$$(u_1, \theta) = 0, \quad (u_2, \theta) = 0, \quad (v_1, \theta) = 0, \ldots, (v_{r-2}, \theta) = 0$$

must be satisfied. The first of these equations is

$$\frac{\partial \theta}{\partial u_2} = 0,$$

so that θ does not involve u_2; the second is

$$-\frac{\partial \theta}{\partial u_1} = 0,$$

so that θ does not involve u_1; and so θ is a function of $v_1, \ldots, v_{r-2}$.

Thus the group of order r possessing m indicial functions has been transformed into another group

$$u_1, u_2, v_1, \ldots, v_{r-2};$$

the quantity $(u_1, u_2) = 1$; and the $r-2$ quantities $v_1, \ldots, v_{r-2}$ constitute a group of order $r-2$, possessing m indicial functions.

If the group $v_1, \ldots, v_{r-2}$ is a system in involution, then, as it possesses m indicial functions, we have

$$m = r-2;$$

and writing

$$P_1 = u_1, \quad X_1 = u_2, \quad X_2 = v_1, \ldots, X_{r-1} = v_{r-2},$$

we have

$$(P_1, X_1) = 1, \quad (P_1, X_i) = 0, \quad (X_1, X_i) = 0, \quad (X_i, X_j) = 0,$$

for i and $j = 2, \ldots, r-1$. The reduction, indicated in the theorem, has been made.

If the group $v_1, \ldots, v_{r-2}$ possessing m indicial functions is not a system in involution, so that $m \leqslant r-4$, then we transform it in the same manner as the original group was transformed. It can be made to take a form

$$X_2, P_2, w_1, \ldots, w_{r-4},$$

where

$$(P_2, X_2) = 1, \quad (P_2, w_i) = 0, \quad (X_2, w_i) = 0,$$

for $i = 1, \ldots, r-4$; and then $w_1, \ldots, w_{r-4}$ constitute a group of order $r-4$, possessing m indicial functions. The original group has thus been changed to the form

$$X_1, P_1, X_2, P_2, w_1, \ldots, w_{r-4},$$

such that

$$\begin{gathered} (P_1, X_1) = 1, \quad (P_2, X_2) = 1, \quad (X_1, X_2) = 0, \quad (X_1, P_2) = 0, \\ (X_2, P_1) = 0, \quad (P_1, P_2) = 0, \\ (X_1, w_i) = 0, \quad (X_2, w_i) = 0, \quad (P_1, w_i) = 0, \quad (P_2, w_i) = 0. \end{gathered}$$

If the group $w_1, \ldots, w_{r-4}$ is a system in involution, the required reduction has been effected: and, as the group possesses m indicial functions, we then have

$$m = r - 4.$$

If the group $w_1, \ldots, w_{r-4}$ is not a system in involution, we proceed as before. At each stage in the successive changes, we isolate two functions X_i and P_i such that

$$(P_i, X_i) = 1,$$

and we are left with a group of order $r - 2i$ possessing m indicial functions. Ultimately, we shall reach a stage when this remaining group is a system in involution, so that

$$r - 2q = m;$$

the isolated pairs of functions are

$$X_1, P_1; \; X_2, P_2; \; \ldots; \; X_q, P_q;$$

the remaining functions, being (as stated) a system in involution, may be represented by $X_{q+1}, \ldots, X_{q+m}$, such that

$$(X_{q+i}, X_{q+j}) = 0.$$

The original group of order r, possessing m indicial functions, has been replaced by

$$X_1, \ldots, X_{q+m}, \quad P_1, \ldots, P_q,$$

such that

$$(P_i, X_i) = 1,$$

for $i = 1, \ldots, q$; also

$$(P_\mu, P_\kappa) = 0, \quad (X_i, X_j) = 0, \quad (P_\mu, X_i) = 0,$$

for μ and $\kappa = 1, \ldots, q$, and for i and $j = 1, \ldots, q + m$, the values of μ and i being unequal. Also

$$r - m = 2q.$$

The proposition is thus established.

Note 1. Obviously $X_1, \ldots, X_{m+q}$, a group of order $m + q$, is a system in involution: hence (§ 138)

$$m + q \leqslant n.$$

Note 2. It is obvious that, on account of the relations

$$(P_1, X_1) = 1, \ldots, (P_q, X_q) = 1,$$

no one of the functions $P_1, X_1, \ldots, P_q, X_q$ can itself be an indicial function. It is equally obvious that, on account of the relations

$$(P_\mu, X_{q+i}) = 0, \quad (X_\mu, X_{q+i}) = 0,$$
$$(X_{q+i}, X_{q+j}) = 0,$$

for $\mu = 1, \ldots, q$, and i and $j = 1, \ldots, m$, the quantities $X_{q+1}, \ldots, X_{q+m}$ are indicial functions.

If it were possible to have any other indicial function, say $g(P_1, \ldots, P_q, X_1, \ldots, X_{q+m})$, it would have to satisfy the equations

$$(P_\mu, g) = 0, \quad (X_\mu, g) = 0, \quad (X_{q+i}, g) = 0,$$

for $\mu = 1, \ldots, q$, and $i = 1, \ldots, m$. The first set of these equations is

$$\frac{\partial g}{\partial X_\mu} = 0;$$

the second set is

$$-\frac{\partial g}{\partial P_\mu} = 0;$$

and the third is identically satisfied. Thus g does not involve $P_1, \ldots, P_q, X_1, \ldots, X_q$: and every indicial function is expressible* in terms of $X_{q+1}, \ldots, X_{q+m}$.

* This is only another way of stating that, as the group $X_{q+1}, \ldots, X_{q+m}$ is in involution, any form of the group is in involution.

Note 3. The relation

$$r - m = 2q,$$

enables us to verify the former result (§ 139, Note 3) that a group of odd order cannot be devoid of indicial forms. If

$$r = 2p + 1,$$

then

$$m = 2p - 2q + 1,$$

so that the number of indicial functions possessed by a group of odd order is certainly odd.

Ex. 1. On p. 352, it was seen that the functions

$$u_1 = p_1p_2 - x_3x_4, \quad u_2 = p_3p_4 - x_1x_2, \quad u_3 = p_1x_1 + p_2x_2 - p_3x_3 - p_4x_4,$$

form a group of order 3: and, as

$$(u_1, u_2) = u_3, \quad (u_1, u_3) = -2u_1, \quad (u_2, u_3) = 2u_2,$$

this group is not a system in involution.

The canonical form can be obtained as in the text. Obviously u_1 is not an indicial function: consequently, we require an integral of the equation

$$(u_1, \theta) = 1,$$

that is, of

$$u_3\frac{\partial\theta}{\partial u_2} - 2u_1\frac{\partial\theta}{\partial u_3} = 1.$$

An integral is given by

$$\theta = -\frac{u_3}{2u_1},$$

so that two functions for the canonical form are

$$P_1 = u_1, \quad X_1 = -\frac{u_3}{2u_1}.$$

One other function is required: it must be a common integral of

$$(P_1, \phi) = 0, \quad (X_1, \phi) = 0.$$

The former equation is

$$u_3\frac{\partial\phi}{\partial u_2} - 2u_1\frac{\partial\phi}{\partial u_3} = 0;$$

and the latter, after reduction in conjunction with this equation, is

$$u_1\frac{\partial\phi}{\partial u_1} - u_2\frac{\partial\phi}{\partial u_2} = 0.$$

A common integral is given by

$$\phi = u_3^2 + 4u_1u_2.$$

Accordingly, we take

$$P_1 = u_1, \quad X_1 = -\frac{u_3}{2u_1}, \quad X_2 = u_3^2 + 4u_1u_2:$$

we have

$$(P_1, X_1) = 1, \quad (P_1, X_2) = 0, \quad (X_1, X_2) = 0,$$

which is a canonical form. Obviously, X_2 is the one indicial function of the group.

Another transformation arises by taking

$$P_1 = u_2, \quad X_1 = -\frac{u_1}{u_3}, \quad X_2 = u_3^2 + 4u_1u_2\,;$$

and again, another, by taking

$$P_1 = u_3, \quad X_1 = \tfrac{1}{2}\log u_1, \quad X_2 = u_3^2 + 4u_1u_2.$$

Both of these are canonical: they satisfy the equations

$$(P_1,\, X_1) = 1, \quad (P_1,\, X_2) = 0, \quad (X_1,\, X_2) = 0.$$

Ex. 2. On p. 354, it was seen that the functions

$$\begin{aligned} v_1 &= p_1p_3 - x_2x_4, \quad v_2 = p_2p_4 - x_1x_3, \\ v_3 &= p_1p_4 - x_2x_3, \quad v_4 = p_2p_3 - x_1x_4, \\ v_5 &= p_1x_1 - p_2x_2 + p_3x_3 - p_4x_4, \end{aligned}$$

form a group, being the reciprocal of the group u_1, u_2, u_3 in the preceding example.

Introducing a quantity t, where

$$t = p_1x_1 - p_2x_2 - p_3x_3 + p_4x_4,$$

so that

$$t^2 = v_5^2 + 4v_1v_2 - 4v_3v_4,$$

obtain a canonical form for the group, given by

$$\begin{aligned} P_1 &= v_1, \\ X_1 &= -\frac{v_5}{2v_1}, \\ P_2 &= v_3, \\ X_2 &= -\frac{t}{2v_3}, \\ X_3 &= v_5^2 + 4v_1v_2, \end{aligned}$$

so that

$$(P_1,\, X_1) = 1, \quad (P_2,\, X_2) = 1,$$

and all other combinations of the functions in this form of the group vanish.

141. Next, *when a group of order $m + 2q$, having m indicial functions, is given in canonical form, it can be amplified, by the association of $2n - (m + 2q)$ appropriately determined functions, so that the $2n$ functions are a group of order $2n$ in canonical form.*

Let the group be

$$X_1,\, P_1,\, \ldots,\, X_q,\, P_q,\, X_{q+1},\, \ldots,\, X_{q+m},$$

with the preceding notation, so that $X_{q+1}, \ldots, X_{q+m}$ are the indicial functions of the group.

Suppose that X_{q+1} is omitted; the surviving $m + 2q - 1$ functions form a group having $m - 1$ indicial functions. Form the reciprocal of this diminished group; this reciprocal contains X_{q+1} and other $2n - (m + 2q)$ functions. Now X_{q+1} does not belong to the diminished group and therefore it cannot be an indicial

function of the reciprocal group; and therefore, as in the preceding investigation, we can determine a function P_{q+1} of the other $2n-(m+2q)$ functions in the reciprocal group such that

$$(P_{q+1},\ X_{q+1})=1.$$

Moreover, P_{q+1} cannot belong to the original unmodified group: if it were expressible in terms of the members of that group, we should have

$$(X_{q+1},\ P_{q+1})=0,$$

because X_{q+1} is an indicial function of the unmodified group. Also, as P_{q+1} belongs to the reciprocal of the diminished group, we have

$$(P_{q+1},\ P_i)=0,\quad (P_{q+1},\ X_i)=0,\quad (P_{q+1},\ X_{q+s})=0,$$

for $i=1,\ \ldots,\ q$, and $s=2,\ \ldots,\ m$. Hence, when P_{q+1} is associated with the original unmodified group, we have a new group

$$X_1,\ P_1,\ \ldots,\ X_q,\ P_q,\ X_{q+1},\ P_{q+1},\ X_{q+2},\ \ldots,\ X_{q+m},$$

which is in a canonical form and possesses $m-1$ indicial functions.

Repeating this process $m-1$ times so as, on each occasion, to associate a new function P with the group and to diminish the number of indicial functions by one unit, we ultimately obtain a group

$$X_1,\ P_1,\ \ldots,\ X_{q+m},\ P_{q+m},$$

which is in a canonical form and possesses no indicial functions. We have seen (§ 140, Note 1) that

$$q+m \leqslant n.$$

If $q+m$ is equal to n, the required amplification of the original group has been effected.

If $q+m$ is less than n, take any member of the group reciprocal to

$$X_1,\ P_1,\ \ldots,\ X_{q+m},\ P_{q+m},$$

and denote it by X_{q+m+1}. Associating it with this group of order $2q+2m$, we have a new group of order $2q+2m+1$, possessing X_{q+m+1} as its one indicial function. We then apply the earlier process so as to determine a new function P_{q+m+1}: and we have a new group

$$X_1,\ P_1,\ \ldots,\ X_{q+m+1},\ P_{q+m+1},$$

which is in a canonical form and possesses no indicial function.

Proceeding in this way, by associating alternately a function X and a function P until the amplified group is of order $2n$, we ultimately obtain a group

$$X_1, P_1, \ldots, X_n, P_n,$$

which is of order $2n$, is in a canonical form, and possesses no indicial functions.

Ex. Let it be required to achieve the first stage in the completion of the group v_1, v_2, v_3, v_4, v_5 which is given in § 140, Ex. 2, so that it shall have a canonical form.

As there indicated, we take

$$P_1 = v_1, \quad X_1 = -\frac{v_5}{2v_1}, \quad P_2 = v_3, \quad X_2 = -\frac{t}{2v_3}, \quad X_3 = v_5^2 + 4v_1v_2;$$

the required first step towards completing the group is to determine a function P_3 such that

$$(P_3, X_3) = 1.$$

For this purpose, we need a set of four independent integrals of

$$(P_1, \theta) = 0, \quad (X_1, \theta) = 0, \quad (P_2, \theta) = 0, \quad (X_2, \theta) = 0,$$

or, what is the same thing, four independent integrals of

$$(v_1, \theta) = 0, \quad (v_5, \theta) = 0, \quad (v_3, \theta) = 0, \quad (t, \theta) = 0.$$

Also, we know that the group v_1, v_2, v_3, v_4, v_5 (or, what is the same thing, the group v_1, v_5, v_3, t, v_2) is the reciprocal of u_1, u_2, u_3; so that three independent integrals of the preceding complete system of four equations are given by u_1, u_2, u_3. Moreover,

$$X_3 = v_5^2 + 4v_1v_2 = u_3^2 + 4u_1u_2;$$

so that what is needed is an integral of those four equations, independent of u_1, u_2, u_3.

Expanding the equations in full, resolving them so as to express $\frac{\partial\theta}{\partial x_1}$, $\frac{\partial\theta}{\partial x_2}$, $\frac{\partial\theta}{\partial x_3}$, $\frac{\partial\theta}{\partial x_4}$ linearly in terms of $\frac{\partial\theta}{\partial p_1}$, $\frac{\partial\theta}{\partial p_2}$, $\frac{\partial\theta}{\partial p_3}$, $\frac{\partial\theta}{\partial p_4}$, and integrating them either by Jacobi's method or by Mayer's method (Chap. IV), we find

$$u_1, u_2, u_3, \frac{p_1}{x_2},$$

as four independent integrals, so that $\frac{p_1}{x_2}$ is the fourth integral required. The quantity P_3 is to be a function of $u_1, u_2, u_3, \frac{p_1}{x_2}$, such that $(P_3, X_3) = 1$.

Now

$$(u_1, v) = v^2, \quad (u_2, v) = -1, \quad (u_3, v) = 2v,$$

so that

$$(X_3, v) = 4(u_2v^2 + u_3v - u_1):$$

and we know that

$$(X_3, u_1) = 0, \quad (X_3, u_2) = 0, \quad (X_3, u_3) = 0.$$

Hence P_3 is given by

$$P_3=\int\frac{dv}{4(u_1-u_3v-u_2v^2)},$$

where u_1, u_2, u_3 are constant in the quadrature: that is,

$$P_3=\tfrac{1}{4}X_3^{-\frac{1}{2}}\log\left(\frac{X_3^{\frac{1}{2}}+u_3+2u_2v}{X_3^{\frac{1}{2}}-u_3-2u_2v}\right).$$

Two more steps are needed in order to obtain the complete group expressed in canonical form. We first require the group of two members which is reciprocal to X_1, P_1, X_2, P_2, X_3, P_3; or, as u_1, u_2, u_3 is the group reciprocal to X_1, P_1, X_2, P_2, X_3, we require a function θ of u_1, u_2, u_3 such that

$$(P_3, \theta)=0.$$

There are two such functions, independent of one another: let them be w_1 and w_2. As

$$(P_3, X_3)=1,$$

w_1, w_2, X_3 are three independent functions of u_1, u_2, u_3. We then take

$$X_4=w_1.$$

The last step is the determination of P_4 as a function of w_2 and the other functions w_1, P_3, X_3, P_2, X_2, P_1, X_1, such that

$$(P_4, X_4)=1:$$

or since

$$(X_4, w_1)=0,\quad (X_4, X_\iota)=0,\quad (X_4, P_\iota)=0,$$

for $\iota=1, 2, 3$, we have

$$P_4=\int\frac{dw_2}{(w_2, w_1)},$$

where (w_2, w_1) should be expressed in terms of w_2, w_1, P_3, X_3, P_2, X_2, P_1, X_1 and, for the quadratures, w_2 should be regarded as the only variable.

Groups of Functions and Contact Transformations.

142. The preceding results can be used to establish an important property of groups, viz. *when a group of functions is subjected to a contact transformation, there are two invariants, being the order of the group and the number of indicial functions; and when two groups in the same variables have the same invariants, they can be transformed into one another by a contact transformation.*

The first part of this proposition is merely a restatement of two results already established. It was seen, in § 137, (vi), that a contact transformation does not alter the order of a group; and, in § 139, Note 1, that a contact transformation does not alter the number of indicial functions.

For the second part, we express each of the groups in a canonical form: as the orders are the same, say $2q+m$, and as the numbers of indicial functions are the same, say m, the canonical forms of the groups may be expressed by

$$X_1, P_1, \ldots, X_q, P_q, X_{q+1}, \ldots, X_{q+m},$$
$$Y_1, Q_1, \ldots, Y_q, Q_q, Y_{q+1}, \ldots, Y_{q+m},$$

respectively. The former group can be amplified into

$$X_1, P_1, \ldots, X_n, P_n,$$

and the latter can be amplified into

$$Y_1, Q_1, \ldots, Y_n, Q_n.$$

Now, on account of the relations

$$(P_i, X_i) = 1, \quad (P_i, P_j) = 0, \quad (P_i, X_j) = 0, \quad (X_i, X_j) = 0,$$

for i and $j = 1, \ldots, n$, with unequal values for i and j, the equations

$$x_\mu' = X_\mu, \quad p_\mu' = P_\mu,$$

for $\mu = 1, \ldots, n$, determine a contact transformation; and the equations

$$x_\mu' = Y_\mu, \quad p_\mu' = Q_\mu,$$

for the same values of μ, similarly determine a contact transformation. Consequently, the equations

$$X_\mu = Y_\mu, \quad P_\mu = Q_\mu,$$

for $\mu = 1, \ldots, n$, determine a contact transformation, which manifestly transforms the one group into the other.

143. We have seen that, when a group of order $2q+m$ possessing m indicial functions is expressed in a canonical form

$$X_1, P_1, \ldots, X_q, P_q, X_{q+1}, \ldots, X_{q+m},$$

the $q+m$ quantities $X_1, \ldots, X_{q+m}$ are such that

$$(X_i, X_j) = 0:$$

that is, the group contains a sub-group of order $q+m$ which is a system in involution. It will now be proved that *any sub-group which is a system in involution is of order not greater than* $q+m$.

Let a sub-group, being a system in involution, be

$$Z_1, \ldots, Z_\mu.$$

Conceive the original group amplified so as to be of order $2n$, expressed in canonical form by the association of $n-q$ functions

$P_{q+1}, \ldots, P_n$ and of $n-q-m$ functions $X_{q+m+1}, \ldots, X_n$. Now these $n-q-m$ new functions X are themselves a system in involution: they are in involution with every member of the original group, and therefore with $Z_1, \ldots, Z_\mu$: and therefore

$$Z_1, \ldots, Z_\mu, \quad X_{q+m+1}, \ldots, X_n$$

is a system in involution. The order of a system in involution cannot be greater than n (§ 138): hence

$$\mu + n - q - m \leqslant n,$$

that is,

$$\mu \leqslant q + m.$$

Further, this result can be used to obtain an upper limit for the tale of indicial functions, when the group is of order greater than n and therefore is not a system in involution. Let

$$r = 2q + m = n + k,$$

where k is positive: then as

$$q + m \leqslant n,$$

we have

$$\begin{aligned} 2n &\geqslant 2q + 2m \\ &\geqslant m + n + k, \end{aligned}$$

and therefore

$$m \leqslant n - k,$$

so that *a group of order $n+k$ cannot possess more than $n-k$ indicial functions.*

144. It is of importance to be able to construct the sub-group of greatest order $q+m$ which is a system in involution. Denoting the group by

$$u_1, \ldots, u_{2q+m},$$

we first determine the m indicial functions as the m functionally independent integrals of the equations

$$(u_1, \theta) = 0, \ldots, (u_{2q+m}, \theta) = 0,$$

making $u_1, \ldots, u_{2q+m}$ the independent variables. This system of equations is equivalent to $2q$ linearly independent equations and is a complete system: it can be integrated by any of the methods in Chapter III. Let the m independent integrals be

$$v_1, \ldots, v_m,$$

which are therefore the m indicial functions and can be taken as m members of the required sub-group.

Now let u_1 denote a function of the group, and suppose that it is not an indicial function so that it cannot be expressed in terms of $v_1, \ldots, v_m$. Then, for the equation

$$(u_1, \theta) = 0,$$

where θ is regarded as a function of the members of the group, we know $m+1$ independent integrals, viz. $u_1, v_1, \ldots, v_m$, and the number of variables is $2q+m$; hence it possesses $2q-2$ other independent integrals. Assuming that q is greater than unity, let w_2 be one of these other $2q-2$ integrals: then $u_1, w_2, v_1, \ldots, v_m$ are in involution with one another.

Again, for the equations

$$(u_1, \theta) = 0, \quad (w_2, \theta) = 0,$$

where θ is regarded as a function of the members of the group, we know that it is a complete system in the $2q+m$ variables; it therefore possesses $2q+m-2$ independent integrals. We already know $m+2$ of these integrals, in the form $u_1, w_2, v_1, \ldots, v_m$; hence there are $2q-4$ other independent integrals. Assuming that $q > 2$, let w_3 be one of these $2q-4$ integrals; then $u_1, w_2, w_3, v_1, \ldots, v_m$ are in involution with one another.

Proceeding in this way, we shall (after $q-1$ similar stages) have obtained $q+m$ functions, independent of one another and in involution with one another; the aggregate is a sub-group of the greatest order that permits it to be a system in involution.

Application of Groups of Functions to the Integration of Systems of Equations.

145. As our main purpose, in connection with these groups of functions, is their application to the integration of a system of differential equations in one dependent variable, we shall not pursue the further development of their properties which will be found in Lie's treatise already quoted (p. 344): we proceed to apply them for the purpose of integration.

Accordingly, let the equations

$$f_1 = 0, \ldots, f_\mu = 0$$

be a system in involution: they may be a system initially given, in which case there is no question of arbitrary constants occurring in

them: or they may be a system in a stage of gradual construction as in Jacobi's method, in which case some at least of the quantities $f_1, \ldots, f_\mu$ will contain additive arbitrary constants. Suppose also that independent integrals $\phi_1, \ldots, \phi_r$ of the complete system of equations

$$(f_1, \phi) = 0, \ldots, (f_\mu, \phi) = 0$$

have been obtained, and that the Poisson-Jacobi theorem has been applied so as to give all the integrals of the type (ϕ_i, ϕ_j) that can thus be constructed. Then the set of functions in the aggregate $f_1, \ldots, f_\mu, \phi_1, \ldots, \phi_r$ constitute a group: and $f_1, \ldots, f_\mu$ certainly are indicial functions of this group. It may happen that $f_1, \ldots, f_\mu$ do not complete the tale of indicial functions of the group: if they do not, let $f_{\mu+1}, \ldots, f_m$ be the other independent indicial functions of the group, so that

$$f_1, \ldots, f_\mu, f_{\mu+1}, \ldots, f_m$$

constitute a system in involution. Moreover, as $r + \mu$ is the order of a group which possesses m indicial functions, we have

$$\begin{aligned} r + \mu - m &= \text{even integer} \\ &= 2q, \end{aligned}$$

say, where q is a whole number.

It has been proved that a group of order $m + 2q$, possessing m indicial functions, contains a sub-group of order $m + q$ which is a system in involution; and consequently, our group of order $r + \mu$ contains a sub-group of order $m + q$, which is in involution and of which m members are given by

$$f_1, \ldots, f_m:$$

let the other members of this sub-group be $f_{m+1}, \ldots, f_{m+q}$. Then the integration of the original system of μ equations in involution is reduced to the integration of the modified system of

$$m + q \; (= \mu + m - \mu + q)$$

equations in involution: as $m \geqslant \mu$, $q \geqslant 0$, the modified form of the problem is usually simpler than the original form.

To complete the integration, we need integrals of the complete system

$$(f_1, \phi) = 0, \ldots, (f_{m+q}, \phi) = 0;$$

by the earlier theory, this is known to possess $2n - m - q$ independent integrals. Of this aggregate $m + q$ integrals are known,

being $f_1, \ldots, f_{m+q}$: so that other $2n-2m-2q$ integrals are required.

If $q+m=n$, the complete aggregate of integrals is possessed. If $q+m<n$, then we can conceive that the proper number of integrals necessary to complete the aggregate has been determined, e.g. by Jacobi's method as amplified by Mayer. Let this aggregate be denoted by

$$f_1=0, \ldots, f_\mu=0, \quad f_{\mu+1}=a_1, \ldots, f_n=a_{n-\mu},$$

where we can take $a_1, \ldots, a_{n-\mu}$ to be arbitrary constants.

The construction of the integral of the system of equations now proceeds as before. If the n equations can be resolved so as to express $p_1, \ldots, p_n$ in terms of $x_1, \ldots, x_n, a_1, \ldots, a_{n-\mu}$, then, when the resolved values are substituted in

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

a single quadrature leads to an equation of the form

$$z + c = F(x_1, \ldots, x_n, a_1, \ldots, a_{n-\mu}),$$

which is the complete integral of the system: and the remaining integrals can be deduced by the known general theory. If, however, the n equations cannot be conveniently resolved for $p_1, \ldots, p_n$, we determine a function Π from the equations

$$(\Pi, f_i) = -\sum_{r=1}^{n} p_r \frac{\partial f_i}{\partial p_r},$$

for $i=1, \ldots, n$, by a quadrature (§ 130): the complete integral of the system of equations is then given by

$$z - c = \Pi,$$

and the other integrals can be deduced as before.

A sufficient indication of the method has been given: for further developments, reference may be made to the authorities quoted (p. 314) at the beginning of this chapter*.

* Special reference should be made to two memoirs by Lie, *Math. Ann.*, t. IX (1876), pp. 245—296, *ib.*, t. XI (1877), pp. 464—557.

CHAPTER X.

THE EQUATIONS OF THEORETICAL DYNAMICS.

THE present chapter is devoted, more to matters cognate with the theory of the integration of partial differential equations than to the theory itself or to processes of integration.

The only process of integration included is that which is commonly called the Jacobi-Hamilton process: in order to make it more easily comprehended and to shew the source of its inspiration, a brief account of Hamilton's investigations in theoretical dynamics is prefixed.

The analysis shews once more, as so often before in the processes already explained, the close relation between the integration of a partial differential equation and the integrals of the set of ordinary equations, sometimes called subsidiary equations, sometimes the equations of the characteristics, here a canonical system. Some properties of canonical systems are given: but there is not an attempt to deal with them exhaustively because, as every property of such a system can be expressed as a result in theoretical dynamics, they really belong to the subject of theoretical dynamics.

The older development of theoretical dynamics was due mainly to Lagrange, Poisson, Hamilton, Jacobi, Donkin, Bertrand; and expositions of that theory will be found in Jacobi's *Vorlesungen über Dynamik*, in Imschenetsky's memoir* *Sur l'intégration des équations aux dérivées partielles du premier ordre*, and in Graindorge's treatise *Intégration des équations de la mécanique.* Further developments have been effected by Routh and are expounded in his *Treatise on the Dynamics of Rigid Bodies.*

The subject has developed in a different direction, since the application of Lie's theory of contact transformations to a quite general canonical system and the discovery of his important proposition that such transformations at once conserve the form of a general canonical system and are the only transformations which do conserve that form—a result that enables many older properties to be seen in an entirely new relation. An account of this theory and of the mode of development will be found in Dziobek's treatise†

* Translated from the original Russian by Hoüel, and published in *Grunert's Archiv*, t. L (1869), pp. 278—474; see, in particular, chapter VII of the memoir.

† An English translation was published in 1892 (The Inland Press, Ann Arbor).

Die mathematischen Theorien der Planetenbewegungen (Leipzig, 1888), Section II, and in Whittaker's *Analytical Dynamics* (Cambridge, 1904). Reference may also be made to an interesting paper by E. O. Lovett*, which gives a critical and historical account of the subject.

While these results hold of quite general systems, they are an incomplete statement of the case (particularly as to contact transformations being the only transformations which conserve the form) for particular given systems. Lie's original memoir† discusses the whole matter. The present chapter purports to give indications of the theory as connected with the theory of partial differential equations; it does not aim at being an introductory account of theoretical dynamics, as developed on the lines of Lie's theory.

HAMILTON'S CHARACTERISTIC EQUATIONS.

146. We have seen that Cauchy's method of integration introduces the notion of initial values of the variables and utilises them in the expression of an integral. The same idea was used by Jacobi in developing some researches of Hamilton on theoretical dynamics where such initial values had been used: and in connection with the idea, he devised a method of integration, which is sometimes called *Jacobi's first method* and more often the *Jacobi-Hamiltonian method.* The details of the method differ from those in Cauchy's method: but on account of the ideas and the results, both Lie and Mansion claim‡ the method as Cauchy's. Some account of the method will be given here, partly because of its close association with methods and results obtained in the region of theoretical dynamics when the equations are taken in their canonical form. Later researches in some branches of this subject have diverged from the earlier course, mainly because of the application of Lie's theory of contact transformations.

In treatises concerned with the dynamics of systems of bodies§, it is shewn that the equations of motion of a holonomic system can be expressed in a form

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial u_i}, \quad \frac{du_i}{dt} = -\frac{\partial H}{\partial \theta_i},$$

* "The theory of perturbations and Lie's theory of contact transformations," *Quart. Journ. Math.*, t. XXX (1899), pp. 47—149.

† "Die Störungstheorie und die Berührungstransformationen," *Arch. f. Math. og Nat.*, t. II (1877), pp. 10—38.

‡ See Part I of this Treatise, p. 183, foot-note.

§ Such as Routh's *Treatise on Rigid Dynamics*; see vol. I, ch. VIII.

for $i=1, \ldots, m$, the quantity H being (in the simplest case) the total energy of the system expressed in terms of t and of the variables $\theta_1, u_1, \ldots$. The form is usually associated with the name of Hamilton, as having been obtained by him*.

The following derivation of this result in the simplest case will give some indication as to the source of the transformation adopted by Jacobi in his method of integration. Denoting, as usual, the kinetic energy of the system by T and its potential energy by V, by $\theta_1, \ldots, \theta_m$ the m independent coordinates of the system, and by $\theta_1', \ldots, \theta_m'$ their derivatives with regard to the time t, we have Lagrange's equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta_i'}\right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} = 0.$$

Introducing a function L, such that

$$L = T - V,$$

and noting that V does not involve $\theta_1', \ldots, \theta_m'$, we have the equations in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_i'}\right) = \frac{\partial L}{\partial \theta_i}.$$

The function H is defined by the equation

$$H = \theta_1' \frac{\partial L}{\partial \theta_1'} + \ldots + \theta_m' \frac{\partial L}{\partial \theta_m'} - L,$$

the form of which has analogies with Legendre's contact transformation; and it is convenient to introduce variables $u_1, \ldots, u_m$ such that

$$\frac{\partial L}{\partial \theta_i'} = u_i,$$

for $i=1, \ldots, m$. Thus

$$H = \theta_1' u_1 + \ldots + \theta_m' u_m - L,$$

and therefore

$$\begin{aligned} dH &= u_1 d\theta_1' + \ldots + u_m d\theta_m' + \theta_1' du_1 + \ldots + \theta_m' du_m \\ &\quad - \left(u_1 d\theta_1' + \ldots + u_m d\theta_m' + \frac{\partial L}{\partial \theta_1} d\theta_1 + \ldots + \frac{\partial L}{\partial \theta_m} d\theta_m + \frac{\partial L}{\partial t} dt\right) \\ &= \theta_1' du_1 + \ldots + \theta_m' du_m - \frac{\partial L}{\partial \theta_1} d\theta_1 - \ldots - \frac{\partial L}{\partial \theta_m} d\theta_m - \frac{\partial L}{\partial t} dt. \end{aligned}$$

* *Phil. Trans.*, (1834), pp. 247—308, (1835), pp. 95—144.

In its initially defined form, H is a function of t, $\theta_1, \ldots, \theta_m$, $\theta_1', \ldots, \theta_m'$; let its expression be supposed changed, by means of the m equations

$$\frac{\partial L}{\partial \theta_1'} = u_1, \;\ldots,\; \frac{\partial L}{\partial \theta_m'} = u_m,$$

so that it becomes a function of $u_1, \ldots, u_m, \theta_1, \ldots, \theta_m, t$, the variables $\theta_1', \ldots, \theta_m'$ being replaced by $u_1, \ldots, u_m$. Then the foregoing differential relation gives

$$\begin{aligned} \frac{\partial H}{\partial u_i} &= \theta_i' = \frac{d\theta_i}{dt}, \\ -\frac{\partial H}{\partial \theta_i} &= \frac{\partial L}{\partial \theta_i} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_i'}\right) \\ &= \frac{du_i}{dt}, \end{aligned}$$

for $i = 1, \ldots, m$: and these equations are frequently called the *canonical form* of the equations of motion.

Now

$$H = \theta_1' u_1 + \ldots + \theta_m' u_m - L,$$

so that, as

$$\frac{\partial L}{\partial \theta_i'} = u_i,$$

for $i = 1, \ldots, m$, we have

$$dH = \theta_1' du_1 + \ldots + \theta_m' du_m - \frac{\partial L}{\partial \theta_1} d\theta_1 - \ldots - \frac{\partial L}{\partial \theta_m} d\theta_m - \frac{\partial L}{\partial t} dt.$$

Hence, when H is expressed as a function of $t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m$ through the removal of $\theta_1', \ldots, \theta_m'$ by means of the equations

$$\frac{\partial L}{\partial \theta_i'} = u_i,$$

we have

$$\frac{\partial H}{\partial u_r} = \theta_r',$$

for $r = 1, \ldots, m$; thus

$$H = u_1 \frac{\partial H}{\partial u_1} + \ldots + u_r \frac{\partial H}{\partial u_r} - L,$$

and therefore

$$L = u_1 \frac{\partial H}{\partial u_1} + \ldots + u_r \frac{\partial H}{\partial u_r} - H,$$

so that the relation between L and H is reciprocal.

This is true when t occurs explicitly in L and in H: it can easily be verified, in the simplest case, when neither T nor V involves t explicitly. Then T is a homogeneous quadratic function of the second order in $\theta_1', \ldots, \theta_m'$, so that

$$\theta_1' \frac{\partial T}{\partial \theta_1'} + \ldots + \theta_m' \frac{\partial T}{\partial \theta_1'} = 2T,$$

that is,

$$\theta_1' \frac{\partial L}{\partial \theta_1'} + \ldots + \theta_m' \frac{\partial L}{\partial \theta_1'} = 2T,$$

that is,

$$\theta_1' u_1 + \ldots + \theta_m' u_m = 2T,$$

and therefore

$$H = 2T - L$$
$$= T + V,$$

so that, in this case, H is the total energy of the system. Also, as the equations

$$\frac{\partial L}{\partial \theta_i'} = u_i$$

in this case determine $\theta_1', \ldots, \theta_m'$ as quantities linear and homogeneous in $u_1, \ldots, u_m$, the quantity T is a homogeneous quadratic function of the second order in $u_1, \ldots, u_m$ after the change of variables is effected, so that

$$u_1 \frac{\partial T}{\partial u_1} + \ldots + u_m \frac{\partial T}{\partial u_m} = 2T,$$

and therefore

$$u_1 \frac{\partial H}{\partial u_1} + \ldots + u_m \frac{\partial H}{\partial u_m} = 2T:$$

consequently,

$$L = 2T - H$$
$$= u_1 \frac{\partial H}{\partial u_1} + \ldots + u_m \frac{\partial H}{\partial u_m} - H,$$

so that, as before, H and L are reciprocal to one another in form. The preceding analysis shews that, when L is derived thus from the function H, and when it is expressed as a function of $\theta_1, \ldots, \theta_m, \theta_1', \ldots, \theta_m'$ by means of the equations

$$\frac{\partial H}{\partial u_1} = \theta_1', \ \ldots, \ \frac{\partial H}{\partial u_m} = \theta_m',$$

the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_i'}\right) = \frac{\partial L}{\partial \theta_i},$$

for $i = 1, \ldots, m$, are satisfied.

The canonical equations

$$\frac{d\theta_i}{dt}=\frac{\partial H}{\partial u_i}, \quad -\frac{\partial H}{\partial \theta_i}=\frac{du_i}{dt},$$

are the general equations of motion, whether H involves t explicitly or not; but there is a substantial difference, as regards the relation of H to the equations, according as t does or does not occur in H.

When t does not occur explicitly in H, we have

$$\begin{aligned}\frac{dH}{dt}&=\sum_{r=1}^{m}\left(\frac{\partial H}{\partial u_r}\frac{du_r}{dt}+\frac{\partial H}{\partial \theta_r}\frac{d\theta_r}{dt}\right)\\&=0,\end{aligned}$$

that is, H is constant throughout the motion: or

$$H=\text{constant}$$

is an integral of the system, being of course the energy integral. Also, if

$$f=f(u_1, \ldots, u_m, \theta_1, \ldots, \theta_m)=\text{constant},$$

be any other integral of the system, we have

$$\begin{aligned}0=\frac{df}{dt}&=\sum_{r=1}^{m}\left(\frac{\partial f}{\partial \theta_r}\frac{d\theta_r}{dt}+\frac{\partial f}{\partial u_r}\frac{du_r}{dt}\right)\\&=\sum_{r=1}^{m}\left(\frac{\partial f}{\partial \theta_r}\frac{\partial H}{\partial u_r}-\frac{\partial f}{\partial u_r}\frac{\partial H}{\partial \theta_r}\right)\\&=(f, H),\end{aligned}$$

in the earlier notation. Conversely, any quantity f, distinct from H, involving the variables but not involving t, and satisfying this equation, is an integral of the canonical system.

But, if t does occur explicitly in H, then in connection with the system of equations we have

$$\begin{aligned}\frac{dH}{dt}&=\frac{\partial H}{\partial t}+\sum_{r=1}^{m}\left(\frac{\partial H}{\partial u_r}\frac{du_r}{dt}+\frac{\partial H}{\partial \theta_r}\frac{d\theta_r}{dt}\right)\\&=\frac{\partial H}{\partial t},\end{aligned}$$

which does not vanish: so that $H=\text{constant}$ is not then an integral of the system. If

$$g=g(t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m)=\text{constant},$$

be an integral of the system, we have

$$0 = \frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_{r=1}^{m} \left(\frac{\partial g}{\partial \theta_r} \frac{d\theta_r}{dt} + \frac{\partial g}{\partial u_r} \frac{du_r}{dt} \right)$$

$$= \frac{\partial g}{\partial t} + \sum_{r=1}^{m} \left(\frac{\partial g}{\partial \theta_r} \frac{\partial H}{\partial u_r} - \frac{\partial g}{\partial u_r} \frac{\partial H}{\partial \theta_r} \right)$$

$$= \frac{\partial g}{\partial t} + (g,\ H);$$

and this is the partial differential equation which is characteristic of every function g leading to an integral of the system. It is one of the equations that occur in Hamilton's theory: and (g, H) is homogeneous and linear in the derivatives of g.

147. Another characteristic partial differential equation is derived by Hamilton through the consideration of the integral

$$S = \int_{t_0}^{t} L\,dt,$$

so that, in the most general case, S is a function of t and t_0 and also of the values* of $\theta_1, \ldots, \theta_m, \theta_1', \ldots, \theta_m'$ at t and at t_0. To obtain some of its properties, imagine a quite general variation and (in order the more simply to allow t also to undergo this variation) introduce a new variable s, so that

$$\theta_i' = \frac{d\theta_i}{dt} = \frac{1}{t'} \frac{d\theta_i}{ds} = \frac{1}{t'} \phi_i,$$

say, for $i = 1, \ldots, m$, where $t' = \dfrac{dt}{ds}$; thus

$$S = \int_{s_0}^{s} t' L \left(t, \theta_1, \ldots, \theta_m, \frac{1}{t'} \phi_1, \ldots, \frac{1}{t'} \phi_m \right) ds$$

$$= \int_{s_0}^{s} \lambda (t, t', \theta_1, \phi_1, \ldots, \theta_m, \phi_m)\, ds,$$

where now all the arguments in λ are assumed functions of s, and s itself is not subject to variation. Taking a variation $t + \delta t$, $\theta_1 + \delta\theta_1, \ldots, \theta_m + \delta\theta_m$, we find, as usual,

$$\delta S = \left[\frac{\partial \lambda}{\partial t'} \delta t + \sum_{r=1}^{m} \frac{\partial \lambda}{\partial \phi_r} \delta\theta_r \right]_{s_0}^{s}$$

$$+ \int_{s_0}^{s} \left\{ \frac{\partial \lambda}{\partial t} - \frac{d}{ds} \left(\frac{\partial \lambda}{\partial t'} \right) \right\} \delta t\, ds$$

$$+ \sum_{\mu=1}^{m} \int_{s_0}^{s} \left\{ \frac{\partial \lambda}{\partial \theta_\mu} - \frac{d}{ds} \left(\frac{\partial \lambda}{\partial \phi_\mu} \right) \right\} \delta\theta_\mu\, ds.$$

* Only half of these $4m$ quantities can be taken as independent variables.

As s is at our choice, we shall choose it so that the equation

$$\frac{\partial \lambda}{\partial t} - \frac{d}{ds}\left(\frac{\partial \lambda}{\partial t'}\right) = 0$$

is satisfied: the actual value of s is not required. Also

$$\frac{\partial \lambda}{\partial \phi_\mu} = t' \left\{\frac{1}{t'}\frac{\partial L}{\partial \theta_\mu'}\right\} = \frac{\partial L}{\partial \theta_\mu'},$$

$$\frac{\partial \lambda}{\partial \theta_\mu} = t' \frac{\partial L}{\partial \theta_\mu},$$

so that

$$\begin{aligned} \frac{\partial \lambda}{\partial \theta_\mu} - \frac{d}{ds}\left(\frac{\partial \lambda}{\partial \phi_\mu}\right) &= t' \frac{\partial L}{\partial \theta_\mu} - \frac{d}{ds}\left(\frac{\partial L}{\partial \theta_\mu'}\right) \\ &= t' \left\{\frac{\partial L}{\partial \theta_\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_\mu'}\right)\right\} \\ &= 0, \end{aligned}$$

for all values of μ. Hence

$$\begin{aligned} \delta S &= \left[\frac{\partial \lambda}{\partial t'}\delta t + \sum_{r=1}^{m} \frac{\partial \lambda}{\partial \phi_r}\delta\theta_r\right]_{s_0}^{s} \\ &= \left[\frac{\partial \lambda}{\partial t'}\delta t + \sum_{r=1}^{m} \frac{\partial \lambda}{\partial \phi_r}\delta\theta_r\right]_{t_0}^{t}. \end{aligned}$$

Now, as above,

$$\frac{\partial \lambda}{\partial \phi_r} = \frac{\partial L}{\partial \theta_r'},$$

and

$$\frac{\partial \lambda}{\partial t'} = L - \Sigma\, \theta_r' \frac{\partial L}{\partial \theta_r'};$$

consequently

$$\begin{aligned} \delta S &= \left[\left(L - \Sigma\, \theta_r' \frac{\partial L}{\partial \theta_r'}\right) dt + \sum_{r=1}^{m} \frac{\partial L}{\partial \theta_r'}\delta\theta_r\right]_{t_0}^{t} \\ &= \left[-H\delta t + \sum_{r=1}^{m} \frac{\partial L}{\partial \theta_r'}\delta\theta_r\right]_{t_0}^{t}. \end{aligned}$$

It is an immediate consequence that in any configuration, as developed from assigned initial conditions, the value of S at any time depends only upon the configuration at that moment and upon the initial conditions.

To make these initial conditions precise, let

$$\begin{aligned} &\theta_\mu = \beta_\mu, \text{ when } t = t_0, \text{ for } \mu = 1, \ldots, m, \\ &L = L_0, \ \ldots\ldots\ldots\ldots, \\ &H = H_0, \ \ldots\ldots\ldots\ldots, \\ &\frac{\partial L_0}{\partial \theta_r'} = c_r, \text{ when } t = t_0, \text{ for } r = 1, \ldots, m: \end{aligned}$$

then we have

$$\left.\begin{aligned}\frac{\partial S}{\partial t} &= -H, & \frac{\partial S}{\partial t_0} &= -H_0\\ \frac{\partial S}{\partial \theta_r} &= u_r, & \frac{\partial S}{\partial \beta_r} &= -c_r\end{aligned}\right\},$$

for $r = 1, \ldots, m$. These are the values of the derivatives of S with regard to each of the quantities it involves.

As our purpose is not the discussion of the organic significance of any property or group of properties of the quantities concerned, but only to indicate so much of the analysis connected with the equations of theoretical dynamics as will throw some light upon the analysis introduced into what is commonly called the Jacobi-Hamilton method, we shall indicate only one inference from the preceding equations. The quantity H can be expressed as a function of $t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m$, say

$$H = \psi(t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m);$$

hence S satisfies the equation

$$\frac{\partial S}{\partial t} + \psi\left(t, \theta_1, \ldots, \theta_m, \frac{\partial S}{\partial \theta_1}, \ldots, \frac{\partial S}{\partial \theta_m}\right) = 0,$$

and this is another of the characteristic equations in Hamilton's theory. Moreover, when H involves t explicitly, H is not homogeneous in the quantities u, so that the equation satisfied by S is not homogeneous in the derivatives.

If, however, H is independent of any explicit occurrence of t, we know that

$$H = h,$$

where h is a constant; and then

$$S = -h(t - t_0) + S_1,$$

so that

$$\frac{\partial S}{\partial \theta_r} = u_r = \frac{\partial S_1}{\partial \theta_r};$$

and the equation satisfied by S_1 is

$$\begin{aligned}h &= H(\theta_1, \ldots, \theta_m, u_1, \ldots, u_m)\\ &= H\left(\theta_1, \ldots, \theta_m, \frac{\partial S_1}{\partial \theta_1}, \ldots, \frac{\partial S_1}{\partial \theta_m}\right).\end{aligned}$$

This is the modified form of Hamilton's characteristic equation when H does not explicitly involve t: the right-hand is homogeneous in the derivatives of S_1.

148. These characteristic equations have been derived from the initial set of equations in the canonical form: the relation between them can be exhibited in another light. By the existence-theorem of a set of ordinary simultaneous equations of the first order*, the canonical equations

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial u_i}, \quad \frac{du_i}{dt} = -\frac{\partial H}{\partial \theta_i},$$

determine the $2m$ quantities $\theta_1, \ldots, \theta_m, u_1, \ldots, u_m$ as functions of t and of parameters, which are the values of those quantities when $t = t_0$: these are

$$\theta_i = \beta_i, \quad u_i = c_i,$$

when $t = t_0$. Now L is a function of $t, \theta_1, \ldots, \theta_m, \theta_1', \ldots, \theta_m'$: as, by means of the equations

$$\theta_i' = \frac{\partial H}{\partial u_i},$$

the quantities $\theta_1', \ldots, \theta_m'$ are expressible in terms of $u_1, \ldots, u_m$, it follows that L can be expressed as a function of $t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m$. When the integrals of the canonical system are used, they can express $u_1, \ldots, u_m, c_1, \ldots, c_m$ in terms of the other quantities: thus L can be expressed in terms of $t, t_0, \theta_1, \ldots, \theta_m, \beta_1, \ldots, \beta_m$, and therefore also S, which is

$$\int_{t_0}^{t} L\,dt,$$

can be expressed in terms of $t, t_0, \theta_1, \ldots, \theta_m, \beta_1, \ldots, \beta_m$. Assuming this expression effected, we have

$$\frac{\partial S}{\partial \theta_r} = u_r, \quad \frac{\partial S}{\partial \beta_r} = -c_r,$$

which are $2m$ equations expressing $u_1, \ldots, u_m$ and $c_1, \ldots, c_m$ in terms of $t, t_0, \theta_1, \ldots, \theta_m, \beta_1, \ldots, \beta_m$: that is, they are equivalent to the $2m$ relations which are the integrals of the canonical system. Combining these results, we have the following theorem:—

The system of ordinary canonical equations

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial u_i}, \quad \frac{du_i}{dt} = -\frac{\partial H}{\partial \theta_i},$$

* See vol. II of this work, § 10.

where $H = H(t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m)$, *is connected in the following manner with the partial differential equation*

$$\frac{\partial S}{\partial t} + H(t, \theta_1, \ldots, \theta_m, u_1, \ldots, u_m) = 0,$$

where

$$u_1 = \frac{\partial S}{\partial \theta_1}, \ldots, u_m = \frac{\partial S}{\partial \theta_m}.$$

Obtain the complete set of integrals of the system of ordinary equations, determining the arbitrary constants so that

$$\theta_1, \ldots, \theta_m = \beta_1, \ldots, \beta_m,$$

$$u_1, \ldots, u_m = c_1, \ldots, c_m,$$

when $t = t_0$. *Construct the function*

$$L = u_1 \frac{\partial H}{\partial u_1} + \ldots + u_m \frac{\partial H}{\partial u_m} - H,$$

and, expressing it in terms of $t, \beta_1, \ldots, \beta_m, c_1, \ldots, c_m$ *by means of the foregoing integrals, obtain the function*

$$I = \int_{t_0}^{t} L\,dt - \sum_{r=1}^{m} c_r \beta_r;$$

and when this is obtained, use the integrals to express I *in terms of* $t, \beta_1, \ldots, \beta_m, \theta_1, \ldots, \theta_m$, *denoting the resulting expression by* s. *Then the relation*

$$S = a + s,$$

when a *is an arbitrary constant, is an integral of the partial differential equation: as it contains* $m + 1$ *arbitrary constants* $a, \beta_1, \ldots, \beta_m$, *it is a complete integral. Moreover, the* $2m$ *equations*

$$\frac{\partial S}{\partial \theta_i} = u_i, \quad \frac{\partial S}{\partial \beta_i} = -c_i, \qquad (i = 1, \ldots, m),$$

can be regarded as an integral equivalent of the system of ordinary equations.

This process has been derived through the general equations of motion of a dynamical system, so that there are limitations on the form of H, quà function of $u_1, \ldots, u_m$, when the theorem is thus obtained. The result, however, is not subject in fact to these limitations: and it is this extension and generalisation of Hamilton's investigation which constitute the method of integration. As it was published* by Jacobi, it is often called the

* *Crelle*, t. XVII (1837), pp. 136 et seq.: see also Part I of this treatise, § 109.

Jacobi-Hamiltonian method; but, as already pointed out, the use of the initial values had been introduced earlier by Cauchy.

149. Other characteristic functions are introduced into the study of theoretical dynamics: among them, one of the most important is Hamilton's function often denoted by A, which represents the dynamical Action when H does not explicitly involve t, and which in general is defined by the equation

$$\begin{aligned} A &= S + \int_{t_0}^{t} \frac{d}{dt}(Ht)\,dt \\ &= S + Ht - H_0 t_0. \end{aligned}$$

With this value, we have

$$\begin{aligned} \delta A &= \delta S + \Big[H\delta t + t\delta H \Big]_{t_0}^{t} \\ &= \Big[t\delta H + \sum_{r=1}^{m} \frac{\partial L}{\partial \theta_r'} \delta\theta_r \Big]_{t_0}^{t} \\ &= \Big[t\delta H + \sum_{r=1}^{m} u_r \delta\theta_r \Big]_{t_0}^{t}. \end{aligned}$$

Now by means of the integrals of the canonical system as associated with initial values, H can be expressed in terms of t, t_0, $\theta_1, \ldots, \theta_m$, $\beta_1, \ldots, \beta_m$, and H_0 can be expressed in terms of t_0, $\beta_1, \ldots, \beta_m$. Also A can be expressed in terms of t, t_0, $\theta_1, \ldots, \theta_m$, $\beta_1, \ldots, \beta_m$: when t and t_0 are eliminated from its expression by means of the expressions for H and H_0, it comes to be a function of H, H_0, $\theta_1, \ldots, \theta_m$, $\beta_1, \ldots, \beta_m$. In this form, the equations

$$\frac{\partial A}{\partial H} = t, \quad \frac{\partial A}{\partial \theta_r} = u_r, \quad \frac{\partial A}{\partial \beta_r} = -c_r, \quad \frac{\partial A}{\partial H_0} = -t_0,$$

are satisfied; they must be equivalent to the integrals of the canonical system.

Ex. 1. Prove that, when the expressions for the kinetic energy T and the potential energy V do not explicitly involve the time, the function A satisfies the partial differential equation

$$T\left(\theta_1, \ldots, \theta_m, \frac{\partial A}{\partial \theta_1}, \ldots, \frac{\partial A}{\partial \theta_m}\right) = h - V(\theta_1, \ldots, \theta_m),$$

where h is a constant. (Jacobi.)

Ex. 2. Shew that, if a complete integral of the partial differential equation satisfied by A be obtained in the form

$$A = g(\theta_1, \ldots, \theta_m, h, a_1, \ldots, a_{m-1}) + a_m,$$

where $a_1, \ldots, a_{m-1}, a_m$ are the arbitrary constants, then a set of integrals of the canonical system is given by

$$\frac{\partial g}{\partial a_1}=b_1, \ldots, \frac{\partial g}{\partial a_{m-1}}=b_{m-1}, \frac{\partial g}{\partial h}=t+\tau,$$

where $a_1, \ldots, a_{m-1}, h, b_1, \ldots, b_{m-1}, \tau$ are the $2m$ arbitrary constants in the integrals. (Jacobi.)

Ex. 3. Let m integrals (in involution) of the canonical system be supposed known, involving $\theta_1, \ldots, \theta_m, u_1, \ldots, u_m$ in such a way that $u_1, \ldots, u_m$ can be expressed in terms of $\theta_1, \ldots, \theta_m, t$, and the m arbitrary constants of the integrals; and let these values be substituted in H, the resulting value being denoted by $\overline{H}$. Prove that

$$u_1 d\theta_1 + \ldots + u_m d\theta_m - \overline{H} dt$$

is an exact differential; and shew how the remaining integrals of the canonical system can be obtained. (Liouville.)

Ex. 4. When the expression for the energy of the system does not involve the time and when $m-1$ integrals (other than $H=h$) of the canonical system have been obtained, so that $u_1, \ldots, u_m$ can be expressed in terms of $\theta_1, \ldots, \theta_m$ by means of those $m-1$ integrals and $H=h$, prove that

$$u_1 d\theta_1 + \ldots + u_m d\theta_m$$

is an exact differential $d\Sigma$. Obtain the other integrals of the canonical system: and shew that the variables in the integrals are connected with t by the relation

$$\frac{\partial \Sigma}{\partial h}=t+\tau.$$

(Liouville.)

Ex. 5. Integrate the equation

$$(p_1^2+p_2^2+p_3^2)(x_1^2+x_2^2+x_3^2)^2=a^4,$$

where a is a constant, by using the theorems in any of the preceding examples.

JACOBI'S GENERALISATION OF HAMILTON'S RESULTS.

150. The preceding brief discussion will sufficiently illustrate the connection between a partial differential equation and a canonical system of ordinary equations, as it arises in the discussion of theoretical dynamics: and each of the methods of integration, which have been expounded in the preceding chapters, shews a similar organic relation. The detailed application of the method, suggested by the processes of theoretical dynamics, differs from the use made in other methods; and though it is somewhat more cumbrous than those methods, its association with the results of theoretical dynamics seems ample justification for its retention among the principal methods of integration.

We proceed from the generalised form of the equation satisfied by S and, making a change in the notation, we suppose that a given partial differential equation has been resolved with regard to one of the derivatives into the form

$$p + H(x, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

where, as usual

$$p = \frac{\partial z}{\partial x}, \quad p_i = \frac{\partial z}{\partial x_i},$$

for $i = 1, \ldots, n$. It will be noticed that z does not occur explicitly: the alternative forms, when z does occur, will be given later.

The equations of the characteristics are

$$\frac{dx}{1} = \frac{dx_1}{\frac{\partial H}{\partial p_1}} = \ldots = \frac{dx_n}{\frac{\partial H}{\partial p_n}} = \frac{dp_1}{-\frac{\partial H}{\partial x_1}} = \ldots = \frac{dp_n}{-\frac{\partial H}{\partial x_n}} = \frac{dp}{-\frac{\partial H}{\partial x}}:$$

but the last fraction can be omitted, because p occurs there only and we have the permanent equation

$$p = -H.$$

Thus the equations can be taken in the form

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i},$$

for $i = 1, \ldots, n$: and these agree with the canonical form of the equations of theoretical dynamics. Assume that these equations have been completely integrated, the arbitrary constants being determined by the conditions that

$$x_1, \ldots, x_n, p_1, \ldots, p_n = a_1, \ldots, a_n, b_1, \ldots, b_n,$$

when $x = a$; and let the results be expressible in the form

$$x_i = \xi_i(x, a_1, \ldots, a_n, b_1, \ldots, b_n),$$
$$p_i = \pi_i(x, a_1, \ldots, a_n, b_1, \ldots, b_n),$$

for $i = 1, \ldots, n$. The determinant

$$J\left(\frac{\xi_1, \ldots, \xi_n}{a_1, \ldots, a_n}\right)$$

is unity when $x = a$, so that it cannot vanish identically; hence the n equations $x_i = \xi_i$ can be resolved for $a_1, \ldots, a_n$.

Now the quantity

$$p_1 dx_1 + \ldots + p_n dx_n + p\,dx$$

is an exact differential: substituting for $dx_1, \ldots, dx_n$ from the ordinary equations, and also $-H$ for p, it takes the form

$$\left(p_1 \frac{\partial H}{\partial p_1} + \ldots + p_n \frac{\partial H}{\partial p_n} - H\right) dx,$$

which also is a form suggested by the analysis connected with the equations of theoretical dynamics. We therefore take a quantity

$$\zeta = \int_a^x \left(\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} - H\right) dx + \sum_{i=1}^n a_i b_i,$$

on the analogy of the dynamical results and, substituting for the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ their values as given by the integrals of the canonical system, we effect the quadrature which then gives ζ as a function of $x, a_1, \ldots, a_n, b_1, \ldots, b_n$. Let c denote any one of these $2n$ constants that occur in ζ; then, taking account of the fact that the values of the variables have been substituted in the initial form of ζ, we have

$$\begin{aligned}
\frac{\partial \zeta}{\partial c} - \frac{\partial}{\partial c}\left(\sum_{i=1}^n a_i b_i\right) \\
&= \int_a^x \sum_{i=1}^n \left\{\pi_i \frac{\partial}{\partial c}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial H}{\partial p_i}\frac{\partial \pi_i}{\partial c} - \frac{\partial H}{\partial x_i}\frac{\partial \xi_i}{\partial c} - \frac{\partial H}{\partial p_i}\frac{\partial \pi_i}{\partial c}\right\} dx \\
&= \int_a^x \sum_{i=1}^n \left\{\pi_i \frac{\partial}{\partial c}\left(\frac{\partial \xi_i}{\partial x}\right) + \frac{\partial \xi_i}{\partial c}\frac{\partial \pi_i}{\partial x}\right\} dx \\
&= \int_a^x \sum_{i=1}^n \left\{\pi_i \frac{\partial}{\partial x}\left(\frac{\partial \xi_i}{\partial c}\right) + \frac{\partial \xi_i}{\partial c}\frac{\partial \pi_i}{\partial x}\right\} dx \\
&= \sum_{i=1}^n \left[\pi_i \frac{\partial \xi_i}{\partial c}\right]_a^x \\
&= \sum_{i=1}^n \left(\pi_i \frac{\partial \xi_i}{\partial c} - b_i \frac{\partial a_i}{\partial c}\right);
\end{aligned}$$

consequently

$$\frac{\partial \zeta}{\partial a_s} = \sum_{i=1}^n \left(\pi_i \frac{\partial \xi_i}{\partial a_s}\right),$$

$$\frac{\partial \zeta}{\partial b_s} = \sum_{i=1}^n \left(\pi_i \frac{\partial \xi_i}{\partial b_s}\right) + a_s.$$

We have seen that the n equations $x_1 = \xi_1, \ldots, x_n = \xi_n$ can be resolved so as to express $a_1, \ldots, a_n$ in terms of $x_1, \ldots, x_n, x, b_1, \ldots,$

b_n; let the values thus obtained be substituted in ζ, and denote the resulting value by Z. Then, as

$$\frac{\partial \zeta}{\partial a_s} = \sum_{i=1}^{n} \frac{\partial Z}{\partial x_i} \frac{\partial \xi_i}{\partial a_s},$$

$$\frac{\partial \zeta}{\partial b_s} = \frac{\partial Z}{\partial b_s} + \sum_{i=1}^{n} \frac{\partial Z}{\partial x_i} \frac{\partial \xi_i}{\partial b_s},$$

we have, on substituting the preceding values for the derivatives of ζ, the equations

$$\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial x_i} - \pi_i \right) \frac{\partial \xi_i}{\partial a_s} = 0,$$

$$\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial x_i} - \pi_i \right) \frac{\partial \xi_i}{\partial b_s} = a_s - \frac{\partial Z}{\partial b_s},$$

for $s = 1, \ldots, n$. The former set of n equations is linear and homogeneous in the quantities $\frac{\partial Z}{\partial x_i} - \pi_i$, and the determinant of the coefficients of these quantities does not vanish; hence

$$\frac{\partial Z}{\partial x_i} - \pi_i = 0,$$

for $i = 1, \ldots, n$, and therefore from the remaining equations

$$\frac{\partial Z}{\partial b_i} - a_i = 0,$$

for $i = 1, \ldots, n$. These relations are not identities, because $\pi_1, \ldots, \pi_n, a_1, \ldots, a_n$ do not occur in Z; and they clearly are independent of one another. Moreover, they are satisfied in connection with the equations $x_1 = \xi_1, \ldots, x_n = \xi_n, p_1 = \pi_1, \ldots, p_n = \pi_n$; hence they are a general integral equivalent of the $2n$ differential equations

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i}.$$

Again, we have

$$\frac{d\zeta}{dx} = \sum_{i=1}^{n} \left(\pi_i \frac{\partial H}{\partial p_i} - H \right),$$

on replacing $x_1, \ldots, x_n, p_1, \ldots, p_n$ by their values: and

$$\frac{dZ}{dx} = \frac{\partial Z}{\partial x} + \sum_{i=1}^{n} \frac{\partial Z}{\partial x_i} \frac{\partial \xi_i}{\partial x}$$

$$= \frac{\partial Z}{\partial x} + \sum_{i=1}^{n} \pi_i \frac{\partial H}{\partial p_i},$$

on using the preceding equations. Hence the equality

$$\frac{d\zeta}{dx} = \frac{dZ}{dx},$$

leads to the relation

$$\begin{aligned}\frac{\partial Z}{\partial x} &= - H(x, \xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n) \\ &= - H\left(x, x_1, \ldots, x_n, \frac{\partial Z}{\partial x_1}, \ldots, \frac{\partial Z}{\partial x_n}\right);\end{aligned}$$

and therefore the equation

$$z = Z + c,$$

where c is an arbitrary constant, is an integral of the equation

$$p + H(x, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0.$$

Accordingly, the process may be stated as follows:—

To obtain an integral of the equation

$$p + H(x, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

form the canonical system

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = - \frac{\partial H}{\partial x_i}, \qquad (i = 1, \ldots, n),$$

of ordinary equations, and construct their complete set of integrals

$$\left.\begin{aligned} x_i &= \xi_i(x, a_1, \ldots, a_n, b_1, \ldots, b_n) \\ p_i &= \pi_i(x, a_1, \ldots, a_n, b_1, \ldots, b_n) \end{aligned}\right\}, \quad (i = 1, \ldots, n),$$

such that $x_1, \ldots, x_n, p_1, \ldots, p_n = a_1, \ldots, a_n, b_1, \ldots, b_n$ *respectively, when* $x = a$. *Take a quantity* ζ *defined by the relation*

$$\zeta = \int_a^x \left(\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} - H\right) dx + \sum_{i=1}^n a_i b_i;$$

and, substituting $\xi_1, \ldots, \xi_n, \pi_1, \ldots, \pi_n$ *for the variables under the sign of integration, effect the quadrature which gives* ζ *as a function of* $x, a_1, \ldots, a_n, b_1, \ldots, b_n$. *From* ζ *eliminate* $a_1, \ldots, a_n$ *by means of the equations* $x_1 = \xi_1, \ldots, x_n = \xi_n$, *and let the resulting function of* $x, x_1, \ldots, x_n, b_1, \ldots, b_n$ *be denoted by* Z*; then*

$$z = Z + c,$$

where c *is an arbitrary constant, is an integral of the partial differential equation, and manifestly it is a complete integral. Moreover, a complete set of integrals of the canonical system is given by*

$$\frac{\partial Z}{\partial x_i} = p_i, \quad \frac{\partial Z}{\partial b_i} = a_i,$$

for $i = 1, \ldots, n$, *the constants* $a_1, \ldots, a_n$ *being arbitrary.*

151. It is easy to see that the complete integral thus obtained is an integral which, when $x = a$, acquires the value

$$b_1 x_1 + \ldots + b_n x_n + c.$$

For Z is the value of ζ when $a_1, \ldots, a_n$ are eliminated from ζ by means of $x_1 = \xi_1, \ldots, x_n = \xi_n$; and therefore the value of Z, when $x = a$, is obtained from that of ζ when $x = a$ (which value is $a_1 b_1 + \ldots + a_n b_n$) by eliminating $a_1, \ldots, a_n$ through the forms of the equations $x_1 = \xi_1, \ldots, x_n = \xi_n$, when $x = a$: and these forms are $x_1 = a_1, \ldots, x_n = a_n$.

The complete integral is therefore somewhat restricted, though it contains the appropriate number of arbitrary constants: its relation to any other complete integral, say

$$z = \phi(x, x_1, \ldots, x_n, k_1, \ldots, k_n) + c,$$

can be simply obtained. In the case of this complete integral, a set of integrals of the canonical system is given by

$$\frac{\partial \phi}{\partial x_i} = p_i, \quad \frac{\partial \phi}{\partial k_i} = \kappa_i,$$

for $i = 1, \ldots, n$. Let

$$\phi_0 = \phi(a, a_1, \ldots, a_n, k_1, \ldots, k_n);$$

then as $a_1, \ldots, a_n, b_1, \ldots, b_n$ are the values of $x_1, \ldots, x_n, p_1, \ldots, p_n$, when $x = a$, we must have

$$\frac{\partial \phi_0}{\partial a_i} = b_i, \quad \frac{\partial \phi_0}{\partial k_i} = \kappa_i.$$

Now

$$\frac{\partial H}{\partial p_i} = \frac{dx_i}{dx}, \quad \frac{\partial \phi}{\partial x_i} = p_i, \quad H = -\frac{\partial \phi}{\partial x};$$

hence

$$\begin{aligned} \zeta &= \sum_{i=1}^{n} a_i b_i + \int_a^x \left(\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} - H \right) dx \\ &= \sum_{i=1}^{n} a_i b_i + \int_a^x \left(\frac{\partial \phi}{\partial x_i} \frac{dx_i}{dx} + \frac{\partial \phi}{\partial x} \right) dx \\ &= \sum_{i=1}^{n} a_i b_i + \phi - \phi_0. \end{aligned}$$

The constants $k_1, \ldots, k_n$ are such that

$$\frac{\partial \phi}{\partial k_i} = \kappa_i, \quad \frac{\partial \phi_0}{\partial k_i} = \kappa_i,$$

that is, such that

$$\frac{\partial \phi}{\partial k_i} = \frac{\partial \phi_0}{\partial k_i},$$

for $i = 1, \ldots, n$: let their values be determined and substituted in ζ. In order to obtain Z from ζ, the constants $a_1, \ldots, a_n$ must be eliminated; their values are given by

$$\frac{\partial \phi_0}{\partial a_i} = b_i,$$

in connection with the preceding values of $k_1, \ldots, k_n$: and so Z is the value of ζ when, from the equation

$$\zeta = \sum_{i=1}^{n} a_i b_i + \phi - \phi_0,$$

the quantities $a_1, \ldots, a_n, k_1, \ldots, k_n$ are removed by means of the equations

$$\frac{\partial \phi}{\partial k_i} = \frac{\partial \phi_0}{\partial k_i}, \quad \frac{\partial \phi_0}{\partial a_i} = b_i,$$

for $i = 1, \ldots, n$. Hence the complete integral in the theorem can be derived from any given complete integral*.

Ex. 1. The detailed working can be shewn by thus solving the equation

$$\frac{x}{p} + \frac{x_1}{p_1} + \frac{x_2}{p_2} = 1,$$

which clearly has an integral

$$z = kx^2 + k_1 x_1^2 + k_2 x_2^2 + c,$$

where

$$\frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} = 2.$$

The value of H for the form $p + H = 0$ is

$$H = \frac{x p_1 p_2}{p_1 x_2 + p_2 x_1 - p_1 p_2}.$$

The canonical system is

$$\frac{dx_1}{dx} = \frac{x x_1 p_2^2}{(p_1 x_2 + p_2 x_1 - p_1 p_2)^2}, \quad \frac{dx_2}{dx} = \frac{x x_2 p_1^2}{(p_1 x_2 + p_2 x_1 - p_1 p_2)^2},$$

$$\frac{dp_1}{dx} = \frac{x p_1 p_2^2}{(p_1 x_2 + p_2 x_1 - p_1 p_2)^2}, \quad \frac{dp_2}{dx} = \frac{x p_1^2 p_2}{(p_1 x_2 + p_2 x_1 - p_1 p_2)^2};$$

and integrals are

$$p_1 = a x_1, \quad p_2 = b x_2, \quad x_1^2 = \frac{c^2}{a^2} x^2 + a', \quad x_2^2 = \frac{c^2}{b^2} x^2 + b',$$

* The whole of the preceding exposition follows that which is given by Mayer, *Math. Ann.*, t. III (1871), pp. 434—452.

where a, b, a', b' are arbitrary constants, and

$$c=\frac{ab}{a+b-ab}.$$

Hence, taking a_1, a_2, b_1, b_2 as initial values when $x=\alpha$, we have

$$a=\frac{b_1}{a_1},\quad b=\frac{b_2}{a_2},\quad \frac{c}{a}=\frac{b_1a_2}{b_2a_1+b_1a_2-b_1b_2},\quad \frac{c}{b}=\frac{b_2a_1}{b_2a_1+b_1a_2-b_1b_2};$$

and the integrals are

$$p_1=\frac{b_1}{a_1}x_1,\quad p_2=\frac{b_2}{a_2}x_2,$$

$$x_1^2-a_1^2=\frac{c^2}{a^2}(x^2-\alpha^2),\quad x_2^2-a_2^2=\frac{c^2}{b^2}(x^2-\alpha^2).$$

If we require the integral which becomes $b_1x_1+b_2x_2+\gamma$, when $x=\alpha$, we take

$$\begin{aligned}\zeta&=a_1b_1+a_2b_2+\int_\alpha^x\left(p_1\frac{\partial H}{\partial p_1}+p_2\frac{\partial H}{\partial p_2}-H\right)dx\\&=a_1b_1+a_2b_2+\tfrac{1}{2}c^2(x^2-\alpha^2);\end{aligned}$$

and the required integral is given by

$$z=Z+\gamma,$$

where Z is given as a function of x, x_1, x_2, by the elimination of a_1 and a_2 between the three equations

$$\left.\begin{aligned}Z&=a_1b_1+a_2b_2+\tfrac{1}{2}c^2(x^2-\alpha^2)\\x_1^2-a_1^2&=\frac{c^2}{a^2}(x^2-\alpha^2)\\x_2^2-a_2^2&=\frac{c^2}{b^2}(x^2-\alpha^2)\end{aligned}\right\},$$

account being taken of the values of a, b, c in terms of a_1, a_2, b_1, b_2.

To derive this integral from the integral

$$z=kx^2+k_1x_1^2+k_2x_2^2+c,$$

where

$$\frac{1}{k}+\frac{1}{k_1}+\frac{1}{k_2}=2,$$

we write

$$\phi=kx^2+k_1x_1^2+k_2x_2^2,$$
$$\phi_0=k\alpha^2+k_1a_1^2+k_2a_2^2;$$

and then we take

$$\zeta=a_1b_1+a_2b_2+\phi-\phi_0.$$

The relations between the quantities a_1, a_2, b_1, b_2, k, k_1, k_2 (other than the single relation between k, k_1, k_2) are

$$\frac{\partial\phi_0}{\partial a_1}=b_1,\quad \frac{\partial\phi_0}{\partial a_2}=b_2,\quad \frac{\partial\phi}{\partial k_1}=\frac{\partial\phi_0}{\partial k_1},\quad \frac{\partial\phi}{\partial k_2}=\frac{\partial\phi_0}{\partial k_2},$$

account being taken of the relation between k, k_1, k_2. These give

$$b_1=2k_1a_1,\quad b_2=2k_2a_2,$$

$$k_1^2(x_1^2-a_1^2)=k^2(x^2-\alpha^2),\quad k_2^2(x_2^2-a_2^2)=k^2(x^2-\alpha^2);$$

hence

$$\zeta = a_1 b_1 + a_2 b_2 + \phi - \phi_0$$
$$= \frac{b_1^2}{2k_1} + \frac{b_2^2}{2k_2} + k(x^2 - a^2) + \frac{k^2}{k_1}(x^2 - a^2) + \frac{k^2}{k_2}(x^2 - a^2)$$
$$= \frac{b_1^2}{2k_1} + \frac{b_2^2}{2k_2} + 2k^2(x^2 - a^2).$$

We eliminate k, k_1, k_2 between the equations

$$\left.\begin{aligned} Z &= \frac{b_1^2}{2k_1} + \frac{b_2^2}{2k_2} + 2k^2(x^2 - a^2) \\ k^2 x_1^2 - \tfrac{1}{4}b_1^2 &= k^2(x^2 - a^2) \\ k^2 x_2^2 - \tfrac{1}{4}b_2^2 &= k^2(x^2 - a^2) \\ 2 &= \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} \end{aligned}\right\};$$

and then the integral is

$$z = Z + \gamma.$$

The verification that Z becomes $b_1 x_1 + b_2 x_2$, when $x = a$, is immediate.

Ex. 2. Obtain, in the preceding manner, the integral of the equation

$$pq - px - qy = 0,$$

such that $q = c$ when $x = a$, in the form

$$z = xy + y(x^2 - 2ac + c^2)^{\frac{1}{2}} + \gamma;$$

where γ is an arbitrary constant.

Deduce it also from the complete integral

$$z = \frac{1}{2a}(y + ax)^2 + \gamma.$$

Another integral is given by

$$z = xy + \{(x^2 - a^2)(y^2 - b^2)\}^{\frac{1}{2}};$$

is there any relation between this integral and the first integral?

Ex. 3. Let

$$z = \phi(x, x_1, \ldots, x_n, b_1, \ldots, b_n) + \gamma$$

be a complete integral of the differential equation in the text. Shew that, if

$$\phi_0 = \phi(a, a_1, \ldots, a_n, b_1, \ldots, b_n),$$

and

$$f_0 = f(a_1, \ldots, a_n),$$

and if $a_1, \ldots, a_n, b_1, \ldots, b_n$ be eliminated between the equations

$$\left.\begin{aligned} Z = \phi - \phi_0 + f_0 \\ \frac{\partial \phi}{\partial b_i} = \frac{\partial \phi_0}{\partial b_i}, \quad \frac{\partial f_0}{\partial a_i} = \frac{\partial \phi_0}{\partial a_i} \end{aligned}\right\},$$

for $i = 1, \ldots, n$, the resulting value of Z is also an integral of the equation and that, when $x = a$, it acquires the value $f(x_1, \ldots, x_n)$. (Mayer.)

152. The discussion in §§ 150, 151 related to a resolved equation in which the dependent variable does not occur explicitly: and the inverse operations required consisted of the integration of a system of $2n$ ordinary equations, followed by a quadrature.

When the partial differential equation involves the dependent variable explicitly, and when it can be resolved so that it may be taken in the form

$$p + f(z, x, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

then the corresponding result is as follows:—

Form and integrate the equations

$$\frac{dx_i}{dx} = \frac{\partial f}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial f}{\partial x_i} - p_i \frac{\partial f}{\partial z}, \quad \frac{dz}{dx} = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i} - f,$$

determining the arbitrary constants by the conditions that $x_1, \ldots, x_n, p_1, \ldots, p_n, z$ *acquire the values*

$$a_1, \ldots, a_n, b_1, \ldots, b_n, c + a_1 b_1 + \ldots + a_n b_n$$

respectively, when $x = a$. *Among this integral system of* $2n+1$ *equations, eliminate* $a_1, \ldots, a_n, p_1, \ldots, p_n$; *and let* Z *denote the resulting value of* z, *which is a function of* $x, x_1, \ldots, x_n, b_1, \ldots, b_n, c$. *Then*

$$z = Z$$

is a complete integral of the partial differential equation; and

$$\frac{\partial Z}{\partial x_i} = p_i, \quad \frac{\partial Z}{\partial b_i} = a_i \frac{\partial Z}{\partial c}, \quad z = Z,$$

for $i = 1, \ldots, n$, *the constants* $a_1, \ldots, a_n$ *being arbitrary, are a set of integrals of the* $2n+1$ *ordinary equations.*

This result may be deduced from the former case, or it may be obtained directly; we shall leave the establishment as an exercise. It will be noticed that, in the present case, the inverse operations required are the integration of a system of $2n+1$ ordinary equations, as contrasted with the slightly simpler inverse operations in the former case constituted by the integration of a system of $2n$ ordinary equations and a quadrature.

Lastly, it may happen that the partial differential equation contains the dependent variable explicitly but that it cannot be resolved, or cannot conveniently be resolved, in terms of any of

the derivatives. In such a case, a similar process exists: and the result has already been stated*.

THE POISSON-JACOBI COMBINANT (ϕ, ψ).

153. The determination of a complete integral of a partial differential equation

$$p + H(x, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

and the determination of a full set of integrals of the associated canonical system

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i},$$

for $i = 1, \ldots, n$, have been shewn to be practically equivalent problems. It is known that, if two equations compatible with the original differential equation have been obtained, the Poisson-Jacobi combination of those equations provides another equation (which may be insignificant or may be evanescent) also compatible with the equation: and naturally therefore a question arises whether the same combination can similarly be effective in assisting the construction of the integrals of the canonical system.

Let

$$\phi = \phi(x, x_1, \ldots, x_n, p_1, \ldots, p_n) = \text{constant}$$

be an integral of the canonical system: then, in connection with that system, we have

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}\frac{dx_i}{dx} + \sum_{i=1}^{n} \frac{\partial \phi}{\partial p_i}\frac{dp_i}{dx} = 0,$$

and therefore

$$\frac{\partial \phi}{\partial x} + \sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial \phi}{\partial p_i}\frac{\partial H}{\partial x_i}\right) = 0,$$

that is, using the Poisson-Jacobi symbol, we have

$$\frac{\partial \phi}{\partial x} + (\phi, H) = 0,$$

analogous with a corresponding equation (§ 146) in theoretical dynamics. Similarly, if

$$\psi = \psi(x, x_1, \ldots, x_n, p_1, \ldots, p_n) = \text{constant}$$

* In Part I, § 109, of the present work. All the results are given in Mayer's memoir quoted on p. 388, foot-note.

be an integral of the canonical system, we have

$$\frac{\partial \psi}{\partial x} + (\psi, H) = 0.$$

It is natural to inquire whether (ϕ, ψ) also is an integral of the canonical system: we have

$$\begin{aligned}
\frac{d}{dx}(\phi, \psi) &= \frac{\partial}{\partial x}(\phi, \psi) + ((\phi, \psi), H) \\
&= \left(\frac{\partial \phi}{\partial x}, \psi\right) + \left(\phi, \frac{\partial \psi}{\partial x}\right) + ((\phi, \psi), H) \\
&= -((\phi, H), \psi) - (\phi, (\psi, H)) + ((\phi, \psi), H) \\
&= ((H, \phi), \psi) + ((\psi, H), \phi) + ((\phi, \psi), H) \\
&= 0,
\end{aligned}$$

on account of the identically satisfied relation of § 52. Hence

$$(\phi, \psi) = \text{constant},$$

as an integral equation, is compatible with the canonical system. Various cases may arise, as in the former investigation.

It may happen that (ϕ, ψ) vanishes identically: no new integral is provided.

It may happen that (ϕ, ψ) is a pure constant not zero; instances have occurred in which (ϕ, ψ) is equal to unity: no new integral is provided.

It may happen that (ϕ, ψ), while a function of the variables, can be expressed in terms of ϕ and ψ alone (and possibly in terms of previously known integrals, if any): no new integral is provided.

And it may happen that (ϕ, ψ) is a function of the variables which cannot be expressed in terms of ϕ and ψ alone (or in terms of these and of previously known integrals, if any): a new integral of the canonical system is then provided.

Note. It may happen that, when the partial differential equation is

$$H(x_1, \ldots, x_n, p_1, \ldots, p_n) = \text{constant} = h,$$

so that p and x have disappeared, care has to be exercised concerning new integrals of the canonical system: such new integrals do not necessarily provide equations compatible with $H = h$ and with equations which coexist with it. The canonical system is effectively the same as before, for it is

$$\frac{dx_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\dfrac{\partial H}{\partial x_i}} = dx,$$

say, for $i=1, \ldots, n$; and if

$$\phi=\phi(x_1, \ldots, x_n, p_1, \ldots, p_n, x)=\text{constant},$$
$$\psi=\psi(x_1, \ldots, x_n, p_1, \ldots, p_n, x)=\text{constant},$$

are two integrals of the system, then

$$(\phi, \psi)=\text{constant}$$

is also an integral of the canonical system; and we have

$$(H, \phi)=0, \quad (H, \psi)=0, \quad (H, (\phi, \psi))=0.$$

But if we are proceeding on the lines of Jacobi's second method, as explained in Chapter IV, for the integration of the partial equation $H=0$, and if we have associated the equations

$$H=0, \quad \phi=a_1,$$

where a_1 is arbitrary, then we can associate

$$\psi=a_2$$

with these, only if

$$(\phi, \psi)=0;$$

and we can associate

$$(\phi, \psi)=a_3$$

with $H=0$, $\phi=a_1$, only if

$$(\phi, (\phi, \psi))=0;$$

and these conditions are not always satisfied.

Thus if the equation be

$$H=p_1^2+p_2^2+p_3^2+p_4^2-f(x_1^2+x_2^2+x_3^2, x_4)=0,$$

integrals of the canonical system are given by

$$\phi=x_2p_3-x_3p_2=a_1,$$
$$\psi=x_3p_1-x_1p_3=a_2;$$

and then

$$\chi=(\phi, \psi)=x_1p_2-x_2p_1,$$

which leads to a new integral of the canonical system. Now

$$H=0, \quad \phi=a_1,$$

can be associated, because

$$(H, \phi)=0.$$

But $\psi=a_2$ cannot be associated with these two equations; for though $(H, \psi)=0$, we have

$$(\phi, \psi)=\chi,$$

different from zero. Again, the equations

$$H=0, \quad \psi=a_2,$$

can be associated, because

$$(H, \psi)=0.$$

But $\phi=a_1$ cannot be associated with these equations because

$$(\psi, \phi)=-\chi,$$

which is not zero. Moreover,

$$\chi=a_3$$

cannot be associated with either pair, because

$$(\chi, \phi)=\psi, \quad (\chi, \psi)=-\phi.$$

But

$$(H, \psi^2+\chi^2)=0, \quad (\phi, \psi^2+\chi^2)=0:$$

and so

$$H=0, \quad \phi=a_1, \quad \psi^2+\chi^2=c,$$

can be associated.

It thus appears that, while the Poisson-Jacobi combination of two integrals of the canonical system can provide a new integral of that system, the new integral cannot necessarily be associated with a retained system compatible with the original equation.

154. It will be convenient, for the sake of brevity, to call the Poisson-Jacobi combination (ϕ, ψ), of two functions ϕ and ψ, their *combinant*. From the preceding results, it is clear that any integral ϕ of the canonical system, which does not involve x, satisfies the equation

$$(\phi, H)=0$$

identically: hence the combinant of H with any such integral ϕ leads to no new integral.

Moreover, when the function H in the canonical system does not explicitly involve x (which corresponds to the case in theoretical dynamics when the total energy of the dynamical system is constant), the combinant of H and of any integral ϕ of the canonical system, that does not explicitly involve x, vanishes identically: for the equation

$$(\phi, H)=0$$

is then satisfied identically, so that (ϕ, H) provides no new integral. Also, with the same supposition concerning H, the integrals of the canonical system can be so taken that $2n-1$ of them are relations among the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$, and the remaining integral can be taken in the form

$$\theta = x - x_0 + \vartheta(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0,$$

where x_0 is an arbitrary quantity. In that case, the equation

$$(\vartheta, H) + 1 = 0,$$

is satisfied identically: that is, (θ, H) provides no new integral. It therefore follows that, when the quantity H in the canonical system does not involve the variable x, no new integral can be derived by combining H with any other integral of the system: in fact, the quantity H is useless for any combinant construction

with any integral ϕ of the system with a view to the derivation of new integrals.

It thus appears that, when an integral of the canonical system has been obtained, no new integral is furnished by the combinant of H with that integral. Clearly, when H is explicitly independent of the variable x, it furnishes an integral of the system: but it is not the only integral of the system which, under the combinant construction, leads to evanescent or unfruitful results. Indeed, earlier results obtained in connection with the development of Jacobi's second method in Chapter IV shew that, when any integral of the canonical system of equations has been obtained, other integrals exist such that their combinants with the given integral provide no new integral but only an evanescent result. For let ϕ be a given integral of the system: and let ψ denote some other integral, distinct from H in case H should not involve x explicitly: then (ϕ, ψ) also satisfies the equations of the system. If (ϕ, ψ) vanishes, or is equal to a pure constant, or is not functionally independent of ϕ and ψ, then (ϕ, ψ) is illusory as providing a new integral. But if no one of these alternatives is valid, we write

$$(\phi, \psi)=\psi_1,$$

and we proceed (as in § 62) to form the series of functions

$$(\phi, \psi_1)=\psi_2, \quad (\phi, \psi_2)=\psi_3, \ \ldots, \ (\phi, \psi_{i-1})=\psi_i.$$

Each of the functions $\psi_1, \psi_2, \ldots$ is an integral of the canonical system; and the set of such functions, that are independent of one another, is limited in number because the canonical system is of finite order. Accordingly, we may assume that the series of functions, derived through combination with ϕ, terminates with ψ_i: the termination can come (§ 62) in one of three ways.

(i) If ψ_i vanishes identically, then ψ_{i-1} is such that

$$(\phi, \psi_{i-1})=0,$$

identically, that is, ψ_{i-1} is an integral of the type indicated.

(ii) If ψ_i is a pure constant, say c, then

$$(\phi, \psi^2_{i-1}-2c\psi_{i-2})=0,$$

so that as i is greater than unity (for otherwise ψ_1 would be an integral of the type indicated), then $\psi^2_{i-1}-2c\psi_{i-2}$ is an integral of the type indicated.

(iii) If ψ_i is a function of the integrals that occur earlier in the series, say θ, then any integral (say Ψ) of the system

$$\frac{d\psi}{\psi_1} = \frac{d\psi_1}{\psi_2} = \ldots = \frac{d\psi_{i-1}}{\theta}$$

is such that

$$(\phi, \Psi) = 0,$$

identically; and so Ψ is an integral of the indicated type belonging to the canonical system.

The process of combination is thus seen to provide a number of integrals: but, account being taken only of integrals that are independent of one another, the process cannot lead to all the independent integrals because, as has been seen, there are integrals which, when combined with a given integral, lead to an illusory result*. We shall not pursue this subject further, and shall be content with referring the reader to an important memoir† by Bertrand.

Ex. Prove that, when the function H in the canonical system does not explicitly involve x and when an integral ϕ other than H is given, the complete set of integrals of the canonical system is given by

$$H = h, \quad \phi = \text{constant},$$

and by

(i) an integral χ, not explicitly involving x and such that

$$(\phi, \chi) = 1,$$

(ii) an integral ψ, where

$$\psi = g(x_1, \ldots, x_n, p_1, \ldots, p_n) - x,$$

such that

$$(\phi, \psi) = 0,$$

(iii) other $2n-4$ integrals $a_1, \ldots, a_{2n-4}$, explicitly independent of x, such that

$$(\phi, a_i) = 0,$$

for $i = 1, \ldots, 2n-4$. (Bertrand.)

* Contrary to the opinion formed by Jacobi according to which it can be seen "in omnibus problematibus mechanicis in quibus virium vivarum conservatio "locum habet, *generaliter* e duobus integralibus præter principium illud inventis "reliqua omnia absque ulla ulteriore integratione inveniri posse": *Ges. Werke*, t. v, p. 49. The original theorem due to Poisson was published at the end of the year 1809: it seems that Jacobi's application and development of Poisson's theorem were made about 1838.

The frequently illusory character of the combinant is one of the causes which limit the number of the general algebraic integrals of the dynamical problem of n bodies to the classical integrals: see vol. III of this work, chapter XVII.

† *Liouville's Journal*, t. XVII (1852), pp. 393—436. Other references will be found in Graindorge's treatise, already quoted.

Contact Transformations and Canonical Systems.

155. In the last chapter, it was seen that the theory of contact transformations could be applied to the integration of a partial differential equation: and it has also been seen, from various points of view, that the integration of such an equation is bound up with the integration of a canonical system of ordinary equations. It is therefore natural to suppose that the theory of contact transformations can be brought into relation with the integration of a canonical system.

Let the canonical system be

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i},$$

for $i = 1, \ldots, n$; and suppose that H does not explicitly involve x. Let a contact transformation be given which passes from $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ to $X_1, \ldots, X_n, P_1, \ldots, P_n$, such that

$$(X_\mu, P_\mu) = 1,$$

$$(X_m, X_\mu) = 0, \quad (P_m, P_\mu) = 0, \quad (X_m, P_\mu) = 0,$$

for μ and $m = 1, \ldots, n$, with unequal values of m and μ; and let it be applied to transform the canonical system. We have

$$\begin{aligned}\frac{dX_i}{dx} &= \sum_{m=1}^{n} \left(\frac{\partial X_i}{\partial x_m}\frac{dx_m}{dx} + \frac{\partial X_i}{\partial p_m}\frac{dp_m}{dx}\right)\\ &= \sum_{m=1}^{n} \left(\frac{\partial X_i}{\partial x_m}\frac{\partial H}{\partial p_m} - \frac{\partial X_i}{\partial p_m}\frac{\partial H}{\partial x_m}\right)\\ &= (X_i, H).\end{aligned}$$

When the variables in H are transformed, let the resulting quantity be denoted by K; then

$$\begin{aligned}(X_i, H) &= \sum_{m=1}^{n} \left\{(X_i, X_m)\frac{\partial K}{\partial X_m} + (X_i, P_m)\frac{\partial K}{\partial P_m}\right\}\\ &= \frac{\partial K}{\partial P_i},\end{aligned}$$

on account of the properties of the contact transformation. Consequently, we have

$$\frac{dX_i}{dx} = \frac{\partial K}{\partial P_i};$$

and, by similar analysis, we also have

$$\frac{dP_i}{dx} = -\frac{\partial K}{\partial X_i}.$$

Hence the contact transformation leaves the form of the canonical system unchanged. But this is not the limit of the property: it is easy to see that any transformation, which leaves the form of the canonical system (supposed perfectly general) unchanged, is of the contact type. For taking any transformation from $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ to $X_1, \ldots, X_n, P_1, \ldots, P_n$, we have

$$\frac{dX_i}{dx} = (X_i, H), \quad \frac{dP_i}{dx} = (P_i, H);$$

if H' be the value of H after transformation has been effected, then

$$(X_i, H) = \sum_{m=1}^{n} \left\{ (X_i, X_m) \frac{\partial H'}{\partial X_m} + (X_i, P_m) \frac{\partial H'}{\partial P_m} \right\},$$

$$(P_i, H) = \sum_{m=1}^{n} \left\{ (P_i, X_m) \frac{\partial H'}{\partial X_m} + (P_i, P_m) \frac{\partial H'}{\partial P_m} \right\}.$$

If the new form of the equations is still canonical, the former of these must be $\dfrac{\partial H'}{\partial P_i}$, and the latter must be $-\dfrac{\partial H'}{\partial X_i}$, for all values of i: hence, as H and H' are supposed quite general functions, we must have

$$(X_m, P_m) = 1,$$

for all values $1, \ldots, n$ of m, and

$$(X_i, X_m) = 0, \quad (X_i, P_m) = -(P_m, X_i) = 0, \quad (P_i, P_m) = 0,$$

for all unequal values $1, \ldots, n$ of i and m. These are the equations which define a contact transformation. Therefore *a canonical system is unchanged in form by a contact transformation; and every transformation, which conserves the form of a quite general canonical system, is of the contact type.*

There is an immediate practical advantage in such a transformation, whenever the form of H' is simpler: the equations may be simpler to integrate.

156. In the next place, suppose that the canonical system is of the form

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i},$$

where H now involves x, as well as $x_1, \ldots, x_n, p_1, \ldots, p_n$. In this case, we have

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + (H, H)$$

$$= \frac{\partial H}{\partial x};$$

hence introducing a new variable p, such that

$$\Theta = H + p = \text{constant} = 0,$$

we have

$$\frac{dp}{dx} = -\frac{\partial H}{\partial x} = -\frac{\partial \Theta}{\partial x},$$

$$\frac{dx}{dx} = 1 \qquad = \frac{\partial \Theta}{\partial p},$$

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i} = \frac{\partial \Theta}{\partial p_i},$$

$$\frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i} = -\frac{\partial \Theta}{\partial x_i},$$

so that the canonical system may be replaced by the amplified system

$$\frac{dx}{\frac{\partial \Theta}{\partial p}} = \frac{dp}{-\frac{\partial \Theta}{\partial x}} = \frac{dx_i}{\frac{\partial \Theta}{\partial p_i}} = \frac{dp_i}{-\frac{\partial \Theta}{\partial x_i}},$$

for $i = 1, \ldots, n$. Now take a contact transformation changing the variables from $x, x_1, \ldots, x_n, p, p_1, \ldots, p_n$ to $X, X_1, \ldots, X_n, P, P_1, \ldots, P_n$: then denoting x, p, X, P by x_0, p_0, X_0, P_0 for convenience, we must have

$$(X_i, P_i) = 1,$$

for $i = 0, 1, \ldots, n$, and

$$(X_i, X_j) = 0, \quad (P_i, P_j) = 0, \quad (X_i, P_j) = 0,$$

for unequal values of i and j from the series $0, 1, \ldots, n$. As in the earlier case, if Φ be the transformed value of Θ, the amplified canonical system can be replaced by

$$\frac{dX}{\frac{\partial \Phi}{\partial P}} = \frac{dP}{-\frac{\partial \Phi}{\partial X}} = \frac{dX_i}{\frac{\partial \Phi}{\partial P_i}} = \frac{dP_i}{-\frac{\partial \Phi}{\partial X_i}},$$

for $i = 1, \ldots, m$. Let $\Phi = 0$ be resolved so as to express P in terms of $X, X_1, \ldots, X_n, P_1, \ldots, P_n$; and let the resolved form be

$$K(X, X_1, \ldots, X_n, P_1, \ldots, P_n) + P = 0.$$

Then

$$\frac{\partial \Phi}{\partial X} - \frac{\partial \Phi}{\partial P}\frac{\partial K}{\partial X} = 0,$$

$$\frac{\partial \Phi}{\partial X_i} - \frac{\partial \Phi}{\partial P}\frac{\partial K}{\partial X_i} = 0,$$

$$\frac{\partial \Phi}{\partial P_i} - \frac{\partial \Phi}{\partial P}\frac{\partial K}{\partial P_i} = 0,$$

for $i = 1, \ldots, n$. The preceding system can now, in its turn, be replaced by the equations

$$\frac{dX_i}{dX} = \frac{\dfrac{\partial \Phi}{\partial P_i}}{\dfrac{\partial \Phi}{\partial P}} = \frac{\partial K}{\partial P_i},$$

$$\frac{dP_i}{dX} = -\frac{\dfrac{\partial \Phi}{\partial X_i}}{\dfrac{\partial \Phi}{\partial P}} = -\frac{\partial K}{\partial X_i},$$

for $i = 1, \ldots, n$, which again are a canonical system: and we also have

$$\frac{dP}{dX} = -\frac{\partial K}{\partial X}.$$

It thus appears that a contact transformation, applied to a canonical system even when the function H involves the variable x, changes the system into another canonical system.

Conversely, any transformation between $x, x_1, \ldots, x_n, p_1, \ldots, p_n$ and $X, X_1, \ldots, X_n, P_1, \ldots, P_n$, which changes a canonical system

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i}, \qquad (i = 1, \ldots, n),$$

where H involves x as well as $x_1, \ldots, x_n, p_1, \ldots, p_n$, into another canonical system

$$\frac{dX_i}{dX} = \frac{\partial K}{\partial P_i}, \quad \frac{dP_i}{dX} = -\frac{\partial K}{\partial X_i}, \qquad (i = 1, \ldots, n),$$

is a contact transformation in an increased number of variables. To establish the result, we introduce a variable p such that

$$\Theta = p + H = 0;$$

hence

$$\frac{dp}{dx} = -\frac{\partial H}{\partial x}.$$

We then consider any transformation which changes the variables from $x, p, x_1, \ldots, x_n, p_1, \ldots, p_n$ to $X, P, X_1, \ldots, X_n, P_1, \ldots, P_n$; and, for convenience, we write x_0, p_0, X_0, P_0 for x, p, X, P respectively. Then

$$\begin{aligned}\frac{dX_i}{dx} &= \sum_{m=0}^{n} \frac{\partial X_i}{\partial x_m}\frac{dx_m}{dx} + \frac{\partial X_i}{\partial p_m}\frac{dp_m}{dx} \\ &= \sum_{m=0}^{n} \left(\frac{\partial X_i}{\partial x_m}\frac{\partial \Theta}{\partial p_m} - \frac{\partial X_i}{\partial p_m}\frac{\partial \Theta}{\partial x_m}\right) \\ &= (X_i, \Theta) \\ &= \sum_{m=0}^{n} \left\{(X_i, X_m)\frac{\partial \Theta}{\partial X_m} + (X_i, P_m)\frac{\partial \Theta}{\partial P_m}\right\},\end{aligned}$$

for $i = 0, 1, \ldots, m$; and similarly

$$\frac{dP_i}{dx} = \sum_{m=0}^{n} \left\{(P_i, X_m)\frac{\partial \Theta}{\partial X_m} + (P_i, P_m)\frac{\partial \Theta}{\partial P_m}\right\}.$$

Let $\Theta = 0$, after substitution has been made for p and in H, be resolved so as to express P_0 in terms of $X_0, X_1, \ldots, X_n, P_1, \ldots, P_n$; let the resolved equivalent be

$$P_0 + K = 0,$$

where K is a function of $X_0, X_1, \ldots, X_n, P_1, \ldots, P_n$; then

$$\frac{\partial \Theta}{\partial X_\mu} - \frac{\partial \Theta}{\partial P_0}\frac{\partial K}{\partial X_\mu} = 0,$$

for $\mu = 0, 1, \ldots, m$, and

$$\frac{\partial \Theta}{\partial P_\mu} - \frac{\partial \Theta}{\partial P_0}\frac{\partial K}{\partial P_\mu} = 0,$$

for $\mu = 1, \ldots, m$. Then, for $i = 1, \ldots, m$, we have

$$\begin{aligned}\frac{dX_i}{dX} &= \frac{dX_i}{dx} \div \frac{dX_0}{dx} \\ &= \frac{\sum_{m=0}^{n} \left\{(X_i, X_m)\frac{\partial \Theta}{\partial X_m} + (X_i, P_m)\frac{\partial \Theta}{\partial P_m}\right\}}{\sum_{m=0}^{n} \left\{(X_0, X_m)\frac{\partial \Theta}{\partial X_m} + (X_0, P_m)\frac{\partial \Theta}{\partial P_m}\right\}} \\ &= \frac{(X_i, P_0) + (X_i, X_0)\frac{\partial K}{\partial X} + \sum_{m=1}^{n} \left\{(X_i, X_m)\frac{\partial K}{\partial X_m} + (X_i, P_m)\frac{\partial K}{\partial P_m}\right\}}{(X_0, P_0) + (X_0, X_0)\frac{\partial K}{\partial X} + \sum_{m=1}^{n} \left\{(X_0, X_m)\frac{\partial K}{\partial X_m} + (X_0, P_m)\frac{\partial K}{\partial P_m}\right\}},\end{aligned}$$

and

$$\frac{dP_i}{dX}=\frac{(P_i, P_0)+(P_i, X_0)\frac{\partial K}{\partial X}+\sum_{m=1}^{n}\left\{(P_i, X_m)\frac{\partial K}{\partial X_m}+(P_i, P_m)\frac{\partial K}{\partial P_m}\right\}}{(X_0, P_0)+(X_0, X_0)\frac{\partial K}{\partial X}+\sum_{m=1}^{n}\left\{(X_0, X_m)\frac{\partial K}{\partial X_m}+(X_0, X_m)\frac{\partial K}{\partial P_m}\right\}}.$$

We are in quest of transformations which will make the new system canonical, and therefore the transformed equations should be of the form

$$\frac{dX_i}{dX}=\frac{\partial \bar{K}}{\partial P_i},\quad \frac{dP_i}{dX}=-\frac{\partial \bar{K}}{\partial X_i},$$

for $i=1, \ldots, m$. Hence, in order that the preceding equations may be of this type, we take

$$\bar{K}=K+a,$$

where a is a constant: and the conditions, necessary and sufficient for the purpose when the system is of the most general type, are

$$(X_0, P_0)=(X_1, P_1)=\ldots=(X_n, P_n),$$
$$(X_i, X_j)=0,\quad (P_i, P_j)=0,\quad (P_i, X_j)=0,$$

for unequal values of i and j from the series 0, 1, ..., m. These equations are characteristic of, and define, a contact transformation in the increased aggregate of variables. Moreover,

$$\begin{aligned}\frac{dP_0}{dX}&=\frac{dP_0}{dx}\div\frac{dX_0}{dx}\\&=-\frac{\partial K}{\partial X}.\end{aligned}$$

Consequently, *even when the function H in the general canonical system*

$$\frac{dx_i}{dx}=\frac{\partial H}{\partial p_i},\quad \frac{dp_i}{dx}=-\frac{\partial H}{\partial x_i},$$

involves the variable x, any contact transformation of the amplified system leads to a new canonical system; and every transformation, which transforms one canonical system of the most general type into another, is a contact transformation in the increased number of variables.

157. But it may be asked whether a contact transformation, involving only the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$, will transform one canonical system into another when H involves the variable x; it is easy to see that such a transformation is possible and that it

is, in fact, a special case of the contact transformation in the amplified number of variables. To verify this statement, we make

$$X_0 = x_0 = x, \quad P_0 = p_0 = p;$$

then, as (X_0, P_0) is unity, we have

$$(X_1, P_1) = 1, \ldots, (X_n, P_n) = 1.$$

Now the equations

$$(X_0, X_i) = 0, \quad (X_0, P_i) = 0, \quad (P_0, X_i) = 0, \quad (P_0, P_i) = 0,$$

give

$$\frac{\partial X_i}{\partial p_0} = 0, \quad \frac{\partial P_i}{\partial p_0} = 0, \quad \frac{\partial X_i}{\partial x_0} = 0, \quad \frac{\partial P_i}{\partial x_0} = 0;$$

and therefore the other equations are

$$(X_i, P_i) = \sum_{m=1}^{n} \left(\frac{\partial X_i}{\partial x_m} \frac{\partial P_i}{\partial p_m} - \frac{\partial X_i}{\partial p_m} \frac{\partial P_i}{\partial x_m} \right) = 1,$$

for $i = 1, \ldots, n$: also

$$(X_i, X_m) = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j} \frac{\partial X_m}{\partial p_j} - \frac{\partial X_i}{\partial p_j} \frac{\partial X_m}{\partial x_j} \right) = 0,$$

$$(X_i, P_m) = 0, \quad (P_i, P_m) = 0.$$

These equations clearly define a contact transformation between $X_1, \ldots, X_n, P_1, \ldots, P_n$ and $x_1, \ldots, x_n, p_1, \ldots, p_n$ alone: and they give a special case of the contact transformation in the amplified number of variables conserving the form of the canonical system.

158. Returning now to the canonical system of equations in the simpler form in which the quantity H does not involve the independent variable of the system explicitly, and denoting that variable by t, we have the system in the form

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i},$$

for $i = 1, \ldots, n$. Here, H is the total energy of the system and it remains constant throughout the motion; and, with the variables adopted for the construction of the canonical system, H is a function of $x_1, \ldots, x_n, p_1, \ldots, p_n$ alone.

We have seen that the most general form of infinitesimal contact transformation is given (§ 129) by

$$\delta z = \epsilon \zeta, \quad \delta x_i = \epsilon \xi_i, \quad \delta p_i = \epsilon \pi_i, \quad (i = 1, \ldots, n),$$

where

$$\xi_i = \frac{\partial U}{\partial p_i}, \qquad -\pi_i = \frac{\partial U}{\partial x_i} + p_i \frac{\partial U}{\partial z},$$

U denoting any arbitrary function of $x_1, \ldots, x_n, p_1, \ldots, p_n, z$. Let U be chosen so as not to involve z explicitly: then the equations become

$$\xi_i = \frac{\partial U}{\partial p_i}, \qquad -\pi_i = \frac{\partial U}{\partial x_i};$$

or, on writing $\epsilon = \delta t$, the equations of the infinitesimal contact transformation may be taken in the form

$$\frac{dx_i}{dt} = \frac{\partial U}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial U}{\partial x_i},$$

for $i = 1, \ldots, n$.

It therefore follows that the equations of the canonical system are the equations of an infinitesimal contact transformation, applied to the variables of the system and derived from the energy H of the originating system; and therefore the changes in the variables of the system can be regarded as the changes caused by the continued application of the infinitesimal contact transformation derived from the energy of the system. It is known, from the theory of groups of transformations, that the infinitesimal contact transformations determine uniquely the finite contact transformations of which they are the infinitesimal expression: moreover, what is the equivalent of this proposition for the present purpose, we have shewn that a finite contact transformation conserves the form of the canonical system. Hence, if we denote the values of the variables of the canonical system at any epoch t_0 by $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$, and their values at the epoch t by $X_1, \ldots, X_n, P_1, \ldots, P_n$, there is a contact transformation between $X_1, \ldots, X_n, P_1, \ldots, P_n$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$; and therefore the variables of the canonical system change continuously from their initial values under the continuous domination of the infinitesimal contact transformation determined by the energy.

This result includes the properties established by Bertrand* as regards canonical constants; for the equations defining these canonical constants are the equations expressing the contact trans-

* *Liouville*, t. XVII (1852), pp. 393 et seq.

formation between $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ and $x_1, \ldots, x_n, p_1, \ldots, p_n$, viz.

$$(\alpha_i, \beta_i) = 1,$$

$$(\alpha_i, \alpha_j) = 0, \quad (\alpha_i, \beta_j) = 0, \quad (\beta_i, \beta_j) = 0,$$

for i and $j = 1, \ldots n$, with unequal values of i and j.

This stage will mark the limit of our discussion of the canonical equations of theoretical dynamics. Their detailed properties constitute a subject, distinct in many of its developments from the theory of partial differential equations; for a fuller discussion, reference may be made to the authorities quoted at the beginning of the chapter.

CHAPTER XI.

Simultaneous Equations of the First Order.

The present chapter is a discussion of systems of simultaneous partial equations of the first order, the number of equations being the same as the number of dependent variables. The operation of integrating such equations is an inverse operation of class greater than unity in general, that is, it cannot generally be resolved into operations of the first order such as the integration of a number of ordinary equations each of the first order. General inverse operations of class greater than unity cannot be performed in finite terms, in the present state of analysis; those particular inverse operations, which can be resolved into operations of the first order, can however be performed, in the sense of the methods given in some of the preceding chapters. Naturally, the simultaneous partial equations involving several dependent variables, which can be integrated by these resoluble operations, are subject to corresponding limitations as regards generality of form: and consequently, owing to this somewhat particularised character, the theory of these equations is not so fully discussed here as has been the theory of equations in a single dependent variable.

The subject appears to have been considered first* by Jacobi: as presented in this form, further developments of Jacobi's theory are given by Natani†, and Zajaçrkowski‡.

A different presentation, and a completely different class of equations, occur in Hamburger's treatment§; cognate investigations have been effected by Königsberger||, who also deals with the existence-theorem for a set of equations, the number of which is equal to the number of dependent variables; and Hamburger's method has been extended by von Weber¶ to the case, when the number of equations is greater than the number of dependent variables.

* *Ges. Werke*, t. IV, pp. 3—15.

† *Die höhere Analysis*, pp. 339—341.

‡ *Grunert's Archiv*, t. LVI (1874), pp. 163—174.

§ *Crelle*, t. LXXXI (1876), pp. 243—280, *ib.*, t. XCIII (1882), pp. 188—214.

|| *Crelle*, t. CIX (1892), pp. 261—340; *Math. Ann.*, t. XLI (1893), pp. 260—285.

¶ *Crelle*, t. CXVIII (1897), pp. 123—157.

A class of equations, the number of which is an exact multiple of the number of dependent variables involved, has been considered by König*: an account of his method and of his results is given in the course of the chapter.

Hamburger has shewn that it is possible to apply the method to partial differential equations of the second order and of higher orders in one dependent variable and two independent variables. The subsidiary equations obtained as ancillary to the integration have substantial similarity with those obtained in the method, devised by Darboux for the integration of such equations and developed by Speckman and others. Accordingly, an account of Hamburger's application of his method to the integration of equations of the second order and of higher orders will not be considered in this chapter but will be deferred until the stage when such equations are being generally considered.

It may be added that some of the geometrical properties that can be associated with the simplest case, viz. when there are two dependent variables and two independent variables, are considered by Bäcklund†. As the processes of integration in this chapter are only applicable to limited classes of equations, these geometrical associations are not discussed in this connection: moreover, they belong more properly to the theory of equations of higher orders and, like the extension of Hamburger's method to such equations, they also will be deferred for consideration in connection with that theory.

159. The investigations in the preceding chapters have been concerned with the integration and the general theory of partial differential equations involving only a single dependent variable; no restriction was laid upon the number of independent variables; and, when more than a single equation occurred, the conditions necessary and sufficient to secure coexistence were obtained. It was shewn how to deduce, from a complete integral, other classes of integrals of various types: the aggregate of these classes was completely comprehensive for some types of equations and largely so (the exceptions being the special integrals) for the remainder. The construction of the complete integral was made to depend upon the integration, complete or incomplete, of a simultaneous system of ordinary equations, formed from the partial differential equations: the integration required depends, in practice, solely upon the possibility of actually effecting general inverse processes of the first order. Speaking broadly, we may say that the theory of partial differential equations of the first order in a single dependent variable can be considered a known theory.

* *Math. Ann.*, t. XXIII (1884), pp. 520—526.

† *Math. Ann.*, t. XVII (1880), pp. 285—328; *ib.* t. XIX (1882), pp. 387—422.

The problems which, in scale of difficulty, lie next to that of partial differential equations of the first order in a single dependent variable, are obtained, on the one hand, by increasing the number of dependent variables and keeping the equations still of the first order, and on the other hand, by taking equations of a higher order than the first, still in a single dependent variable.

As concerns partial differential equations of the second order (and of higher orders) in a single dependent variable, there is a considerable body of theory: moreover, the frequent occurrence of such differential equations, in subjects such as geometry and many of the developed branches of mathematical physics, has led to the discussion of detailed properties of particular equations which, once known, have pointed the way to further developments of the general theory.

But as concerns sets of partial differential equations of the first order in several dependent variables, when these sets are not the equivalent of a single equation of higher order in a single dependent variable, the amount of finished theory that has been obtained is comparatively slight. Thus, when the number of equations is equal to the number of dependent variables and when these equations have a special form which, among other limitations, is linear in the derivatives, it is known (§§ 9—14) that integrals of the equations do exist, satisfying assigned conditions of a given type. But when there is a question of constructing an integral in some form other than a multiple power-series as it occurs in the establishment of the existence-theorem, methods even only theoretically effective for the purpose have been devised solely for very restricted classes of systems of equations. Accordingly, before passing to equations of higher order in a single dependent variable, we shall deal with systems of equations of the first order in several variables, so as to indicate such general methods and results as have been obtained.

As in the early stages of the development of the theory of equations of the first order in a single dependent variable, some indications of results, which may be expected to hold frequently in simple cases though far from universally, can be obtained by proceeding from a set of integral equations. Let the independent variables be

$$x_1, \ldots, x_n,$$

and the dependent variables be

$$z_1, \ldots, z_m;$$

then the m dependent variables will be given by m integral equations. These equations may contain a number of arbitrary constants: let this number be N, and suppose that these are essential constants, so that they cannot be expressed by a number smaller than N.

When the first derivatives of these equations are formed, by differentiating with respect to the independent variables in turn, and are associated with the integral system, the total number of equations then possessed is $m(n+1)$. Suppose that all the arbitrary constants can be eliminated and that no peculiarities* occur during the processes of elimination; then the number of differential equations of the first order, emerging after the elimination, is $m(n+1)-N$. If these differential equations are to be conceived as capable of determining the m dependent variables, their number cannot be less than m; hence

$$m(n+1) - N \geqslant m,$$

that is,

$$N \leqslant mn,$$

thus giving an upper limit for N.

If $N = m(n+1-r)$, where $1 \leqslant r \leqslant n$, and if the same suppositions be made concerning the integral system in the passage to the differential equations, the number of emerging differential equations is rm.

But conversely, unless conditions equivalent to the reversibility of the preceding process are satisfied by a given system of simultaneous equations, it does not follow that their integral is of the assumed initial form: indeed, if the number of equations in the simultaneous system be greater than the number of dependent variables, it does not follow that the system possesses any integral at all. In order that the equations in such a system may coexist, conditions will have to be satisfied.

* Such, for instance, as occur when a partial differential equation in a single dependent variable is thus constructed from its general integral which may contain any number of arbitrary constants. The supposition, adopted in the face of such an instance, is enough to destroy any confidence as to more than possibility in the inferences that can be drawn.

KÖNIG'S COMPLETELY INTEGRABLE EQUATIONS.

160. The conditions just indicated can be set out in the case of certain classes of equations of simple types: one such class is discussed* by König. Let

$$p_{ij} = \frac{\partial z_i}{\partial x_j},$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n$: suppose that the system contains rm equations and that they can be resolved so as to express the derivatives of the m dependent variables with regard to one and the same set of r independent variables, in terms of the remaining quantities of the system. Let these r independent variables be $x_1, \ldots, x_r$; then the rm equations may be taken in the form

$$p_{ij} = f_{ij},$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, r$; the arguments of f_{ij} are the variables $x_1, \ldots, x_n, z_1, \ldots, z_m$, and also the derivatives $p_{\lambda\mu}$, where

$$\lambda = 1, \ldots, m, \quad \mu = r+1, \ldots, n.$$

Then König's theorem is as follows:—

When appropriate formal conditions are satisfied, the system of equations

$$p_{ij} = f_{ij}$$

possesses an integral equivalent $z_1, \ldots, z_m$ such that, when initial values $c_1, \ldots, c_r$ are assigned to $x_1, \ldots, x_r$ respectively, the functions $z_1, \ldots, z_m$ become functions of $x_{r+1}, \ldots, x_n$, which are regular functions in a certain domain and otherwise can be arbitrarily assigned.

Denote by S_j the aggregate of the m differential equations in which the second suffix is j; and let Σ_j denote the set made up of the aggregates

$$S_j, S_{j+1}, \ldots, S_r.$$

Consider the aggregate S_r: it contains no derivatives with regard to $x_1, \ldots, x_{r-1}$, which therefore may be regarded as parameters during processes of integration. It thus is a system of m equations involving the m dependent variables and the $n - r + 1$ independent variables $x_r, x_{r+1}, \ldots, x_n$; and it is resolved with respect to

$$\frac{\partial z_1}{\partial x_r}, \ldots, \frac{\partial z_m}{\partial x_r}.$$

* *Math. Ann.*, t. XXIII (1884), pp. 520—526.

It thus is of the class to which Madame Kowalevsky's proof of Cauchy's existence-theorem can be applied, if the formal conditions imposed for the theorem are satisfied. Assuming that these conditions are satisfied, the system possesses a set of integrals $z_1, \ldots, z_m$ which, when $x_r = c_r$, become assigned functions of $x_{r+1}, \ldots, x_n$, taken to be regular in a definite domain and otherwise arbitrarily assigned. The variables $x_1, \ldots, x_{r-1}$ are parametric throughout; the integrals of the aggregate S_r are a set of functions satisfying the conditions assigned in the theorem as stated, when we make $x_1, \ldots, x_{r-1}$ equal to $c_1, \ldots, c_{r-1}$.

Next, consider the aggregate of equations represented by S_{r-1}: it contains no derivatives with regard to $x_1, \ldots, x_{r-2}, x_r$, which therefore may be regarded as parameters during processes of integration. It is a system of m equations in the m dependent variables and the $n-r+1$ independent variables $x_{r-1}, x_{r+1}, \ldots, x_n$; and it is resolved with respect to

$$\frac{\partial z_1}{\partial x_{r-1}}, \ldots, \frac{\partial z_m}{\partial x_{r-1}}.$$

Applying Cauchy's existence-theorem to this system, on the assumption that the formal conditions are satisfied, we infer that the system possesses a set of integrals $z_1, \ldots, z_m$ which, when $x_{r-1} = c_{r-1}$, become the assigned functions of $x_{r+1}, \ldots, x_n$. The variables $x_1, \ldots, x_{r-2}, x_r$ are parametric throughout: the integrals of the aggregate S_{r-1} are a set of functions satisfying the conditions assigned in the theorem as stated, when we make $x_1, \ldots, x_{r-2}, x_r$ equal to $c_1, \ldots, c_{r-2}, c_r$.

And so on, for each of the aggregates in turn: in the case of each of them, we obtain a set of integrals which satisfy the initial conditions assigned in König's theorem as stated.

But though the integrals of the aggregate S_{r-1} satisfy the same initial conditions as the integrals of the aggregate S_r, it does not follow that they are the same functions of the variables; and, *à fortiori*, it does not follow that the integrals of the aggregate S_j are integrals of all the succeeding aggregates, that is, are integrals of the set Σ_j.

It may however happen that the integrals determined for the aggregate S_j are integrals for the set Σ_j. When this is the case for all values of j and, in particular, for $j = 1$, it is clear that the original system of equations possesses a set of integrals with the

properties stated in the theorem. In that case, the system is said to be *completely integrable*: it will therefore be necessary to determine the conditions which are necessary and sufficient to secure the complete integrability of the system.

161. We have assumed that the formal conditions, which justify the application of Cauchy's theorem, are satisfied. These conditions relate to the form of the functions f_{ij}, and require those functions to be regular within a domain, which belongs to the values $c_1, \ldots, c_r$ of $x_1, \ldots, x_r$ and to initial values assignable at will to $x_{r+1}, \ldots, x_n$: it is within such a domain that the functions, postulated in the initial conditions, are regular. Moreover, the determination of $z_1, \ldots, z_m$, for any aggregate S_j, as regular functions of the variables is unique under the assigned initial conditions; so that integrals are, or are not, possessed by the system in accordance with the initial conditions according as, for all values of j, the integrals of the aggregate S_j are, or are not, integrals of the set Σ_{j+1}. And, in particular, it is sufficient, in order to secure that the integrals of the aggregate S_j are the same as the integrals (if any) of the set Σ_{j+1}, that the integrals of S_j should satisfy the equations in the set Σ_{j+1}.

The conditions of complete integrability are therefore such that the integrals of S_j should satisfy the equations in Σ_{j+1}, for all values $1, \ldots, r-1$ of j. In order that the integrals of S_j may satisfy the equations in the set Σ_{j+1}, it is necessary and sufficient that, when they are substituted in those equations, they should make each of the equations an identity. Let $E=0$ be any one of the equations in the set Σ_{j+1}, thus made an identity: then we have

$$\frac{\partial E}{\partial x_1}=0, \ \ldots, \ \frac{\partial E}{\partial x_n}=0,$$

in virtue of those integrals and of the equations of the system. Now in S_j and Σ_{j+1} there are no derivatives with regard to $x_1, \ldots, x_{j-1}$; consequently, the conditions

$$\frac{\partial E}{\partial x_{j-1}}=0, \quad \frac{\partial E}{\partial x_{j-2}}=0, \ \ldots, \ \frac{\partial E}{\partial x_1}=0,$$

can be held over for consideration with the aggregates S_{j-1}, $S_{j-2}, \ldots, S_1$ respectively. Also, the conditions

$$\frac{\partial E}{\partial x_{j+1}}=0, \quad \frac{\partial E}{\partial x_{j+2}}=0, \ \ldots, \ \frac{\partial E}{\partial x_{n-1}}=0, \quad \frac{\partial E}{\partial x_n}=0,$$

can be regarded as satisfied, on the hypothesis that Σ_{j+1} possesses integrals; and therefore, at the stage of considering whether the integrals of S_j satisfy the equations in the set Σ_{j+1}, it is sufficient to take the conditions

$$\frac{\partial E}{\partial x_j} = 0,$$

where E is any equation of the set Σ_{j+1}. These conditions must be satisfied in virtue of the equations of the system: and they are

$$\frac{\partial (p_{i\alpha} - f_{i\alpha})}{\partial x_j} = 0,$$

for $i = 1, \ldots, n$, and $\alpha = j+1, \ldots, r$.

Now, when the integrals are substituted, we have

$$\begin{aligned} 0 = \frac{\partial (p_{i\alpha} - f_{i\alpha})}{\partial x_j} &= \frac{\partial p_{i\alpha}}{\partial x_j} - \frac{df_{i\alpha}}{dx_j} \\ &= \frac{\partial p_{ij}}{\partial x_\alpha} - \frac{df_{i\alpha}}{dx_j} \\ &= \frac{df_{ij}}{dx_\alpha} - \frac{df_{i\alpha}}{dx_j}. \end{aligned}$$

Here

$$\frac{df_{ij}}{dx_\alpha} = \frac{\partial f_{ij}}{\partial x_\alpha} + \sum_{\lambda=1}^{m} \frac{\partial f_{ij}}{\partial z_\lambda} p_{\lambda\alpha} + \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial p_{\rho\mu}}{\partial x_\alpha},$$

and

$$\begin{aligned} \frac{\partial p_{\rho\mu}}{\partial x_\alpha} &= \frac{\partial p_{\rho\alpha}}{\partial x_\mu} \\ &= \frac{df_{\rho\alpha}}{dx_\mu} \\ &= \frac{\partial f_{\rho\alpha}}{\partial x_\mu} + \sum_{\lambda=1}^{m} \frac{\partial f_{\rho\alpha}}{\partial z_\lambda} p_{\lambda\mu} + \sum_{\sigma=1}^{m} \sum_{\tau=r+1}^{n} \frac{\partial f_{\rho\alpha}}{\partial p_{\sigma\tau}} \frac{\partial p_{\sigma\tau}}{\partial x_\mu}, \end{aligned}$$

while

$$p_{\lambda\alpha} = f_{\lambda\alpha};$$

hence

$$\begin{aligned} \frac{df_{ij}}{dx_\alpha} = \frac{\partial f_{ij}}{\partial x_\alpha} &+ \sum_{\lambda=1}^{m} \frac{\partial f_{ij}}{\partial z_\lambda} f_{\lambda\alpha} + \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial x_\mu} \\ &+ \sum_{\lambda=1}^{m} \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial z_\lambda} p_{\lambda\mu} \\ &+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \sum_{\sigma=1}^{m} \sum_{\tau=r+1}^{n} \frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial p_{\sigma\tau}} \frac{\partial p_{\sigma\tau}}{\partial x_\mu}. \end{aligned}$$

Similarly,

$$\frac{df_{i\alpha}}{dx_j} = \frac{\partial f_{i\alpha}}{\partial x_j} + \sum_{\lambda=1}^{m} \frac{\partial f_{i\alpha}}{\partial z_\lambda} f_{\lambda j} + \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial x_\mu}$$

$$+ \sum_{\lambda=1}^{m} \sum_{\rho=1}^{n} \sum_{\mu=r+1}^{n} \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\mu j}}{\partial z_\lambda} p_{\lambda\mu}$$

$$+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \sum_{\sigma=1}^{m} \sum_{\tau=r+1}^{n} \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial p_{\sigma\tau}} \frac{\partial p_{\sigma\tau}}{\partial x_\mu}.$$

Consequently, as our condition is

$$\frac{df_{ij}}{dx_\alpha} - \frac{df_{i\alpha}}{dx_j} = 0,$$

we have

$$\frac{\partial f_{ij}}{\partial x_\alpha} - \frac{\partial f_{i\alpha}}{\partial x_j} + \sum_{\lambda=1}^{m} \left(f_{\lambda\alpha} \frac{\partial f_{ij}}{\partial z_\lambda} - f_{\lambda j} \frac{\partial f_{i\alpha}}{\partial z_\lambda} \right)$$

$$+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \left(\frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial x_\mu} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial x_\mu} \right)$$

$$+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \sum_{\lambda=1}^{m} \left\{ \left(\frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial z_\lambda} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial z_\lambda} \right) p_{\lambda\mu} \right\}$$

$$+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \sum_{\sigma=1}^{m} \sum_{\tau=r+1}^{n} \left\{ \left(\frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial p_{\sigma\tau}} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial p_{\sigma\tau}} \right) \frac{\partial p_{\sigma\tau}}{\partial x_\mu} \right\} = 0.$$

Now

$$\frac{\partial p_{\sigma\tau}}{\partial x_\mu} = \frac{\partial p_{\sigma\mu}}{\partial x_\tau},$$

while μ and $\tau = r+1, \ldots, n$ in the last summation; and the preceding condition is to be satisfied, either identically or in connection with the equations of the original series

$$p_{ij} = f_{ij},$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, r$. The quantities f_{ij} involve the quantities $p_{\lambda\mu}$, $p_{\sigma\tau}$, but they do not involve $\dfrac{\partial p_{\sigma\tau}}{\partial x_\mu}$; and no derivatives of the equations in the original system involve only derivatives of $p_{\sigma\tau}$ for $\tau = r+1, \ldots, n$, with respect to x_μ or only derivatives of $p_{\sigma\mu}$, for $\mu = r+1, \ldots, n$, with respect to x_τ, because such derivatives of the equations would introduce derivatives of $p_{\sigma\alpha}$, where α is less than $r+1$. The preceding condition therefore

must be satisfied without the assistance of the equations of the original system: and therefore the relations

$$\frac{\partial f_{ij}}{\partial x_\alpha} - \frac{\partial f_{i\alpha}}{\partial x_j} + \sum_{\lambda=1}^{m} \left(f_{\lambda\alpha} \frac{\partial f_{ij}}{\partial z_\lambda} - f_{\lambda j} \frac{\partial f_{i\alpha}}{\partial z_\lambda} \right)$$

$$+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \left(\frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial x_\mu} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial x_\mu} \right)$$

$$+ \sum_{\rho=1}^{m} \sum_{\mu=r+1}^{n} \sum_{\lambda=1}^{m} \left\{ \left(\frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial z_\lambda} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial z_\lambda} \right) p_{\lambda\mu} \right\} = 0,$$

and

$$\sum_{\rho=1}^{m} \left(\frac{\partial f_{ij}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho\alpha}}{\partial p_{\sigma\tau}} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\mu}} \frac{\partial f_{\rho j}}{\partial p_{\sigma\tau}} + \frac{\partial f_{ij}}{\partial p_{\rho\tau}} \frac{\partial f_{\rho\alpha}}{\partial p_{\sigma\mu}} - \frac{\partial f_{i\alpha}}{\partial p_{\rho\tau}} \frac{\partial f_{\rho j}}{\partial p_{\sigma\mu}} \right) = 0,$$

the latter relation arising from the combination of the coefficients of the equal quantities $\dfrac{\partial p_{\sigma\tau}}{\partial x_\mu}$ and $\dfrac{\partial p_{\sigma\mu}}{\partial x_\tau}$, must be satisfied identically.

The first of these identical relations holds for

$$\alpha = j+1, \ldots, r; \quad i = 1, \ldots, m.$$

The second of these identical relations holds for

$$\alpha = j+1, \ldots, r; \quad i \text{ and } \sigma = 1, \ldots, m;$$
$$\mu \text{ and } \tau = r+1, \ldots, n;$$

the subscripts μ and τ may have the same value, in which case there is a superfluous factor 2; or they may have different values, and then only the pair of values from the series $r+1, \ldots, n$ need be taken. Lastly, as α is greater than j, the preceding tale of relations holds for

$$j = 1, \ldots, r-1.$$

Note 1. There are three extreme cases.

(i) Let $r=1$: there is no possible value of j, and so there are no conditions. In this case, we have a system of m equations in m dependent variables: they are of the form

$$\frac{\partial z_1}{\partial x_1} = \phi_1, \quad \frac{\partial z_2}{\partial x_1} = \phi_2, \ldots, \frac{\partial z_m}{\partial x_1} = \phi_m,$$

where $\phi_1, \ldots, \phi_m$ involve all the variables and all the derivatives except those on the left-hand sides of the equations. It is clear that such equations can coexist without the necessity of submitting $\phi_1, \ldots, \phi_m$ to conditions.

(ii) Let $m=1$, so that there is a single dependent variable: the system of equations is

$$p_j = f_j(z, x_1, \ldots, x_n, p_{r+1}, \ldots, p_n),$$

for $j=1, \ldots, r$, and thus it belongs to the type of Jacobian systems.

The first set of conditions is

$$\frac{\partial f_j}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_j} + f_\alpha \frac{\partial f_j}{\partial z} - f_j \frac{\partial f_\alpha}{\partial z}$$
$$+ \sum_{\mu=r+1}^{n} \left\{ \frac{\partial f_j}{\partial p_\mu}\left(\frac{\partial f_\alpha}{\partial x_\mu} + p_\mu \frac{\partial f_\alpha}{\partial z}\right) - \frac{\partial f_\alpha}{\partial p_\mu}\left(\frac{\partial f_j}{\partial x_\mu} + p_\mu \frac{\partial f_j}{\partial z}\right)\right\} = 0,$$

for $\alpha = 1, \ldots, j-1$, and $j=1, \ldots, r$; and the second set of conditions, being

$$\frac{\partial(f_j, f_\alpha)}{\partial(p_\mu, p_\tau)} + \frac{\partial(f_j, f_\alpha)}{\partial(p_\tau, p_\mu)} = 0,$$

is evanescent. The aggregate of these conditions constitutes the aggregate for a complete Jacobian system.

(iii) Let $r=n$, so that there is a system of mn equations of the form

$$p_{ij} = f_{ij}(z_1, \ldots, z_m, x_1, \ldots, x_n),$$

for $i=1, \ldots, m$, and $j=1, \ldots, n$. As the functions f_{ij} involve no derivatives, the second set of conditions does not appear; and the first set becomes

$$\frac{\partial f_{ij}}{\partial x_\alpha} - \frac{\partial f_{i\alpha}}{\partial x_j} + \sum_{\lambda=1}^{m}\left(f_{\lambda\alpha}\frac{\partial f_{ij}}{\partial z_\lambda} - f_{\lambda j}\frac{\partial f_{i\alpha}}{\partial z_\lambda}\right) = 0,$$

for $i=1, \ldots, m$; $\alpha=1, \ldots, j-1$; and $j=1, \ldots, n$. This is Mayer's system of completely integrable equations*.

Note 2. The first set of conditions is sufficient to secure that

$$\frac{\partial z_i}{\partial x_j} = p_{ij}, \text{ for } i=1, \ldots, m, \text{ and } j=1, \ldots, n;$$

$$\frac{\partial p_{i\rho}}{\partial x_\beta} = \frac{\partial p_{i\beta}}{\partial x_\rho}, \text{ for } i=1, \ldots, m;\ \beta=1, \ldots, n;\ \rho=1, \ldots, r;$$

and the second set of conditions is sufficient to secure that

$$\frac{\partial p_{i\tau}}{\partial x_\mu} = \frac{\partial p_{i\mu}}{\partial x_\tau},$$

for $i=1, \ldots, m$; μ and $\tau = r+1, \ldots, n$.

* They are discussed fully in vol. I of this work, §§ 34—42.

162. Suppose that all the conditions for complete integrability are satisfied; then the theorem is established, according to which the completely integrable system of rm equations possesses a set of integrals $z_1, \ldots, z_m$; when $x_1, \ldots, x_r$ are made equal to $c_1, \ldots, c_r$, these integrals become functions of $x_{r+1}, \ldots, x_n$ which, subject solely to the condition of being regular within an assigned domain, may be arbitrarily assumed. As already stated, it is necessary, in addition to the conditions for complete integrability, that the quantities f_{ij} should be regular functions of their arguments within the domains considered.

Moreover, the argument shews that, in order to obtain the integrals required, it is necessary to integrate an aggregate of m equations. In practice, instead of beginning with a selected aggregate, it is convenient to effect Mayer's transformation adopted (§ 43, Note 1) for a complete Jacobian system in a single dependent variable. For this purpose, we write

$$\begin{aligned} x_1 &= y_1, \\ x_2 - a_2 &= (y_1 - a_1)\, y_2, \\ &\cdots\cdots\cdots\cdots \\ x_r - a_r &= (y_1 - a_1)\, y_r, \end{aligned}$$

leaving the other variables unaltered: then, taking

$$\frac{\partial z_i}{\partial y_j} = p'_{ij},$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, r$, we have an equivalent set of equations in the form

$$\begin{aligned} p'_{i1} &= f_{i1} + y_2 f_{i2} + \ldots + y_r f_{ir} = g_{i1}, \\ p'_{i\rho} &= (y_1 - a_1) f_{i\rho}, \end{aligned}$$

for $\rho = 2, \ldots, r$, and $i = 1, \ldots, n$. The first aggregate is

$$p'_{i1} = g_{i1},$$

for $i = 1, \ldots, n$: suppose it possible to obtain a set of integrals of this set of m equations such that, when $y_1 = a_1$, the integrals become functions of $x_{r+1}, \ldots, x_n$ only. These integrals satisfy the other equations, by the preceding argument: and as regards initial conditions for those equations, we see that

$$p'_{i\rho} = 0, \qquad (\rho = 2, \ldots, r),$$

when $y_1 = a_1$, that is, when $y_1 = a_1$, the integrals are not to involve $y_2, \ldots, y_n$—a set of conditions actually satisfied by the form of the functions assigned to the integrals when $y_1 = a_1$.

Hence the integration of a set of rm completely integrable equations of the resolved type indicated can be effected, if the integration of a set of only m equations of that resolved type can be effected.

Different kinds of Integrals: their Relations.

163. Before proceeding to the discussion of systematic attempts at the integration of simultaneous equations involving more than one dependent variable, it is worth noting that two kinds of integrals of simultaneous equations have been indicated. In one kind, the arbitrary element consists of arbitrary constants; in the other, it consists of arbitrary functions introduced through initial conditions. We have seen that, in the case of equations in a single dependent variable, it is possible to connect the two kinds of integrals organically; and it is natural to inquire whether the organic relation can be extended to the integrals of systems of equations involving several dependent variables.

It has appeared that, in a general sense, mn is the greatest number of arbitrary constants that can be eliminated from a system of m integral equations, in m dependent and n independent variables, so as to lead to m partial differential equations of the first order: and conversely, it is natural to expect, also in a general sense, that mn is the greatest number of arbitrary constants that can occur in the integral equivalent of the system of m partial differential equations. On the analogy of the case when $m=1$, such an integral involving this greatest number mn of arbitrary constants is called the *complete integral*. Let mn be denoted by μ, and suppose that the integral equations are resolved so as to express each of the dependent variables explicitly in terms of the independent variables and the arbitrary constants: the complete integral then* is of the form

$$\left.\begin{array}{c} z_1 = g_1(x_1, \ldots, x_n, a_1, \ldots, a_\mu) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_m = g_m(x_1, \ldots, x_n, a_1, \ldots, a_\mu) \end{array}\right\}.$$

To deduce other integrals, if possible, from the complete integral, let the customary Lagrangian method be adopted. The

* The following discussion is partly based upon that which is given by Königsberger, *Crelle*, t. CIX (1892), pp. 303 et seq.

quantities $a_1, \ldots, a_\mu$ are made variable, subject to the condition that the m differential equations are unaltered in form; and this condition will be satisfied if the expressions for the values of the derivatives p_{ij}, for $i=1, \ldots, m$, and $j=1, \ldots, n$, are unaltered. Hence we must have

$$\sum_{r=1}^{\mu} \frac{\partial g_i}{\partial a_r} \frac{\partial a_r}{\partial x_j} = 0,$$

for the specified values of i and j; and these are the equations, mn in number, to be satisfied by the mn quantities a. As these quantities $a_1, \ldots, a_\mu$ are variable, being functions of $x_1, \ldots, x_n$, we may take n of them as equivalent to $x_1, \ldots, x_n$; let these be $a_1, \ldots, a_n$, and let the remainder $a_{n+1}, \ldots, a_\mu$ be regarded as functions of $a_1, \ldots, a_n$. The preceding equations now become

$$\sum_{r=1}^{n} \left[\frac{\partial a_r}{\partial x_j} \left\{\frac{\partial g_i}{\partial a_r} + \sum_{s=n+1}^{\mu} \left(\frac{\partial g_i}{\partial a_s} \frac{\partial a_s}{\partial a_r}\right)\right\}\right] = 0,$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n$.

Out of this set of mn equations, let that aggregate be selected which is obtained by taking one value of i and all the n values of j; it is

$$\left.\begin{array}{l} P_{i1} \dfrac{\partial a_1}{\partial x_1} + P_{i2} \dfrac{\partial a_2}{\partial x_1} + \ldots + P_{in} \dfrac{\partial a_n}{\partial x_1} = 0 \\ P_{i1} \dfrac{\partial a_1}{\partial x_2} + P_{i2} \dfrac{\partial a_2}{\partial x_2} + \ldots + P_{in} \dfrac{\partial a_n}{\partial x_2} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{i1} \dfrac{\partial a_1}{\partial x_n} + P_{i2} \dfrac{\partial a_2}{\partial x_n} + \ldots + P_{in} \dfrac{\partial a_n}{\partial x_n} = 0 \end{array}\right\},$$

where

$$P_{ij} = \frac{\partial g_i}{\partial a_j} + \sum_{s=n+1}^{\mu} \left(\frac{\partial g_i}{\partial a_s} \frac{\partial a_s}{\partial a_j}\right).$$

The n equations, contained in this aggregate, are homogeneous and linear in the quantities $P_{i1}, P_{i2}, \ldots, P_{in}$; hence, either

$$P_{i1} = 0, \quad P_{i2} = 0, \ldots, P_{in} = 0,$$

or else

$$\frac{\partial (a_1, \ldots, a_n)}{\partial (x_1, \ldots, x_n)} = 0.$$

The latter result implies a functional relation between $a_1, \ldots, a_n$ without the intervention of quantities other than pure constants: let it be

$$a_n = F(a_1, \ldots, a_{n-1}).$$

Multiply the n equations in the aggregate by $dx_1, \ldots, dx_n$ respectively, and add: then

$$P_{i1} da_1 + P_{i2} da_2 + \ldots + P_{in} da_n = 0,$$

so that, in conjunction with $a_n = F(a_1, \ldots, a_{n-1})$, we must have

$$P_{ij} + P_{in} \frac{\partial F}{\partial a_j} = 0,$$

for $j = 1, \ldots, n-1$, and $i = 1, \ldots, m$. There are thus two ways of satisfying the selected aggregate of equations: either

$$P_{i1} = 0, \quad P_{i2} = 0, \ldots, P_{in} = 0,$$

or

$$\left.\begin{aligned} a_n &= F(a_1, \ldots, a_{n-1}) \\ 0 &= P_{ij} + P_{in} \frac{\partial F}{\partial a_j} \end{aligned}\right\},$$

where $j = 1, \ldots, n-1$ in the latter set. And then, taking all the aggregates which can thus be selected so that we use the full set of mn equations, we see that the two sets of equations in virtue of which the mn equations can be satisfied are (i) the system of equations

$$P_{ij} = 0,$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n$; and (ii) the system of equations

$$a_n = F(a_1, \ldots, a_{n-1}),$$

$$0 = P_{ij} + P_{in} \frac{\partial F}{\partial a_j},$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n$.

The alternatives must be considered separately.

164. I. For the first alternative, we have the system of mn equations

$$P_{ij} = 0,$$

where

$$P_{ij} = \frac{\partial g_i}{\partial a_j} + \sum_{s=n+1}^{\mu} \left(\frac{\partial g_i}{\partial a_s} \frac{\partial a_s}{\partial a_j} \right).$$

These equations contain the variables $x_1, \ldots, x_n$ and the mn parameters $a_1, \ldots, a_\mu$. When the variables are eliminated, $n(m-1)$ equations survive as the eliminant; and these are partial differential equations of the first order, in which $a_{n+1}, \ldots, a_\mu$ are $n(m-1)$ dependent variables and $a_1, \ldots, a_n$ are n independent variables. Now

$$n(m-1) > m,$$

except in three cases; in the first case of exception, $m=1$, and then we have the usual Jacobian theory of partial differential equations in a single dependent variable; in the second case of exception, $n=1$, and then we have a system of ordinary equations; the third case of exception is given by the equality of the numbers $n(m-1)$ and m, and then the only possible values are $n=2$, $m=2$.

Hence, when $m \geqslant 2$ and $n \geqslant 2$, the derivation of the values of $a_{n+1}, \ldots, a_\mu$ through $P_{ij}=0$ exacts the integration of a system of simultaneous partial equations in a number of dependent variables greater than the number in the original system of equations; and therefore the process of deducing new integrals from the complete integral by means of the equations $P_{ij}=0$ is of more elaborate extent than the process of integrating the original system. There is one exception to this result, and it is given by $m=2$, $n=2$; in that case, the two processes are of the same degree of difficulty.

II. For the second alternative, we have the $m(n-1)$ equations

$$P_{ij} + P_{in}\frac{\partial F}{\partial a_j} = 0,$$

for $j=1, \ldots, n-1$, and $i=1, \ldots, m$, together with the relation

$$a_n = F(a_1, \ldots, a_{n-1}).$$

As $a_1, \ldots, a_n$ are now connected by a relation, they are no longer eligible as a set of n independent quantities. We therefore choose some other set, say $a_{n+1}, a_1, \ldots, a_{n-1}$, as the n independent quantities equivalent to $x_1, \ldots, x_n$; and, instead of using the $m(n-1)$ equations associated with $a_n = F$, we return to the equations, which secure the absence of change of form in the derivatives and therefore conserve the form of the differential equations. The quantities $a_n, a_{n+2}, a_{n+3}, \ldots, a_\mu$ are now functions of $a_1, \ldots, a_{n-1}, a_{n+1}$: so that, writing

$$Q_{ij} = \frac{\partial g_i}{\partial a_j} + \frac{\partial g_i}{\partial a_n}\frac{\partial F}{\partial a_j} + \sum_{s=n+2}^{\mu} \frac{\partial g_i}{\partial a_s}\frac{\partial a_s}{\partial a_j},$$

for $j=1, \ldots, n-1$, and

$$Q_{ij'} = \frac{\partial g_i}{\partial a_{j'}} + \sum_{s=n+2}^{\mu} \frac{\partial g_i}{\partial a_s}\frac{\partial a_s}{\partial a_{j'}},$$

for $j' = n + 1$, we have these equations in the form

$$\sum_{j=1}^{n-1} Q_{ij} \frac{\partial a_j}{\partial x_k} + Q_{ij'} \frac{\partial a_{j'}}{\partial x_k} = 0,$$

for $i = 1, \ldots, m$, and $k = 1, \ldots, n$.

Proceeding as before, these equations can be satisfied in two different ways. We may have

$$Q_{ij} = 0,$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n-1, n+1$, being a system of mn equations; or we may have a relation of the form

$$a_{n+1} = G(a_1, \ldots, a_{n-1}),$$

coupled with the equations

$$Q_{ij} + Q_{ij'} \frac{\partial G}{\partial a_j} = 0,$$

where $j' = n + 1$; $j = 1, \ldots, n-1$; $i = 1, \ldots, m$.

In the former case, we eliminate $x_1, \ldots, x_n$ from the system of mn equations: when the elimination has been effected, there remain $n(m-1)$ differential equations in the n independent variables $a_1, \ldots, a_{n-1}, a_{n+1}$ and the $n(m-1)-1$ dependent variables $a_{n+2}, a_{n+3}, \ldots, a_\mu$, the value of a_n being already known. As the number of equations is greater than the number of dependent variables by unity, and as the equations are formally independent of one another, the system can coexist only if conditions are satisfied: it will not unconditionally determine the dependent variables.

In the latter case, the quantities $a_1, \ldots, a_{n-1}, a_{n+1}$, being connected by a relation, are not eligible as independent variables; we proceed to choose a set of quantities independent of one another as equivalent to $x_1, \ldots, x_n$, say $a_1, \ldots, a_{n-1}, a_{n+2}$, and construct the corresponding equations. The equations can be satisfied, as before, in two ways: either by a system of mn equations which, on the elimination of $x_1, \ldots, x_n$, give a set of $n(m-1)$ differential equations, involving $n(m-1)-2$ dependent variables and therefore not unconditionally determining those variables: or by a relation

$$a_{n+2} = H(a_1, \ldots, a_{n-1}),$$

with associated equations.

Pursuing the latter alternative to the extreme end, we have, at that end, a number of relations

$$a_s = F_s(a_1, \ldots, a_{n-1}),$$

for $s = n, n+1, \ldots, \mu$. Writing

$$T_{ij} = \frac{\partial g_i}{\partial a_j} + \sum_{s=n}^{\mu} \frac{\partial g_i}{\partial a_s}\frac{\partial F_s}{\partial a_j},$$

for $i = 1, \ldots, m$, and $j = 1, \ldots, n-1$, the equations to be associated with the $n(m-1)+1$ relations are

$$T_{ij} = 0.$$

These $m(n-1)$ equations, together with the $n(m-1)+1$ relations, make up $2mn - m - n + 1$ equations to be associated with the original m equations: they are more in number than the mn unknown quantities a; and therefore they cannot unconditionally be satisfied, for they are formally independent.

Summing up the various results we see that, *except in the single case $m = 2$, $n = 2$, it is not possible with unconditioned equations to use the Lagrangian process of variation of constants for the derivation of integrals from the complete integral without integrating a set of differential equations of more elaborate extent than the original system: when this set can be integrated, the integrals thus provided are, in general, the only integrals that can be derived from the complete integral. In the case of exception, the equations to be integrated are of the same order of difficulty as the original system.*

165. It is perhaps superfluous to point out that such integrals as can be obtained may belong to various classes. When they are derived through the $n(m-1)$ partial differential equations which determine the $n(m-1)$ quantities $a_{n+1}, \ldots, a_\mu$ in terms of a and b, there will be as many kinds of integrals of the original equations thus provided as there are different types of integrals of these new equations. Thus some integrals will involve arbitrary functional forms: these will correspond to one or other of the classes of general integrals. We know already that all the demands are satisfied by having $a_1, \ldots, a_\mu$ constant: we then have the complete integral. There may be equations of intermediate types in which some arbitrary functional forms occur and, at the same time, some of the varied arbitrary constants may survive merely as constants, by arising as trivial constant integrals satisfying the partial differential equations.

Ex. 1. Such systematic processes as have been devised for the integration of particular systems of simultaneous equations will be discussed almost immediately. Meanwhile, the special case can be illustrated by an example given* by Königsberger in the form

$$yp_1 - xq_2 = 0, \quad xq_1 - yp_2 = 0,$$

where z_1, z_2 are the dependent variables, x, y are the independent variables, and

$$\frac{\partial z_1}{\partial x} = p_1, \quad \frac{\partial z_1}{\partial y} = q_1, \quad \frac{\partial z_2}{\partial x} = p_2, \quad \frac{\partial z_2}{\partial y} = q_2.$$

The actual integration of the equations happens to be easy. We have

$$\begin{aligned} dz_1 &= p_1 dx + q_1 dy \\ &= \frac{x}{y} q_2 dx + \frac{y}{x} p_2 dy; \end{aligned}$$

the right hand side must be a perfect differential, so that

$$\frac{\partial}{\partial x}\left(\frac{y}{x} p_2\right) = \frac{\partial}{\partial y}\left(\frac{x}{y} q_2\right),$$

that is,

$$\frac{1}{x}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z_2}{\partial x}\right) = \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z_2}{\partial y}\right),$$

and therefore

$$z_2 = F(x^2 + y^2) - G(x^2 - y^2),$$

where f and g are arbitrary functions of their arguments. It is then easy to deduce, by means of the values of p_1 and q_1, that the value of z_1 is

$$z_1 = F(x^2 + y^2) + G(x^2 - y^2).$$

Now a set of integrals containing four arbitrary constants (and therefore constituting a complete integral) is

$$\left.\begin{aligned} z_1 &= \alpha + ax^2 + by^2 \\ z_2 &= \beta + bx^2 + ay^2 \end{aligned}\right\},$$

where a, b, α, β are arbitrary constants. To deduce other integrals if possible, we make α, β, a, b variable quantities, being functions of x and y; and, in the first place, we make a and b equivalent to x and y, and can therefore use them as independent variables. Then if

$$A_1 = \frac{\partial \alpha}{\partial a} + x^2, \quad A_2 = \frac{\partial \beta}{\partial a} + y^2,$$

$$B_1 = \frac{\partial \alpha}{\partial b} + y^2, \quad B_2 = \frac{\partial \beta}{\partial b} + x^2,$$

the values of p_1, q_1, p_2, q_2 will be unaltered if

$$A_1 \frac{\partial a}{\partial x} + B_1 \frac{\partial b}{\partial x} = 0, \quad A_2 \frac{\partial a}{\partial x} + B_2 \frac{\partial b}{\partial x} = 0,$$

$$A_1 \frac{\partial a}{\partial y} + B_1 \frac{\partial b}{\partial y} = 0, \quad A_2 \frac{\partial a}{\partial y} + B_2 \frac{\partial b}{\partial y} = 0,$$

* *Crelle*, t. CIX, p. 319: the integral selected is simpler than Königsberger's.

which are equations for the determination of a and b. They can be satisfied in various ways.

(i) They are satisfied if

$$\frac{\partial a}{\partial x}=0,\quad \frac{\partial a}{\partial y}=0,\quad \frac{\partial b}{\partial x}=0,\quad \frac{\partial b}{\partial y}=0:$$

then a, b (and therefore α and β, which are functions of a and b) are constants. We return to the complete integral.

(ii) They are satisfied if

$$A_1=0,\quad B_1=0,\quad A_2=0,\quad B_2=0.$$

From $A_1=0$, $B_2=0$, we have

$$\frac{\partial \alpha}{\partial a}=\frac{\partial \beta}{\partial b};$$

and from $B_1=0$, $A_2=0$, we have

$$\frac{\partial \alpha}{\partial b}=\frac{\partial \beta}{\partial a}.$$

Hence

$$\frac{\partial^2 \alpha}{\partial a^2}-\frac{\partial^2 \alpha}{\partial b^2}=0,$$

so that

$$\alpha=f(a+b)+g(a-b),$$

where f and g are arbitrary functions of their arguments; and then it easily follows that

$$\beta=f(a+b)-g(a-b).$$

Also from $A_1=0$, $B_1=0$, we now have

$$-x^2=\frac{\partial \alpha}{\partial a}=f'(a+b)+g'(a-b),$$

$$-y^2=\frac{\partial \alpha}{\partial b}=f'(a+b)-g'(a-b),$$

so that

$$-(x^2+y^2)=2f'(a+b),\quad -(x^2-y^2)=2g'(a-b).$$

Hence $a+b$ is an arbitrary function of x^2+y^2, and $a-b$ is an arbitrary function of x^2-y^2, say

$$a+b=\theta(x^2+y^2),\quad a-b=\phi(x^2-y^2).$$

Thus

$$\begin{aligned}
z_1&=\alpha+ax^2+by^2\\
&=f\{\theta(x^2+y^2)\}+g\{\phi(x^2-y^2)\}+\tfrac{1}{2}(x^2+y^2)\,\theta(x^2+y^2)+\tfrac{1}{2}(x^2-y^2)\,\phi(x^2-y^2)\\
&=\Theta(x^2+y^2)+\Phi(x^2-y^2),\\
z_2&=\beta+bx^2+ay^2\\
&=f\{\theta(x^2+y^2)\}-g\{\phi(x^2-y^2)\}+\tfrac{1}{2}(x^2+y^2)\,\theta(x^2+y^2)-\tfrac{1}{2}(x^2-y^2)\,\phi(x^2-y^2)\\
&=\Theta(x^2+y^2)-\Phi(x^2-y^2),
\end{aligned}$$

where Θ and Φ are arbitrary functions of their arguments.

(iii) The equations necessary in order that the forms of p_1, q_1, p_2, q_2 may be conserved can be satisfied by

$$\frac{\partial(a, b)}{\partial(x, y)}=0,$$

that is, by

$$b=h(a),$$

where h is any function of its argument, together with

$$A_1+B_1\frac{db}{da}=0,$$

$$A_2+B_2\frac{db}{da}=0.$$

As a and b are not now independent, we return to the equations; and we make α and a the independent variables.

Proceeding as before, we find that the alternative to the integral obtained in the last case is

$$\alpha=k(a),$$

where k is any function of its argument, with associated equations.

To deal with the latter alternative, we again return to the initial equations; and we make β and a the independent variables. A similar process leads to the result that the alternative to the integral already obtained is given by

$$\beta=l(a),$$

with the associated equations.

We thus have

$$b=h(a),\quad \alpha=k(a),\quad \beta=l(a);$$

and the associated equations are

$$\left.\begin{array}{r} k'(a)+x^2+y^2h'(a)=0 \\ l'(a)+x^2h'(a)+y^2=0 \end{array}\right\}.$$

These can coexist only in two cases. In the first case,

$$h'(a)=1,$$

and

$$k'(a)=l'(a)=-(x^2+y^2);$$

the corresponding integrals are

$$z_1=\Omega(x^2+y^2),\quad z_2=\Omega(x^2+y^2),$$

being particular forms of the integrals already obtained. In the second case,

$$h'(a)=-1,$$

$$k'(a)=-l'(a)=-(x^2-y^2);$$

the corresponding integrals are

$$z_1=\Psi(x^2-y^2),\quad z_2=-\Psi(x^2-y^2),$$

being particular forms of the integrals already obtained.

Hence the most general integrals that can thus be derived from the complete integrals are

$$z_1=\Theta(x^2+y^2)+\Phi(x^2-y^2),$$
$$z_2=\Theta(x^2+y^2)-\Phi(x^2-y^2).$$

Ex. 2. Generalise similarly the integrals

$$z_1=\quad a+(a+\gamma)x^2+(a-\gamma)y^2+\beta(x^2+y^2)^2,$$
$$z_2=-a+(a-\gamma)x^2+(a+\gamma)y^2+\beta(x^2+y^2)^2,$$

of the same equations

$$yp_1-xq_2=0,\quad xq_1-yp_2=0.$$

(Königsberger.)

Ex. 3. Construct the differential equations of the first order satisfied by

$$z_1=a^2x+by+c^2xy+kx^2,$$
$$z_2=ay+b^2x+cx+k^2y,$$

where a, b, c, k are arbitrary constants.

Generalise the integrals so as to deduce others from this complete integral. In particular, shew that another integral is given by keeping c and k constant, and by making a and b functions of x and y such that

$$\left.\begin{array}{r} 4abx^2=y^2 \\ 8a^3-\alpha a^{\frac{3}{2}}+\dfrac{y^3}{x^3}=0 \end{array}\right\},$$

where α is an arbitrary constant. (Königsberger.)

HAMBURGER'S LINEAR EQUATIONS.

166. It has appeared, from the discussion of the classes of simultaneous equations already considered, that the construction of a system of integrals can be made to depend on the construction of the integrals of a set of simultaneous equations the number of which is the same as the number of dependent variables involved. The only indication of any systematic method of obtaining the integrals is furnished in the proof of the existence theorem; they are obtained in the form of converging power-series in the independent variables. What is usually desired for the purpose is an expression for the integrals in some form more compact than multiple power-series.

A method, which has been found effective for a limited number of classes of equations, has been devised* by Hamburger.

* *Crelle*, t. LXXXI (1876), pp. 243—280, the equations being linear in the derivatives of the dependent variables; *ib.*, t. XCIII (1882), pp. 188—214, the equations not being necessarily linear in those derivatives.

See also a paper by Königsberger, *Math. Ann.*, t. XLI (1893), pp. 260—285.

Denoting the independent variables by $x_1, \ldots, x_n$, the dependent variables by $z_1, \ldots, z_m$, and the derivatives of the dependent variables by p_{ij} as before, where

$$p_{ij} = \frac{\partial z_i}{\partial x_j},$$

we first consider a set of m algebraically independent equations which are linear in the derivatives. We also assume that they can be resolved so as to express the derivatives of the m dependent variables with regard to one and the same independent variable: let this variable be x_1, so that the system may be taken in the form

$$p_{i1} = \pi_i + \sum_{j=1}^{m} \sum_{s=2}^{n} p_{js}\theta_{jsi},$$

for $i = 1, \ldots, m$: the quantities π_i and θ_{jsi}, for all values of i, j, s, are functions of the variables $z_1, \ldots, z_m, x_1, \ldots, x_n$. Multiplying the equations by $\lambda_1, \ldots, \lambda_m$, a set of provisionally indeterminate multipliers, and adding, we have

$$\sum_{i=1}^{m} \lambda_i \pi_i - \sum_{i=1}^{m} \lambda_i p_{i1} + \sum_{j=1}^{m} \sum_{s=2}^{n} p_{js} \left(\sum_{i=1}^{m} \lambda_i \theta_{jsi} \right) = 0.$$

The values of the derivatives must be such that the differential relations

$$dz_i - p_{i1}dx_1 - p_{i2}dx_2 - \ldots - p_{in}dx_n = 0,$$

for $i = 1, \ldots, m$, must be satisfied: consequently, the relation

$$\sum_{i=1}^{m} \lambda_i dz_i - dx_1 \sum_{i=1}^{n} \lambda_i p_{i1} - \sum_{j=1}^{m} \sum_{s=2}^{n} p_{js}\lambda_j dx_s = 0$$

also must be satisfied. Comparing this differential relation with the preceding composite equation and having regard to the ordinary subsidiary equations constructed in connection with a single partial differential equation, we construct the set

$$\frac{\sum_{i=1}^{m} \lambda_i dz_i}{\sum_{i=1}^{m} \lambda_i \pi_i} = dx_1 = \frac{-\lambda_j dx_s}{\sum_{i=1}^{m} \lambda_i \theta_{jsi}}$$

of ordinary equations, to hold for all values of j and s.

In this system of ordinary equations, let

$$\frac{dx_s}{dx_1} = \mu_s,$$

so that

$$\sum_{i=1}^{m} \lambda_i \theta_{jsi} + \lambda_j \mu_s = 0,$$

for all values of j and s. Selecting those of the equations given by one value of s and the m values of j, we can eliminate $\lambda_1, \ldots, \lambda_m$ determinantally; and we obtain an equation satisfied by μ_s. For each root μ_s of this equation we obtain a set of ratios $\lambda_1 : \lambda_2 : \ldots : \lambda_m$, the values of these ratios depending upon the coefficients θ_{jsi}, where s is the same for all the coefficients in the tableau.

If there be more than one value of s, say if σ be another value, then certain combinations of the coefficients $\theta_{j\sigma i}$ must be the same as those combinations of the coefficients θ_{jsi}, in order to secure the same values for the ratios $\lambda_1 : \lambda_2 : \ldots : \lambda_m$. This requirement would impose conditions upon the equations which would not, in general, be satisfied: though it might be of interest to construct classes of equations for which the appropriate conditions are satisfied, we shall assume that our equations are not thus conditioned. Accordingly, there will be only one value of s, say $s = 2$.

167. Thus for the present purpose, we restrict ourselves to the consideration of equations in two independent variables, which will be denoted by x and y. Writing

$$p_i = \frac{\partial z_i}{\partial x}, \quad q_i = \frac{\partial z_i}{\partial y},$$

we may take the equations in the form

$$p_i = \pi_i + \sum_{s=1}^{m} a_{is} q_s,$$

for $i = 1, \ldots, m$. In connection with the differential relations

$$dz_i - p_i dx - q_i dy = 0,$$

we form the set of ordinary equations

$$\frac{\sum\limits_{i=1}^{m} \lambda_i dz_i}{\sum\limits_{i=1}^{m} \lambda_i \pi_i} = dx = -\frac{\lambda_s dy}{\sum\limits_{i=1}^{m} \lambda_i a_{is}},$$

for $s = 1, \ldots, m$. Take

$$dy = \mu dx,$$

so that

$$\sum_{i=1}^{m} \lambda_i a_{is} + \mu\lambda_s = 0,$$

for $s = 1, \ldots, m$, and eliminate the quantities $\lambda_1, \ldots, \lambda_m$ among these equations: then μ satisfies the equation

$$\Theta = \begin{vmatrix} a_{11}+\mu, & a_{21} \;, & a_{31} \;, & \ldots, & a_{m1} \\ a_{12} \;, & a_{22}+\mu, & a_{32} \;, & \ldots, & a_{m2} \\ a_{13} \;, & a_{23} \;, & a_{33}+\mu, & \ldots, & a_{m3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1m} \;, & a_{2m} \;, & a_{3m} \;, & \ldots, & a_{mm}+\mu \end{vmatrix} = 0,$$

which is of order m.

When μ is a simple root of this equation $\Theta = 0$, the preceding equations determine a unique set of values of $\lambda_1 : \lambda_2 : \ldots : \lambda_m$; then, when we take

$$\frac{\lambda_i}{\sum\limits_{i=1}^{m} \lambda_i \pi_i} = \alpha_i, \qquad (i = 1, \ldots, m),$$

the quantities $\alpha_1, \ldots, \alpha_m$ are uniquely determinate. Hence we have

$$\left.\begin{array}{r} \alpha_1 dz_1 + \ldots + \alpha_m dz_m = dx \\ dy = \mu dx \end{array}\right\};$$

that is, there are two linear equations for each simple root of $\Theta = 0$.

Next, let μ be a multiple root of the equation $\Theta = 0$, and suppose that it occurs θ times; then t of the m equations that lead to $\Theta = 0$ are deducible from the remainder, where

$$1 \leqslant t \leqslant \theta \leqslant m.$$

The ratios $\lambda_1 : \lambda_2 : \ldots : \lambda_m$ are no longer determinate: the quantities $\lambda_1, \lambda_2, \ldots, \lambda_m$ are expressible in terms of t arbitrary quantities $\kappa_1, \ldots, \kappa_t$ by equations of the form

$$\lambda_r = \gamma_{r1}\kappa_1 + \ldots + \gamma_{rt}\kappa_t,$$

for $r = 1, \ldots, m$, the quantities $\gamma_{r1}, \ldots, \gamma_{rt}$ being determinate. The equation

$$\sum_{r=1}^{m} \lambda_r dz_r = \left(\sum_{r=1}^{m} \lambda_r \pi_r\right) dx$$

now becomes

$$\sum_{r=1}^{m} \sum_{i=1}^{t} (\gamma_{ri} dz_r)\, \kappa_i = \sum_{r=1}^{m} \sum_{i=1}^{t} (\gamma_{ri} \pi_r)\, \kappa_i dx;$$

and therefore, as the quantities $\kappa_1, \ldots, \kappa_t$ are arbitrary, we have the $t+1$ linear equations

$$dy = \mu dx,$$

$$\sum_{r=1}^{m} \gamma_{ri} dz_r = \left(\sum_{r=1}^{m} \gamma_{ri} \pi_r \right) dx,$$

for $i = 1, \ldots, t$.

168. The extreme case among multiple roots is that in which

$$\theta = m:$$

the quantity Θ is then the mth power of a linear factor: the coefficients $a_{11}, a_{22}, \ldots, a_{mm}$ have a common value, say α; and all the coefficients a_{ij}, for unequal values of i and j, are zero. The equations are

$$p_i = \pi_i + \alpha q_i;$$

and the associated subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-\alpha} = \frac{dz_1}{\pi_1} = \ldots = \frac{dz_m}{\pi_m}.$$

The equations in this extreme case were considered by Jacobi*, independently of the preceding mode of origin: he approached them from the stage of integral relations with any number of dependent variables and any number of independent variables, and the particular set just given are his equations when there are two independent variables.

When μ is a multiple root of order θ, the most important case is that in which $t = \theta$; the conditions† are connected with the elementary divisors (or elementary factors) of the determinant Θ. They will not be set out in detail because the method devised by Hamburger will be sufficiently illustrated for the equations, to which it can be applied, by a discussion of the simplest cases.

The integral of the Jacobian set can easily be obtained. Let $u_1, \ldots, u_{m+1}$ be a complete set of independent integrals of the $m+1$ ordinary equations

$$\frac{dx}{1} = \frac{dy}{-\alpha} = \frac{dz_1}{\pi_1} = \ldots = \frac{dz_m}{\pi_m},$$

* *Ges. Werke*, t. IV, p. 7; *Crelle*, t. II (1827), p. 321.

† For this theory, see Weierstrass, *Ges. Werke*, t. II, pp. 19—44. Other references are given in vol. IV of the present work, p. 42, foot-note.

which are subsidiary to the system

$$p_i = \pi_i + \alpha q_i,$$

for $i = 1, \ldots, m$: then the differential relation

$$\frac{\partial u_r}{\partial x} dx + \frac{\partial u_r}{\partial y} dy + \frac{\partial u_r}{\partial z_1} dz_1 + \ldots + \frac{\partial u_r}{\partial z_m} dz_m = 0$$

is satisfied in virtue of the above set of ordinary equations, and consequently

$$\frac{\partial u_r}{\partial x} - \alpha \frac{\partial u_r}{\partial y} + \pi_1 \frac{\partial u_r}{\partial z_1} + \ldots + \pi_m \frac{\partial u_r}{\partial z_m} = 0,$$

holding for $r = 1, \ldots, m+1$.

Now take any equation

$$\psi_i (u_1, \ldots, u_{m+1}) = 0,$$

regarded as one of a system of m equations to express m dependent variables $z_1, \ldots, z_m$, in terms of x and y. We have

$$\sum_{r=1}^{m+1} \frac{\partial \psi_i}{\partial u_r} \left\{ \frac{\partial u_r}{\partial x} + p_1 \frac{\partial u_r}{\partial z_1} + \ldots + p_m \frac{\partial u_r}{\partial z_m} \right\} = 0,$$

$$\sum_{r=1}^{m+1} \frac{\partial \psi_i}{\partial u_r} \left\{ \frac{\partial u_r}{\partial y} + q_1 \frac{\partial u_r}{\partial z_1} + \ldots + q_m \frac{\partial u_r}{\partial z_m} \right\} = 0;$$

multiplying the latter by α, subtracting from the former, and using the partial equation satisfied by the quantity u_r, we have

$$\sum_{r=1}^{m+1} \frac{\partial \psi_i}{\partial u_r} \left\{ (p_1 - \pi_1 - \alpha q_1) \frac{\partial u_r}{\partial z_1} + \ldots + (p_m - \pi_m - \alpha q_m) \frac{\partial u_r}{\partial z_m} \right\} = 0.$$

Now, when we write

$$\frac{d\psi_i}{dz_s} = \sum_{r=1}^{m+1} \frac{\partial \psi_i}{\partial u_r} \frac{\partial u_r}{\partial z_s},$$

so that $\dfrac{d\psi_i}{dz_s}$ is the complete derivative of ψ_i with regard to z_s, this equation is

$$(p_1 - \pi_1 - \alpha q_1) \frac{d\psi_i}{dz_1} + \ldots + (p_m - \pi_m - \alpha q_m) \frac{d\psi_i}{dz_m} = 0.$$

Accordingly, take m equations

$$\psi_i (u_1, \ldots, u_{m+1}) = 0,$$

for $i = 1, \ldots, m$, independent of one another and determining* the m quantities $z_1, \ldots, z_m$; then the equation

$$(p_1 - \pi_1 - \alpha q_1)\frac{d\psi_i}{dz_1} + \ldots + (p_m - \pi_m - \alpha q_m)\frac{d\psi_i}{dz_m} = 0$$

is satisfied for each of the values $i = 1, \ldots, m$. Also as the equations are independent of one another and determine the quantities $z_1, \ldots, z_m$, the determinant of the quantities $\frac{d\psi_i}{dz_j}$, for i and $j = 1, \ldots, m$, does not vanish: consequently

$$p_j - \pi_i - \alpha q_j = 0,$$

for $j = 1, \ldots, m$.

Hence the m equations

$$\psi_i(u_1, \ldots, u_{m+1}) = 0,$$

for $i = 1, \ldots, m$, give integrals of the Jacobian set; and this is true however arbitrary the functions ψ may be, provided only that they are independent of one another.

It can be proved, as in the case of a single dependent variable (§§ 31—33), that the preceding integrals (when $\psi_1, \ldots, \psi_m$ are kept as arbitrary as possible) include all integrals that are not of the type called *special* in the simpler case. Such special integrals could occur in connection with zeros, with branch-values, and with singularities of the quantities $\alpha, \pi_1, \ldots, \pi_m$.

* There cannot be an identical relation between $u_1, \ldots, u_{m+1}$ which leads to an equation independent of $z_1, \ldots, z_m$. If such an equation were possible in a form

$$\theta(u_1, \ldots, u_{m+1}) = 0,$$

then as

$$\sum_{r=1}^{m+1}\frac{\partial\theta}{\partial u_r}\frac{\partial u_r}{\partial x} + p_1\frac{d\theta}{dz_1} + \ldots + p_m\frac{d\theta}{dz_m} = 0,$$

$$\sum_{r=1}^{m+1}\frac{\partial\theta}{\partial u_r}\frac{\partial u_r}{\partial y} + q_1\frac{d\theta}{dz_1} + \ldots + q_m\frac{d\theta}{dz_m} = 0,$$

we should then have

$$\sum_{r=1}^{m+1}\frac{\partial\theta}{\partial u_r}\frac{\partial u_r}{\partial x} = 0, \quad \sum_{r=1}^{m+1}\frac{\partial\theta}{\partial u_r}\frac{\partial u_r}{\partial y} = 0.$$

Also

$$0 = \frac{d\theta}{dz_i} = \sum_{r=1}^{m+1}\frac{\partial\theta}{\partial u_r}\frac{\partial u_r}{\partial z_i} = 0,$$

for $i = 1, \ldots, m$: consequently, if some of the derivatives $\frac{\partial\theta}{\partial u_1}, \ldots, \frac{\partial\theta}{\partial u_{m+1}}$ do not vanish,

$$\frac{\partial(u_1, \ldots, u_{m+1})}{\partial(z_1, \ldots, z_m, x)} = 0, \quad \frac{\partial(u_1, \ldots, u_{m+1})}{\partial(z_1, \ldots, z_m, y)} = 0,$$

which are not true because $u_1, \ldots, u_{m+1}$ are a set of independent integrals of the subsidiary system.

Ex. 1. Integrate the equations

$$\left.\begin{aligned} p_1+q_1&=\frac{1-z_1(x+y)}{z_1-z_2}\\ p_2+q_2&=\frac{1-z_2(x+y)}{z_2-z_1}\end{aligned}\right\}.$$

The subsidiary equations, taken according to the preceding explanations, are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz_1}{\dfrac{1-z_1(x+y)}{z_1-z_2}}=\frac{dz_2}{\dfrac{1-z_2(x+y)}{z_2-z_1}}.$$

Three independent integrals of this system are easily obtained in the form

$$\begin{aligned} u_1&=x-y=a,\\ u_2&=z_1+z_2+xy=b,\\ u_3&=z_1z_2+x=c,\end{aligned}$$

where a, b, c are arbitrary constants; and therefore a set of integrals of the partial differential equations is given by the equations

$$\left.\begin{aligned} z_1+z_2+xy&=\phi(x-y)\\ z_1z_2+x&=\psi(x-y)\end{aligned}\right\},$$

where ϕ and ψ are arbitrary functions. These equations constitute a general integral.

Ex. 2. Integrate the equations

$$\left.\begin{aligned} p_1-q_1&=-\tfrac{1}{2}z_2\left(\frac{y-x}{z_2^2}+\frac{y+x}{y^2}\right)\\ p_2-q_2&=\tfrac{1}{2}\frac{z_2^2}{z_1}\left(\frac{y+x}{y^2}-\frac{y-x}{z_2^2}\right)\end{aligned}\right\},$$

obtaining a general integral in the form

$$\frac{z_1}{z_2}+\frac{x}{y}=f(x+y),\quad z_1z_2+xy=g(x+y),$$

where f and g are arbitrary functions.

State also a complete integral: and from it deduce the general integral.

Hamburger's Equations in Two Dependent Variables.

169. The simplest case of the general problem occurs when there are two dependent variables and two independent variables, so that the equations may be taken to be

$$\left.\begin{aligned} p_1&=\gamma_1+a_1q_1+b_1q_2\\ p_2&=\gamma_2+a_2q_1+b_2q_2\end{aligned}\right\}.$$

The subsidiary ordinary equations are

$$\frac{\lambda_1dz_1+\lambda_2dz_2}{\lambda_1\gamma_1+\lambda_2\gamma_2}=dx=\frac{-\lambda_1dy}{\lambda_1a_1+\lambda_2a_2}=\frac{-\lambda_2dy}{\lambda_1b_1+\lambda_2b_2}.$$

Hence, if

$$dy = \mu dx,$$

we have

$$\lambda_1 (a_1 + \mu) + \lambda_2 a_2 = 0,$$

$$\lambda_1 b_1 + \lambda_2 (b_2 + \mu) = 0:$$

consequently

$$\mu^2 + \mu (a_1 + b_2) + a_1 b_2 - a_2 b_1 = 0,$$

and therefore

$$2\mu + a_1 + b_2 = \{(a_1 - b_2)^2 + 4a_2 b_1\}^{\frac{1}{2}}.$$

Thus, if the radical does not vanish, there are two values of μ. Denoting either of these values by μ, we have the subsidiary equations in the form

$$\left.\begin{aligned} \alpha_1 dz_1 + \alpha_2 dz_2 &= dx \\ dy &= \mu dx \end{aligned}\right\},$$

where

$$\frac{\alpha_1}{a_2} = \frac{\alpha_2}{-(a_1 + \mu)} = \frac{1}{a_2\gamma_1 - (a_1 + \mu)\gamma_2}.$$

Let

$$u(x, y, z_1, z_2) = \text{constant}$$

be an integral of these differential relations: then the relation

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2} dz_2$$
$$= \rho (\alpha_1 dz_1 + \alpha_2 dz_2 - dx) + \sigma (dy - \mu dx)$$

must be identically satisfied, so that

$$\left.\begin{aligned} \frac{\partial u}{\partial z_1} + \alpha_1 \frac{\partial u}{\partial x} + \alpha_1 \mu \frac{\partial u}{\partial y} = 0 \\ \frac{\partial u}{\partial z_2} + \alpha_2 \frac{\partial u}{\partial x} + \alpha_2 \mu \frac{\partial u}{\partial y} = 0 \end{aligned}\right\}.$$

It is easy to see that these two equations are not generally a complete system: for if they were, and if

$$u(x, y, z_1, z_2) = \text{constant}$$

were an integral, we should have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z_1} p_1 + \frac{\partial u}{\partial z_2} p_2 = 0,$$

that is,

$$(1 - p_1\alpha_1 - p_2\alpha_2) \frac{\partial u}{\partial x} - \mu (p_1\alpha_1 + p_2\alpha_2) \frac{\partial u}{\partial y} = 0,$$

an equation which, in general, cannot be satisfied identically. Writing

$$A_1 u = \frac{\partial u}{\partial z_1} + \alpha_1 \frac{\partial u}{\partial x} + \alpha_1 \mu \frac{\partial u}{\partial y},$$

$$A_2 u = \frac{\partial u}{\partial z_2} + \alpha_2 \frac{\partial u}{\partial x} + \alpha_2 \mu \frac{\partial u}{\partial y},$$

we have

$$\begin{aligned} &A_1(A_2 u) - A_2(A_1 u) \\ &\quad = \{A_1(\alpha_2) - A_2(\alpha_1)\} \frac{\partial u}{\partial x} + \{A_1(\alpha_2 \mu) - A_2(\alpha_1 \mu)\} \frac{\partial u}{\partial y}; \end{aligned}$$

and therefore, if a common integral of the two partial equations for u exists, we must have

$$\{A_1(\alpha_2) - A_2(\alpha_1)\} \frac{\partial u}{\partial x} + \{A_1(\alpha_2 \mu) - A_2(\alpha_1 \mu)\} \frac{\partial u}{\partial y} = 0.$$

When this is associated with the other two, the three may make a complete system: in that case, there is one integral of the complete system of three equations in four variables, which may be denoted by

$$u = u(x, y, z_1, z_2).$$

Similarly, from the other value of μ, there may be an integral: let it be denoted by

$$v = v(x, y, z_1, z_2).$$

Then the equations

$$u = \text{constant}, \quad v = \text{constant},$$

give an integral system of the original equations.

It may however happen that only one of the values of μ may lead to an integral equation of the form

$$u = u(x, y, z_1, z_2) = \text{constant}.$$

In that case, we can use the equation thus obtained to eliminate one of the dependent variables and its derivatives from the original equations: and it appears as follows that, if one of the original equations is then satisfied, the other also is satisfied so that, in fact, the integral can be used to reduce the two original equations to one only.

The integral $u = \text{constant}$ is the one integral common to a complete system of three equations, which may be taken in the form

$$\frac{1}{\theta} \frac{\partial u}{\partial y} = \frac{1}{\phi} \frac{\partial u}{\partial z_1} = \frac{1}{\psi} \frac{\partial u}{\partial z_2} = \frac{\partial u}{\partial x};$$

and it also is an integral of the system

$$\left.\begin{aligned} a_1 dz_1 + a_2 dz_2 &= dx \\ dy &= \mu dx \end{aligned}\right\},$$

so that

$$\frac{\partial u}{\partial z_1} + a_1 \frac{\partial u}{\partial x} + a_1 \mu \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial u}{\partial z_2} + a_2 \frac{\partial u}{\partial x} + a_2 \mu \frac{\partial u}{\partial y} = 0;$$

and therefore

$$\phi + a_1(1 + \mu\theta) = 0, \quad \psi + a_2(1 + \mu\theta) = 0.$$

Now if $u = u(x, y, z_1, z_2)$ is to be used to eliminate z, p_1, q_1 from the original equations, we have

$$\frac{\partial u}{\partial x} + p_1 \frac{\partial u}{\partial z_1} + p_2 \frac{\partial u}{\partial z_2} = 0,$$

$$\frac{\partial u}{\partial y} + q_1 \frac{\partial u}{\partial z_1} + q_2 \frac{\partial u}{\partial z_2} = 0,$$

as the equations giving the values of the derivatives: and therefore

$$1 + \phi p_1 + \psi p_2 = 0,$$

$$\theta + \phi q_1 + \psi q_2 = 0,$$

that is,

$$1 - (a_1 p_1 + a_2 p_2)(1 + \mu\theta) = 0,$$

$$\theta - (a_1 q_1 + a_2 q_2)(1 + \mu\theta) = 0.$$

When by means of these two relations, we eliminate p_1 and q_1 from the original equations, they become

$$\left(\frac{1}{1 + \mu\theta} - a_2 p_2\right)\frac{1}{a_1} = \gamma_1 + \frac{a_1}{a_1}\left(\frac{\theta}{1 + \mu\theta} - a_2 q_2\right) + b_1 q_2,$$

$$p_2 = \gamma_2 + \frac{a_2}{a_1}\left(\frac{\theta}{1 + \mu\theta} - a_2 q_2\right) + b_2 q_2,$$

respectively; and these are easily proved to be one and the same equation, in virtue of the relations between the quantities a, α, γ, μ. Eliminating z_1 from either of them, we then have a single partial equation of the first order involving only z_2 and its derivatives; its integral can be associated with

$$u = u(x, y, z_1, z_2) = \text{constant},$$

and the two equations constitute an integral equivalent of the original equations.

It is an immediate consequence of this analytical investigation that, if the two equations can be combined in any way with

$$u = u(x, y, z_1, z_2) = \text{constant},$$

so as to lead to a new integral equation independent of $u=$ constant, then the new integral can be combined with $u=$ constant as above to provide an integral equivalent of the original system.

Ex. 1. As an example*, consider the equations

$$\left.\begin{aligned} p_1 &= \frac{2x+2y(z_1-z_2)}{z_2-2z_1} - z_1 q_1 - z_2 q_2 \\ p_2 &= \frac{-x+y(5z_1-2z_2)}{z_2-2z_1} - 2z_1 q_1 - 2z_2 q_2 \end{aligned}\right\}.$$

The equation for μ is

$$\begin{vmatrix} -z_1+\mu, & -2z_1 \\ -z_2\ \ , & -2z_2+\mu \end{vmatrix} = 0,$$

so that there are two values for μ, viz.

$$\mu = z_1 + 2z_2, \quad \mu = 0.$$

Taking the value $\mu = z_1 + 2z_2$, the associated values of a_1 and a_2 are

$$a_1 = \frac{-z_1}{x+y(z_1+2z_2)},$$

$$a_2 = \frac{-z_2}{x+y(z_1+2z_2)};$$

and then the equations for u are

$$A_1 u = \frac{\partial u}{\partial z_1} - \frac{z_1}{x+y(z_1+2z_2)}\left\{\frac{\partial u}{\partial x} + (z_1+2z_2)\frac{\partial u}{\partial y}\right\} = 0,$$

$$A_2 u = \frac{\partial u}{\partial z_2} - \frac{z_2}{x+y(z_1+2z_2)}\left\{\frac{\partial u}{\partial x} + (z_1+2z_2)\frac{\partial u}{\partial y}\right\} = 0.$$

Also

$$A_1(A_2 u) - A_2(A_1 u) = \frac{2z_1 - z_2}{\{x+y(z_1+2z_2)\}^2}\left(-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}\right),$$

so that

$$\frac{\partial u}{\partial y} = \frac{y}{x}\frac{\partial u}{\partial x};$$

and then the other two equations are

$$\frac{\partial u}{\partial z_1} = \frac{z_1}{x}\frac{\partial u}{\partial x},$$

$$\frac{\partial u}{\partial z_2} = \frac{z_2}{x}\frac{\partial u}{\partial x}.$$

The system for u is complete: it has the single integral

$$u = x^2 + y^2 + z_1^2 + z_2^2.$$

* It is given by Königsberger, *Math. Ann.*, t. XLI (1893), p. 264.

Next, taking the value $\mu=0$, (and this implies $y=$constant, as an integral of the subsidiary system), we have

$$a_1=\frac{2(z_2-2z_1)}{5x-y(z_1+2z_2)},$$

$$a_2=\frac{-(z_2-2z_1)}{5x-y(z_1+2z_2)};$$

and the equations for u are

$$B_1u=\frac{\partial u}{\partial z_1}-\frac{2(z_2-2z_1)}{5x-y(z_1+2z_2)}\frac{\partial u}{\partial x}=0,$$

$$B_2u=\frac{\partial u}{\partial z_2}+\frac{z_2-2z_1}{5x-y(z_1+2z_2)}\frac{\partial u}{\partial x}=0,$$

clearly satisfied by $u=y$, which however is not an effective integral for our purpose. Also, the equation

$$B_1(B_2u)-B_2(B_1u)=0$$

is satisfied only in virtue of

$$\frac{\partial u}{\partial x}=0;$$

and the other two equations then are

$$\frac{\partial u}{\partial z_1}=0,\quad \frac{\partial u}{\partial z_2}=0.$$

Clearly no integral that is effective can be derived through $\mu=0$.

We thus have only one integral of the original system, viz.

$$u=x^2+y^2+z_1{}^2+z_2{}^2=\text{constant}.$$

As explained in the text, this integral can be used to eliminate one of the dependent variables and its derivatives from the two original equations: when this elimination has been effected, the resulting equations are one and the same: and the integral of this last equation will complete the integral equivalent of the original equations. Or, also as explained in the text, it can be used so as, in combination with the original equations, to construct a new integral, independent of $u=$ constant. The latter process happens, in the present example, to be the simpler. We have

$$y+z_1q_1+z_2q_2=0.$$

When this is combined with the first of the equations, we find

$$p_1+\frac{yz_2-2x}{z_2-2z_1}=0;$$

and, when it is combined with the second of the equations, we find

$$p_2+\frac{x-yz_1}{z_2-2z_1}=0.$$

These two are equivalent to one another in virtue of

$$x+z_1p_1+z_2p_2=0,$$

as derived from the integral already obtained: and so they can be replaced by any relation which is a combination of the two. Such a relation is

$$p_1+2p_2+y=0,$$

an integral of which will serve to complete the integral system. An integral is

$$z_1+2z_2+xy=\phi(y),$$

where ϕ is an arbitrary function.

Consequently, an integral equivalent of the two original equations is

$$\left.\begin{aligned} x^2+y^2+z_1^2+z_2^2&=a \\ z_1+2z_2+xy&=\phi(y) \end{aligned}\right\},$$

where a is an arbitrary constant, and ϕ is an arbitrary function.

It is also possible to obtain the integral from

$$p_2+\frac{x-yz_1}{z_2-2z_1}=0,$$

by substituting $(a-x^2-y^2-z_2^2)^{\frac{1}{2}}$ for z_1 and integrating.

Ex. 2. Obtain an integral system of the equations

$$\left.\begin{aligned} p_1-\frac{x^2+y}{x+y^2}q_1+x^2q_2&=0 \\ p_2-\frac{1}{x+y^2}q_1+q_2&=0 \end{aligned}\right\},$$

in the form

$$z_1-\tfrac{1}{3}x^3-2xy-\tfrac{2}{3}y^3=a, \quad z_2-x-y=b,$$

where a and b are arbitrary constants. (Königsberger.)

Ex. 3. Obtain an integral system of the equations

$$\left.\begin{aligned} p_1+x^2q_1-\tfrac{1}{4}x(x-y)^2q_2&=0 \\ p_2+xq_1+xyq_2&=0 \end{aligned}\right\},$$

in the form

$$z_1-\tfrac{1}{3}x^3+y=a, \quad z_2-\tfrac{1}{2}x^2=b,$$

where a and b are arbitrary constants. (Königsberger.)

Ex. 4. Shew that, if u_1, u_2, u_3 are any three functions of x, y, z_1, z_2, the differential equations for z_1 and z_2 that correspond to the integral relations

$$\phi(u_1, u_2, u_3)=0, \quad \psi(u_1, u_2, u_3)=0,$$

where ϕ and ψ are arbitrary, are

$$\left.\begin{aligned} a_1(p_1q_2-p_2q_1)+\beta_1p_1+\gamma_1q_1+\delta_1p_2+\epsilon_1q_2&=\zeta_1 \\ a_2(p_1q_2-p_2q_1)+\beta_2p_1+\gamma_2q_1+\delta_2p_2+\epsilon_2q_2&=\zeta_2 \end{aligned}\right\}.$$

(Hamburger.)

What are the limitations on u_1, u_2, u_3, in order that these equations may reduce to Jacobi's set?

Hamburger's Process when there are more than Two Equations.

170. When the number of dependent variables is greater than two, and μ is a simple root of the critical equation, we proceed in a similar manner. The subsidiary equations are

$$\left.\begin{array}{r} \alpha_1 dz_1 + \dots + \alpha_m dz_m = dx \\ dy = \mu dx \end{array}\right\},$$

where the quantities $\alpha_1, \dots, \alpha_m$ are determinate. Let

$$u(x, y, z_1, \dots, z_m) = \text{constant}$$

be an integral of these relations; then the differential relation

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \sum_{i=1}^{m} \frac{\partial u}{\partial z_i} dz_i$$
$$= \rho\left(\sum_{i=1}^{m} \alpha_i dz_i - dx\right) + \sigma(dy - \mu dx)$$

must be satisfied identically, where ρ and σ are independent of the differential elements: hence

$$\frac{\partial u}{\partial z_i} + \alpha_i \frac{\partial u}{\partial x} + \alpha_i \mu \frac{\partial u}{\partial y} = 0,$$

for $i = 1, \dots, m$. This is a system of m equations in $m+2$ variables; according to its character in respect of completeness, it may possess two independent integrals, or only one, or none. The most general integral, which it possesses and which involves any of the dependent variables, provides an integral of the original system.

It is possible that such an integral may be provided by each simple root of the critical equation. If each root of the critical equation is simple, and if an integral can be determined in association with each of the roots, the aggregate of all the integrals thus obtained is an integral equivalent of the original system of equations.

But an integral equivalent will not thus be provided if it should not be possible to obtain an integral in connection with a simple root of the critical equation. In that case, we take such integrals, say $m - \mu$, as are thus provided: and we use them to eliminate, from the m original equations, $m - \mu$ of the dependent

variables with their derivatives; there will then be left a system of μ equations, which are of the same form as before and which involve only μ dependent variables. The problem now is similar to the problem in its initial stage: but it is simpler, because the number of dependent variables has been decreased.

Next, consider a multiple root of the critical equation, and let it give rise to a system of differential relations represented by

$$\left.\begin{aligned} dy &= \mu dx \\ \sum_{i=1}^{m} \gamma_{is} dz_i &= \left(\sum_{i=1}^{m} \gamma_{is}\pi_i\right) dx \end{aligned}\right\},$$

for $s = 1, \ldots, t$, the system thus containing $t+1$ relations.

Let the last t relations be resolved so as to express t of the elements dz, say $dz_1, \ldots, dz_m$, in terms of the remainder, so that the system may be replaced by a system of the form

$$\left.\begin{aligned} dy &= \mu dx \\ dz_s &= \pi_s dx + \sum_{\sigma=t+1}^{m} \epsilon_{s\sigma}(\pi_\sigma dx - dz_\sigma) \end{aligned}\right\},$$

for $s = 1, \ldots, t$. The modified system implies, as the former system implied, that t of the equations

$$E_\rho = \sum_{i=1}^{m} \lambda_i a_{i\rho} + \mu\lambda_\rho = 0,$$

for $\rho = 1, \ldots, m$, are deducible from the remainder, the quantities $a_{i\rho}$ being the coefficients in the original set of equations

$$p_i = \pi_i + \sum_{\rho=1}^{m} a_{i\rho} q_\rho, \qquad (i = 1, \ldots, m),$$

and the quantities $\lambda_1, \ldots, \lambda_m$ are such that the t differential relations arise from

$$\sum_{i=1}^{m} \lambda_i (dz_i - \pi_i dx) = 0.$$

In order to express the interdependence of some of the equations $E_1 = 0, \ldots, E_m = 0$, we write

$$E_i + \sum_{j=t+1}^{m} \epsilon_{ij} E_j = 0,$$

for $i = 1, \ldots, t$; and therefore the quantities ϵ_{ij} are such that

$$a_{i\mu} + \sum_{j=t+1}^{m} \epsilon_{ij} a_{j\mu} = 0,$$

for all values of μ different from i, $t+1$, ..., m, together with

$$a_{ii} + \mu + \sum_{j=t+1}^{m} \epsilon_{ij} a_{ji} = 0,$$

$$a_{ij} + \sum_{\rho=t+1}^{m} \epsilon_{i\rho} a_{\rho j} + \epsilon_{ij}(a_{jj} + \mu) = 0,$$

where the summation with regard to ρ excludes $\rho = j$, while i has the values 1, ..., t, and j has the values $t+1$, ..., m. With these values, we have

$$\begin{aligned} p_i + \mu q_i - \pi_i + \sum_{j=t+1}^{m} \epsilon_{ij}(p_j + \mu q_j - \pi_j) \\ = a_{i1} q_1 + a_{i2} q_2 + \ldots + (a_{ii} + \mu) q_i + \ldots + a_{im} q_m \\ + \sum_{j=t+1}^{m} \epsilon_{ij} \{a_{j1} q_1 + a_{j2} q_2 + \ldots + (a_{jj} + \mu) q_j + \ldots + a_{jm} q_m\} \\ = 0, \end{aligned}$$

because the coefficient of each of the quantities q_1, ..., q_m vanishes on account of the above equations in the quantities ϵ_{ij}.

Thus the t equations

$$p_i + \mu q_i - \pi_i + \sum_{j=t+1}^{m} \epsilon_{ij}(p_i + \mu q_j - \pi_j) = 0$$

can be regarded as replacing t of the equations in the original system: other $m-t$ equations would be required to have a complete equivalent of that system.

Now let

$$\phi(x, y, z_1, \ldots, z_m) = \text{constant}$$

be one of the equations in the integral equivalent of the subsidiary differential relations; and assume that the differential relations are completely integrable*, so that there are $t+1$ such integrals. Then the relation

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \sum_{i=1}^{m} \frac{\partial \phi}{\partial z_i} dz_i = 0$$

is consistent with the $t+1$ differential relations

$$dy = \mu dx,$$

$$dz_j = \pi_j dx + \sum_{\sigma=t+1}^{m} \epsilon_{j\sigma}(\pi_\sigma dx - dz_\sigma);$$

* The assumption is a distinct limitation, as its justification requires that conditions should be satisfied. It must be remembered, however, that we are dealing with equations, restricted in form and in the number of independent variables, so that the method does not claim to be general ; it is therefore hardly necessary to deal generally with all the minutiæ of alternatives, when these could be dealt with in any particular case.

and therefore

$$\frac{\partial \phi}{\partial x} + \mu \frac{\partial \phi}{\partial y} + \sum_{s=1}^{t} \left(\pi_s + \sum_{\sigma=t+1}^{m} \epsilon_{s\sigma}\pi_\sigma\right) \frac{\partial \phi}{\partial z_s} = 0,$$

$$\frac{\partial \phi}{\partial z_\rho} - \sum_{s=1}^{t} \epsilon_{s\rho} \frac{\partial \phi}{\partial z_s} = 0,$$

for $\rho = t+1, \ldots, m$. And these equations must be satisfied by each of the functions ϕ.

171. Let the $t+1$ functions, supposed to be thus obtained, be denoted by $\phi_1, \ldots, \phi_{t+1}$; and let t arbitrary combinations of these functions be taken which, when equated to zero, may be regarded as t equations helping to express $z_1, \ldots, z_m$ in terms of x and y. If

$$f_i(\phi_1, \ldots, \phi_{t+1}) = 0$$

be any one of these integral equations, then, on multiplying by $\dfrac{\partial f}{\partial \phi_i}$ the differential equations which determine ϕ_i and on adding the results, we have

$$\left.\begin{array}{r} \dfrac{\partial f_i}{\partial x} + \mu \dfrac{\partial f_i}{\partial y} + \displaystyle\sum_{s=1}^{t} \left(\pi_s + \sum_{\sigma=t+1}^{m} \epsilon_{s\sigma}\pi_\sigma\right) \dfrac{\partial f_i}{\partial z_s} = 0 \\ \dfrac{\partial f_i}{\partial z_\rho} - \displaystyle\sum_{s=1}^{t} \epsilon_{s\rho} \dfrac{\partial f_i}{\partial z_s} = 0 \end{array}\right\},$$

for $\rho = t+1, \ldots, m$. But when $f_i = 0$ is regarded as an integral equation, we have

$$\frac{\partial f_i}{\partial x} + \sum_{s=1}^{m} p_s \frac{\partial f_i}{\partial z_s} = 0,$$

$$\frac{\partial f_i}{\partial y} + \sum_{s=1}^{m} q_s \frac{\partial f_i}{\partial z_s} = 0,$$

and therefore

$$\frac{\partial f_i}{\partial x} + \mu \frac{\partial f_i}{\partial y} + \sum_{s=1}^{m} (p_s + \mu q_s) \frac{\partial f_i}{\partial z_s} = 0.$$

Substituting in this equation for $\dfrac{\partial f_i}{\partial x} + \mu \dfrac{\partial f_i}{\partial y}$, and for $\dfrac{\partial f_i}{\partial z_\rho}$ (for $\rho = t+1, \ldots, m$), we have

$$\sum_{s=1}^{t} \left\{p_s + \mu q_s - \pi_s + \sum_{\sigma=t+1}^{m} \epsilon_{s\sigma}(p_\sigma + \mu q_\sigma - \pi_\sigma)\right\} \frac{\partial f_i}{\partial z_s} = 0.$$

This relation holds for $i = 1, \ldots, t$; and the functions $f_1, \ldots, f_t$ are independent, so that the quantity

$$J\left(\frac{f_1, \ldots, f_t}{z_1, \ldots, z_t}\right)$$

is not evanescent; consequently, the equations

$$p_s + \mu q_s - \pi_s + \sum_{\sigma=t+1}^{m} \epsilon_{s\sigma}(p_\sigma + \mu q_\sigma - \pi_\sigma) = 0,$$

for $s = 1, \ldots, t$, are satisfied. And these equations are part of the system equivalent to the original system: the set of integrals thus obtained effectively satisfy t of the equations of the original system.

When we proceed in this way with all the multiple roots of the critical equation and obtain the integrals associated with them in turn, and when we similarly retain all the integrals associated with the simple roots of that equation, we obtain an aggregate of integrals of the differential equations: let the number of these be τ. Then these τ equations can be resolved so as to express τ of the dependent variables, say $z_1, \ldots, z_\tau$, in terms of the remainder and of x, y; and they are such as to satisfy an appropriate set of τ combinations of the original equations. When the values of $z_1, \ldots, z_\tau$ and of their derivatives are substituted in the $m - \tau$ other combinations, which (with the τ just satisfied) constitute an algebraic equivalent of the original system, then we have a simultaneous set of $m - \tau$ equations having $z_{\tau+1}, \ldots, z_\tau$ for the dependent variables, and x, y for the independent variables. The problem of obtaining the integrals of this new set of equations is similar to the initial problem: but it is simpler, because the number of dependent variables has been reduced from m to $m - \tau$.

As already remarked, the easiest case is that in which each root μ of the critical equation is simple. With each such root, two differential relations are satisfied: let u_i=constant, v_i=constant be their integral equivalent. Then

$$g_i(u_i, v_i) = 0,$$

where g_i is arbitrary, is an integral of the original system; and a full set of integrals of the original equations is given by

$$g_1(u_1, v_1) = 0, \quad g_2(u_2, v_2) = 0, \ldots, g_m(u_m, v_m) = 0,$$

where $g_1, \ldots, g_m$ are arbitrary functions.

If, connected with a simple root μ_j of the critical equation, only one integral (say u_j) can be obtained, then

$$u_j = \text{constant}$$

takes the place of $g_j(u_j, v_j) = 0$. And if no integral can be obtained, then (as already explained) we use the known integrals to reduce the order of the system and adopt the process for the integration of the reduced system.

It is to be noticed that what is wanted at each stage, in connection with the differential relations of the form

$$\left.\begin{aligned} \alpha_1 dz_1 + \dots + \alpha_m dz_m &= dx \\ dy &= \mu dx \end{aligned}\right\}$$

for a simple root μ of the critical equation, and of similar relations for a multiple root, is not a complete equivalent of each set regarded as a set of Pfaffian equations but only those integral equations (if any), which arise by forming an exactly integrable combination of the differential relations or which can be obtained by some equivalent process.

Further it is clear from the general argument that, if circumstances make the use of an obtained integral convenient at any stage, the integral can be used to reduce the order of the system at once without determining any further integral or integrals connected with the root in question, or with any other root, of the critical equation in μ.

Ex. 1. Integrate the equations

$$\left.\begin{aligned} p_1 &= \frac{1}{2x}(3z_1 + 2z_2 + z_3) - \frac{y}{x}(q_1 + q_2 + q_3) \\ p_2 &= \frac{1}{2x}(z_1 + 2z_2 - z_3) + \frac{y}{x} q_3 \\ p_3 &= \frac{1}{2x}(-z_1 - 2z_2 + z_3) + \frac{y}{x} q_2 \end{aligned}\right\}.$$

The critical equation for the determination of μ is

$$\begin{vmatrix} \mu - \frac{y}{x}, & -\frac{y}{x}, & -\frac{y}{x} \\ 0, & \mu, & \frac{y}{x} \\ 0, & \frac{y}{x}, & \mu \end{vmatrix} = 0,$$

that is,

$$\left(\mu - \frac{y}{x}\right)\left(\mu^2 - \frac{y^2}{x^2}\right) = 0,$$

so that $\mu = -\frac{y}{x}$ is a simple root and $\mu = \frac{y}{x}$ is a repeated root.

Taking $\mu=-\frac{y}{x}$, we have

$$-\lambda_1\frac{y}{x}=\frac{y}{x}\lambda_1,$$

$$-\lambda_1\frac{y}{x}+\lambda_3\frac{y}{x}=\frac{y}{x}\lambda_2,$$

$$-\lambda_1\frac{y}{x}+\lambda_2\frac{y}{x}=\frac{y}{x}\lambda_3,$$

so that

$$\lambda_1=0, \quad \lambda_2=\lambda_3.$$

The subsidiary equations are

$$\frac{dz_2+dz_3}{0}=dx,$$

$$dy=-\frac{y}{x}dx:$$

two integrals of these are

$$z_2+z_3=\text{constant}, \quad xy=\text{constant}:$$

hence an integral of the original system of equations is

$$z_2+z_3=f(xy),$$

where f is an arbitrary function.

Next, taking the repeated root $\mu=\frac{y}{x}$, we find that there is only a single relation among the three quantities λ: it is

$$\lambda_2+\lambda_3=\lambda_1.$$

The subsidiary equations, on the substitution of this value of λ_1, take the form

$$\frac{\lambda_2(dz_1+dz_2)+\lambda_3(dz_1+dz_3)}{\frac{\lambda_2}{2x}(4z_1+4z_2)+\frac{\lambda_3}{2x}(2z_1+2z_3)}=dx,$$

$$dy=\frac{y}{x}dx;$$

hence as $\lambda_2:\lambda_3$ is undetermined, we take the subsidiary equations in the form

$$\left.\begin{aligned}\frac{x}{2z_1+2z_2}(dz_1+dz_2)&=dx\\ \frac{x}{z_1+z_3}(dz_1+dz_3)&=dx\\ \frac{x}{y}dy&=dx\end{aligned}\right\}.$$

Three independent integrals of these equations are

$$\frac{z_1+z_2}{x^2}=\text{constant},$$

$$\frac{z_1+z_3}{x}=\text{constant},$$

$$\frac{y}{x}=\text{constant};$$

hence two integrals of the original system are given by

$$\phi\left(\frac{z_1+z_2}{x^2},\ \frac{z_1+z_3}{x},\ \frac{y}{x}\right)=0,$$

$$\psi\left(\frac{z_1+z_2}{x^2},\ \frac{z_1+z_3}{x},\ \frac{y}{x}\right)=0,$$

where ϕ and ψ are arbitrary functions or, what is the same thing, two integrals are given by

$$\frac{z_1+z_2}{x^2}=g\left(\frac{y}{x}\right),\quad \frac{z_1+z_3}{x}=h\left(\frac{y}{x}\right),$$

where g and h are arbitrary functions.

Hence an integral equivalent of the system of differential equations is given by the three equations

$$\left.\begin{array}{l} z_2+z_3=f(xy) \\ z_1 \quad\ +z_3=xh\left(\dfrac{y}{x}\right) \\ z_1+z_2 \quad\ =x^2g\left(\dfrac{y}{x}\right) \end{array}\right\},$$

where f, g, h are arbitrary functions.

Ex. 2. Integrate the equations

$$\left.\begin{array}{l} 7xp_1=7z_1+z_2+2z_3-y\,(7q_1+2q_2+4q_3) \\ 7xp_2=6z_2-2z_3-y\,(5q_2-4q_3) \\ 7xp_3=-3z_2+z_3+y\,(6q_2+5q_3) \end{array}\right\}.$$

Alternative Method, with Partial Subsidiary Equations.

172. In the preceding investigation, the construction of the integrals of the system

$$p_i=\pi_i+\sum_{s=1}^{m} a_{is}q_s, \qquad (i=1,\ \ldots,\ m)$$

was made to depend upon the integration of a subsidiary set of equations homogeneous and linear in differential elements. As is well known from many discussions in earlier parts of this treatise, the integration of such a set can be replaced by the integration of a system of simultaneous partial differential equations in a single dependent variable: and indeed, in §§ 169, 170, the problem was thus actually transferred from the region of ordinary equations to that of partial equations. The construction of these partial equations can be effected, without the intervention of the subsidiary equations as follows. Let

$$u=u\,(x,\ y,\ z_1,\ \ldots,\ z_m)=\text{constant}$$

be a member of the integral equivalent of the above system of partial differential equations: then the equations

$$\frac{\partial u}{\partial x} + \sum_{i=1}^{m} \frac{\partial u}{\partial z_i} p_i = 0,$$

$$\frac{\partial u}{\partial y} + \sum_{i=1}^{m} \frac{\partial u}{\partial z_i} q_i = 0,$$

not merely coexist with the system of m equations but they are not independent of them, quà equations in the derivatives. Consequently, when substitution in the prior of the last two equations is made for $p_1, \dots, p_m$, so that the original system has been completely used for purposes of inter-relation of the whole set, the two equations

$$\frac{\partial u}{\partial x} + \sum_{i=1}^{m} \left(\pi_i + \sum_{s=1}^{m} a_{is} q_s\right) \frac{\partial u}{\partial z_i} = 0,$$

$$\frac{\partial u}{\partial y} + \sum_{i=1}^{n} q_i \frac{\partial u}{\partial z_i} = 0,$$

are not independent of one another so far as concerns the derivatives. Comparing the coefficients of $q_1, \dots, q_m$, and the quantities not involving these derivatives, in the two equations, we have

$$\sum_{j=1}^{m} a_{ji} \frac{\partial u}{\partial z_j} = -\mu \frac{\partial u}{\partial z_i},$$

$$\frac{\partial u}{\partial x} + \sum_{j=1}^{m} \pi_j \frac{\partial u}{\partial z_j} = -\mu \frac{\partial u}{\partial y},$$

where μ is an unknown quantity and the former equation holds for $i = 1, \dots, m$.

The m equations

$$\sum_{j=1}^{m} a_{ji} \frac{\partial u}{\partial z_j} = -\mu \frac{\partial u}{\partial z_i},$$

for $i = 1, \dots, m$, are homogeneous and linear in the m derivatives of u with regard to $z_1, \dots, z_m$: and these do not all vanish, for $u =$ constant is postulated as an integral equation. Hence μ is given by the equation

$$\Theta = \begin{vmatrix} a_{11} + \mu, & a_{21}, & \dots, & a_{m1} \\ a_{12}, & a_{22} + \mu, & \dots, & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1m}, & a_{2m}, & \dots, & a_{mm} + \mu \end{vmatrix} = 0,$$

being the same equation for μ as in the other investigation (§ 167). According to the character of μ as a root of this equation $\Theta = 0$, the form of the system of equations for u alters.

Let $\mu = \sigma$ be a simple root of $\Theta = 0$; then the former set of $m+1$ equations involving the then unknown quantity μ and the derivatives of u can be replaced by the set of m equations

$$\frac{\partial u}{\partial z_i} + \alpha_i \left(\frac{\partial u}{\partial x} + \sigma \frac{\partial u}{\partial y} \right) = 0,$$

for $i = 1, \ldots, m$, the quantity μ now being known; and the quantities $\alpha_1, \ldots, \alpha_m$ are given by the equation

$$\sum_{j=1}^{m} \pi_j \alpha_j = -1,$$

and by any $m-1$ of the m equations

$$\sum_{j=1}^{m} a_{ji} \alpha_j = -\sigma \alpha_i,$$

for $i = 1, \ldots, m$. The set of m equations for u involves $m+2$ variables. It may possess no integral at all or no integral involving any one of the variables $z_1, \ldots, z_m$; in that case, no integral of the original system of equations is derivable through the root σ of $\Theta = 0$. Or it may possess one integral involving at least one of the variables $z_1, \ldots, z_m$; in that case,

$$U = \text{constant},$$

where U is the integral in question, is an integral of the original system. Or it may possess two integrals U and V, one at least of which involves one or more than one of the variables $z_1, \ldots, z_m$; in that case,

$$\phi(U, V) = 0$$

where ϕ is arbitrary, is the most general integral of the original system thus obtainable.

Similarly for each simple root of $\Theta = 0$.

Next, let μ be a multiple root of $\Theta = 0$; then the m equations

$$E_i = \sum_{j=1}^{m} a_{ji} \alpha_j + \mu \alpha_i = 0, \qquad (i = 1, \ldots, m),$$

are not independent of one another. Let them be such that t (and not more than t) of them can be deduced from the remainder, so that there will be t relations of the form

$$E_s + \sum_{j=1}^{m} \epsilon_{sj} E_j = 0,$$

for $s = 1, \ldots, t$: thus the quantities ϵ are given by

$$a_{ss} + \mu + \sum_{j=t+1}^{m} \epsilon_{sj} a_{js} = 0,$$

$$\epsilon_{sj}(a_{jj} + \mu) + a_{sj} + \sum_{\rho=t+1}^{m} \epsilon_{s\rho} a_{\rho j} = 0,$$

$$a_{s\mu} + \sum_{j=t+1}^{m} \epsilon_{sj} a_{j\mu} = 0,$$

where the first equation holds for $s = 1, \ldots, t$: the second holds for $s = 1, \ldots, t$, and $j = t+1, \ldots, m$, and in it the summation with regard to ρ is for all values $t+1, \ldots, m$ except $\rho = j$; and the third holds for the values $s = 1, \ldots, t$, and for the values $1, \ldots, t$ except $\mu = s$. Proceeding as for the simple root, we find that the equations for u are

$$\frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial y} + \sum_{s=1}^{t} \left(\pi_s + \sum_{\sigma=t+1}^{m} \epsilon_{s\sigma} \pi_\sigma \right) \frac{\partial u}{\partial z_s} = 0,$$

$$\frac{\partial u}{\partial z_\rho} - \sum_{s=1}^{t} \epsilon_{s\rho} \frac{\partial u}{\partial z_s} = 0,$$

for $\rho = t+1, \ldots, m$, being the same equations as in § 171. This set of equations, being $m - t + 1$ in number and involving $m + 2$ variables, can have any number* of integrals from 0 up to $t + 1$; let these integrals be

$$U_1, \ldots, U_\kappa,$$

where

$$0 \leqslant \kappa \leqslant t + 1.$$

Not more than one of these integrals can be independent of all the variables $z_1, \ldots, z_m$: if there be one such, let it be U_κ.

If $\kappa > 1$, then the equations

$$U_1 = f_1(U_\kappa), \ldots, U_{\kappa-1} = f_{\kappa-1}(U_\kappa),$$

where $f_1, \ldots, f_{\kappa-1}$ are arbitrary functions, constitute $\kappa - 1$ integrals of the original system; they are associated with the multiple root μ of the equation $\Theta = 0$.

If $\kappa = 1$, and if U_1 involves one at least of the variables $z_1, \ldots, z_m$, then

$$U_1 = \text{constant}$$

is an integral of the original system: it is associated with the multiple root μ.

* The conditions as to number depend solely upon the number of equations in the system when rendered complete. If this number be κ', where $\kappa' \leqslant m + 2$, then the number of integrals is $m + 2 - \kappa'$.

When $\kappa=1$ and U_1 does not involve any of the variables $z_1, \ldots, z_m$, and when $\kappa=0$, no integral is provided for the original system in association with the multiple root.

Similarly for each multiple root of the critical equation $\Theta=0$.

As before, the integrals can be used to eliminate some of the dependent variables and so to reduce the order of the original system.

Ex. The preceding process can be illustrated by being applied to the equations in Ex. 1, § 171, viz.

$$\left.\begin{aligned} p_1 &= \frac{1}{2x}(3z_1+2z_2+z_3)-\frac{y}{x}(q_1+q_2+q_3) \\ p_2 &= \frac{1}{2x}(z_1+2z_2-z_3)+\frac{y}{x}q_3 \\ p_3 &= \frac{1}{2x}(-z_1-2z_2+z_3)+\frac{y}{x}q_2 \end{aligned}\right\}.$$

Let it be supposed that

$$u(x, y, z_1, z_2, z_3)=0$$

is an integral of these equations: then the preceding explanations shew that the equations

$$\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z_1}&\left\{\frac{1}{2x}(3z_1+2z_2+z_3)-\frac{y}{x}(q_1+q_2+q_3)\right\} \\ &+\frac{\partial u}{\partial z_2}\left\{\frac{1}{2x}(z_1+2z_2-z_3)+\frac{y}{x}q_3\right\} \\ &+\frac{\partial u}{\partial z_3}\left\{\frac{1}{2x}(-z_1-2z_2+z_3)+\frac{y}{x}q_2\right\}=0, \end{aligned}$$

and

$$\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z_1}q_1+\frac{\partial u}{\partial z_2}q_2+\frac{\partial u}{\partial z_3}q_3=0,$$

quà equations in q_1, q_2 and q_3, are one and the same. Hence

$$\begin{aligned} \theta\frac{\partial u}{\partial z_1}&=-\frac{y}{x}\frac{\partial u}{\partial z_1}, \\ \theta\frac{\partial u}{\partial z_2}&=-\frac{y}{x}\frac{\partial u}{\partial z_1}+\frac{y}{x}\frac{\partial u}{\partial z_3}, \\ \theta\frac{\partial u}{\partial z_3}&=-\frac{y}{x}\frac{\partial u}{\partial z_1}+\frac{y}{x}\frac{\partial u}{\partial z_2}, \\ \theta\frac{\partial u}{\partial y}&=\frac{\partial u}{\partial x}+\frac{1}{2x}\left\{(3z_1+2z_2+z_3)\frac{\partial u}{\partial z_1}+(z_1+2z_2-z_3)\left(\frac{\partial u}{\partial z_2}-\frac{\partial u}{\partial z_3}\right)\right\}. \end{aligned}$$

From the first three of these equations, we have

$$\begin{vmatrix} \theta+\frac{y}{x}, & 0, & 0 \\ \frac{y}{x}, & \theta, & -\frac{y}{x} \\ \frac{y}{x}, & -\frac{y}{x}, & \theta \end{vmatrix}=0,$$

that is,

$$\left(\theta+\frac{y}{x}\right)\left(\theta^2-\frac{y^2}{x^2}\right)=0,$$

so that a simple root is given by

$$\theta=\frac{y}{x},$$

and a double root is given by

$$\theta=-\frac{y}{x}.$$

I. Let $\theta=\frac{y}{x}$. The first equation gives

$$\frac{\partial u}{\partial z_1}=0;$$

and the other equations then are

$$\frac{\partial u}{\partial z_2}-\frac{\partial u}{\partial z_3}=0,$$

$$\frac{y}{x}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}=0.$$

These three equations are a complete system: they possess two independent integrals, in the form

$$u=z_2+z_3, \quad u=xy;$$

hence the equation

$$z_2+z_3=F(xy),$$

where F is an arbitrary function, is part of an integral of the original equations.

II. Let $\theta=-\frac{y}{x}$. The first equation becomes evanescent: the next two equations both become

$$\frac{\partial u}{\partial z_1}-\frac{\partial u}{\partial z_2}-\frac{\partial u}{\partial z_3}=0,$$

and the last equation is

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2(z_1+z_2)\frac{\partial u}{\partial z_2}+(z_1+z_3)\frac{\partial u}{\partial z_3}=0.$$

These two equations are a complete system: they possess three independent integrals, in the form

$$\frac{z_1+z_2}{x^2}, \quad \frac{z_1+z_3}{x}, \quad \frac{y}{x};$$

hence the equations

$$z_1+z_2=x^2G\left(\frac{y}{x}\right),$$

$$z_1+z_3=xH\left(\frac{y}{x}\right),$$

where G and H are arbitrary functions, are part of an integral of the original equations.

The full integral of the original equations is given by combining all the parts obtained: it is

$$z_2+z_3=F(xy),$$

$$z_1+z_2=x^2G\left(\frac{y}{x}\right),$$

$$z_1+z_3=xH\left(\frac{y}{x}\right),$$

being the same as by the other process.

HAMBURGER'S METHOD APPLIED TO NON-LINEAR EQUATIONS.

173. The method of constructing the integral of a system of simultaneous equations in a number of dependent variables, as just expounded, depends apparently on the formal property that the equations in question are linear in the derivatives of the dependent variable. It was only natural to expect that the method could be extended so as to apply to equations not restricted to being linear in those derivatives; and this extension, due* initially to Hamburger, was effected by a device, (successful specially in connection with equations of the second order, as will be seen later), which replaces the non-linear system by an amplified linear system.

Adopting the same notation as before for the independent variables, for the dependent variables and for their derivatives, and assuming that the number of partial differential equations algebraically independent of one another is the same as the number of dependent variables, we take these equations in the form

$$f_i(x, y, z_1, \ldots, z_n, p_1, \ldots, p_n, q_1, \ldots, q_n)=0,$$

for $i=1, \ldots, n$. Were the integrals known, and the values of $z_1, \ldots, z_n$ and of their derivatives substituted in the differential equations, the latter would become identities; accordingly, when we take

$$\frac{df_i}{dx}=\frac{\partial f_i}{\partial x}+\sum_{r=1}^{n}\frac{\partial f_i}{\partial z_r}p_r,\quad \frac{df_i}{dy}=\frac{\partial f_i}{\partial y}+\sum_{r=1}^{n}\frac{\partial f_i}{\partial z_r}q_r,$$

the integrals of the equations are in accord with the further equations

$$\sum_{\kappa=1}^{n}\frac{\partial f_i}{\partial p_\kappa}\frac{\partial p_\kappa}{\partial x}+\sum_{\kappa=1}^{n}\frac{\partial f_i}{\partial q_\kappa}\frac{\partial p_\kappa}{\partial y}=-\frac{df_i}{dx},$$

$$\sum_{\kappa=1}^{n}\frac{\partial f_i}{\partial p_\kappa}\frac{\partial q_\kappa}{\partial x}+\sum_{\kappa=1}^{n}\frac{\partial f_i}{\partial q_\kappa}\frac{\partial q_\kappa}{\partial y}=-\frac{df_i}{dy},$$

* For references, see p. 407.

deduced from derivatives of the identities by using the necessary relations

$$\frac{\partial q_\kappa}{\partial x} = \frac{\partial p_\kappa}{\partial y}, \qquad (\kappa = 1, \ldots, n).$$

Also, because of the necessary relations

$$\frac{\partial z_\kappa}{\partial x} = p_\kappa, \quad \frac{\partial z_\kappa}{\partial y} = q_\kappa,$$

we have

$$\sum_{\kappa=1}^{n} \frac{\partial f_i}{\partial p_\kappa}\frac{\partial z_\kappa}{\partial x} + \sum_{\kappa=1}^{n} \frac{\partial f_i}{\partial q_\kappa}\frac{\partial z_\kappa}{\partial y} = \sum_{\kappa=1}^{n}\left(p_\kappa \frac{\partial f_i}{\partial p_\kappa} + q_\kappa \frac{\partial f_i}{\partial q_\kappa}\right).$$

These equations hold for $i = 1, \ldots, n$: they thus constitute a system of $3n$ equations, involving $3n$ dependent variables $z_1, \ldots, z_n$, $p_1, \ldots, p_n$, $q_1, \ldots, q_n$; and they are linear in the derivatives of those $3n$ dependent variables. Hence this system of equations is amenable to the Hamburger method for linear equations already expounded.

To apply the method, we introduce n quantities $l_1, \ldots, l_n$, which are functions of all the variables and which (as to their ratios) will be determined subsequently; and we write

$$\sum_{i=1}^{n} l_i \frac{\partial f_i}{\partial p_\kappa} = P_\kappa,$$

$$\sum_{i=1}^{n} l_i \frac{\partial f_i}{\partial q_\kappa} = Q_\kappa,$$

$$\sum_{i=1}^{n} l_i \frac{df_i}{dx} = X,$$

$$\sum_{i=1}^{n} l_i \frac{df_i}{dy} = Y.$$

Then, multiplying the preceding typical equations by l_i and adding for all values of i, we have

$$\sum_{\kappa=1}^{n} P_\kappa \frac{\partial p_\kappa}{\partial x} + \sum_{\kappa=1}^{n} Q_\kappa \frac{\partial p_\kappa}{\partial y} = -X,$$

$$\sum_{\kappa=1}^{n} P_\kappa \frac{\partial q_\kappa}{\partial x} + \sum_{\kappa=1}^{n} Q_\kappa \frac{\partial q_\kappa}{\partial y} = -Y,$$

$$\sum_{\kappa=1}^{n} P_\kappa \frac{\partial z_\kappa}{\partial x} + \sum_{\kappa=1}^{n} Q_\kappa \frac{\partial z_\kappa}{\partial y} = \sum_{\kappa=1}^{n} (P_\kappa p_\kappa + Q_\kappa q_\kappa)$$

Again, we have

$$dp_\kappa = \frac{\partial p_\kappa}{\partial x}\,dx + \frac{\partial p_\kappa}{\partial y}\,dy,$$

$$dq_\kappa = \frac{\partial q_\kappa}{\partial x}\,dx + \frac{\partial q_\kappa}{\partial y}\,dy,$$

$$dz_\kappa = \frac{\partial z_\kappa}{\partial x}\,dx + \frac{\partial z_\kappa}{\partial y}\,dy;$$

and therefore, if $\lambda_1, \ldots, \lambda_n$ denote another series of quantities, which are functions of all the variables and which (also as to their ratios) will be determined subsequently, we have

$$\sum_{\kappa=1}^{n} \lambda_\kappa \frac{\partial p_\kappa}{\partial x}\,dx + \sum_{\kappa=1}^{n} \lambda_\kappa \frac{\partial p_\kappa}{\partial y}\,dy = \sum_{\kappa=1}^{n} \lambda_\kappa dp_\kappa,$$

$$\sum_{\kappa=1}^{n} \lambda_\kappa \frac{\partial q_\kappa}{\partial x}\,dx + \sum_{\kappa=1}^{n} \lambda_\kappa \frac{\partial q_\kappa}{\partial y}\,dy = \sum_{\kappa=1}^{n} \lambda_\kappa dq_\kappa,$$

$$\sum_{\kappa=1}^{n} \lambda_\kappa \frac{\partial z_\kappa}{\partial x}\,dx + \sum_{\kappa=1}^{n} \lambda_\kappa \frac{\partial z_\kappa}{\partial y}\,dy = \sum_{\kappa=1}^{n} \lambda_\kappa dz_\kappa.$$

In connection with these two sets of equations and as a generalisation of the corresponding step in the earlier process, we construct a subsidiary system of equations

$$\frac{\sum\limits_{\kappa=1}^{n} \lambda_\kappa dp_\kappa}{-X} = \frac{\sum\limits_{\kappa=1}^{n} \lambda_\kappa dq_\kappa}{-Y} = \frac{\sum\limits_{\kappa=1}^{n} \lambda_\kappa dz_\kappa}{\sum\limits_{\kappa=1}^{n} (P_\kappa p_\kappa + Q_\kappa q_\kappa)}$$

$$= \frac{\lambda_1 dx}{P_1} = \ldots = \frac{\lambda_n dx}{P_n}$$

$$= \frac{\lambda_1 dy}{Q_1} = \ldots = \frac{\lambda_n dy}{Q_n}.$$

Take

$$dy = \mu dx;$$

then

$$\mu = \frac{Q_r}{P_r},$$

for $r = 1, \ldots, n$, that is,

$$l_1\left(\frac{\partial f_1}{\partial q_r} - \mu \frac{\partial f_1}{\partial p_r}\right) + \ldots + l_n\left(\frac{\partial f_n}{\partial q_r} - \mu \frac{\partial f_n}{\partial p_r}\right) = 0,$$

for $r=1, \ldots, n$. Hence μ must satisfy the equation

$$\Delta = \begin{vmatrix} \frac{\partial f_1}{\partial q_1} - \mu \frac{\partial f_1}{\partial p_1}, \ldots, \frac{\partial f_n}{\partial q_1} - \mu \frac{\partial f_n}{\partial p_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial f_1}{\partial q_n} - \mu \frac{\partial f_1}{\partial p_n}, \ldots, \frac{\partial f_n}{\partial q_n} - \mu \frac{\partial f_n}{\partial p_n} \end{vmatrix} = 0.$$

When μ is determined as a simple root of $\Delta = 0$, the ratios $l_1 : l_2 : \ldots : l_n$ are those of first minors of the determinant. When μ is determined as a multiple root of $\Delta = 0$, say of multiplicity θ, then the quantities $l_1, \ldots, l_n$ can be expressed linearly and homogeneously in terms of t independent quantities, where $t \leqslant \theta$. Moreover, when μ and the ratios of the quantities $l_1, \ldots, l_n$ are known, the ratios of the quantities $\lambda_1, \ldots, \lambda_n$ can be considered known; for we have

$$\frac{\lambda_1}{P_1} = \ldots = \frac{\lambda_n}{P_n},$$

and

$$\frac{\lambda_1}{Q_1} = \ldots = \frac{\lambda_n}{Q_n},$$

these two sets being the same in virtue of the relations

$$Q_r = \mu P_r,$$

for $r=1, \ldots, n$. Being concerned only with ratios of $\lambda_1, \ldots, \lambda_n$, for it is only the ratios that occur in the subsidiary system, we take

$$\lambda_1 = P_1, \ldots, \lambda_n = P_n;$$

and then we have the subsidiary system in the form

$$\begin{aligned} \Delta &= 0, \\ dy &= \mu\, dx, \\ \sum_{\kappa=1}^{n} P_\kappa dp_\kappa &= -X dx, \\ \sum_{\kappa=1}^{n} P_\kappa dq_\kappa &= -Y dx, \\ \sum_{\kappa=1}^{n} P_\kappa dz_\kappa &= \sum_{\kappa=1}^{n} \{P_\kappa p_\kappa + Q_\kappa q_\kappa\}\, dx \\ &= \sum_{\kappa=1}^{n} P_\kappa (p_\kappa + \mu q_\kappa)\, dx \\ &= \sum_{\kappa=1}^{n} P_\kappa (p_\kappa dx + q_\kappa dy). \end{aligned}$$

The last equation can evidently be replaced by

$$dz_\kappa = p_\kappa dx + q_\kappa dy,$$

for $\kappa = 1, \ldots, n$: and so we take the final form of the equations of the subsidiary system to be

$$\left.\begin{aligned} &\Delta = 0, \quad dy = \mu dx \\ &dz_\kappa = p_\kappa dx + q_\kappa dy, \quad (\kappa = 1, \ldots, n), \\ &\sum_{\kappa=1}^{n} P_\kappa dp_\kappa = -X dx \\ &\sum_{\kappa=1}^{n} P_\kappa dq_\kappa = -Y dx \end{aligned}\right\}.$$

This set includes $n+3$ total differential relations, which involve the $3n+2$ variables $x, y, z_1, \ldots, z_n, p_1, \ldots, p_n, q_1, \ldots, q_n$. For each simple root of the equation $\Delta = 0$, one such set arises; and there is a corresponding set, similar in form but larger in number, for any multiple root of $\Delta = 0$. The application of the method will be sufficiently indicated for a system of equations such that all the roots of $\Delta = 0$ are simple: it is given by the theorem:—

Assuming that all the n roots of $\Delta = 0$ are simple, let $v = c$ be an integral of the subsidiary system, distinct from $f_1 = 0, \ldots, f_n = 0$, and associated with a root μ; and suppose that, on taking the n roots μ in succession, the successive subsidiary systems give integrals

$$v_1 = c_1, \ldots, v_n = c_n,$$

these equations being distinct from $f_1 = 0, \ldots, f_n = 0$, quà functions of $p_1, \ldots, p_n, q_1, \ldots, q_n$. When the $2n$ equations

$$f_1 = 0, \ldots, f_n = 0, \quad v_1 = c_1, \ldots, v_n = c_n$$

are resolved for $p_1, \ldots, p_n, q_1, \ldots, q_n$, and the deduced values are substituted in

$$dz_\kappa = p_\kappa dx + q_\kappa dy,$$

for $\kappa = 1, \ldots, n$, the latter n equations become a completely integrable aggregate. The integral of this aggregate is an integral of the original system which, as it contains $2n$ arbitrary constants, is a complete integral.

174. This theorem, which is due to Hamburger, can be established as follows.

When $v=c$ is an integral of the subsidiary system associated with a root μ of $\Delta=0$, then the relation

$$dv=0$$

is satisfied in consequence of that system: that is, the relation

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\sum_{s=1}^{n}\left(\frac{\partial v}{\partial z_s}dz_s+\frac{\partial v}{\partial p_s}dp_s+\frac{\partial v}{\partial q_s}dq_s\right)=0$$

must be a consequence of the $n+3$ equations. When the $n+3$ equations are used to remove dy, dz_1, ..., dz_n, dp_1, dq_1 from the differential relation $dv=0$, the coefficients of the remaining differential elements must vanish; and therefore

$$\frac{\partial v}{\partial p_s}-\frac{P_s}{P_1}\frac{\partial v}{\partial p_1}=0,$$

$$\frac{\partial v}{\partial q_s}-\frac{P_s}{P_1}\frac{\partial v}{\partial q_1}=0,$$

for $s=2$, ..., n, together with

$$\left(\frac{dv}{dx}+\mu\frac{dv}{dy}\right)P_1-X\frac{\partial v}{\partial p_1}-Y\frac{\partial v}{\partial q_1}=0,$$

where

$$\frac{dv}{dx}=\frac{\partial v}{\partial x}+\sum_{i=1}^{n}\frac{\partial v}{\partial z_i}p_i,$$

$$\frac{dv}{dy}=\frac{\partial v}{\partial y}+\sum_{i=1}^{n}\frac{\partial v}{\partial z_i}q_i.$$

We thus have $2n-1$ equations, homogeneous and linear in the derivatives of v, and the number of arguments occurring is $3n+2$: hence the number of integrals common to the system may be anything from zero to $n+3$, according to the extra number of equations required to make the system complete. We shall assume that the conditions securing the existence of a variable non-trivial integral are satisfied: and we shall make this assumption for each of the roots of $\Delta=0$: so that there thus will arise n equations

$$v_1=c_1,\ \ldots,\ v_n=c_n.$$

Now let the $2n$ equations

$$f_1=0,\ \ldots,\ f_n=0,\quad v_1=c_1,\ \ldots,\ v_n=c_n$$

be resolved so as to express $p_1, \ldots, p_n, q_1, \ldots, q_n$ in terms of $x, y, z_1, \ldots, z_n$. When their values are substituted in the $2n$ equations, each of these becomes an identity; and therefore, from each equation $v_i = c_i$ thus changed, we have

$$\frac{\partial v_i}{\partial x} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial p_s}\frac{\partial p_s}{\partial x} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial q_s}\frac{\partial q_s}{\partial x} = 0,$$

$$\frac{\partial v_i}{\partial y} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial p_s}\frac{\partial p_s}{\partial y} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial q_s}\frac{\partial q_s}{\partial y} = 0,$$

$$\frac{\partial v_i}{\partial z_j} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial p_s}\frac{\partial p_s}{\partial z_j} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial q_s}\frac{\partial q_s}{\partial z_j} = 0,$$

for $j = 1, \ldots, n$. Writing

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \sum_{j=1}^{n} p_j \frac{\partial}{\partial z_j}, \quad \frac{d}{dy} = \frac{\partial}{\partial y} + \sum_{j=1}^{n} q_j \frac{\partial}{\partial z_j},$$

we have

$$\frac{dv_i}{dx} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial p_s}\frac{dp_s}{dx} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial q_s}\frac{dq_s}{dx} = 0,$$

$$\frac{dv_i}{dy} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial p_s}\frac{dp_s}{dy} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial q_s}\frac{dq_s}{dy} = 0,$$

and therefore

$$\frac{dv_i}{dx} + \mu\frac{dv_i}{dy} + \sum_{s=1}^{n} \frac{\partial v_i}{\partial p_s}\left(\frac{dp_s}{dx} + \mu\frac{dp_s}{dy}\right) + \sum_{s=1}^{n} \frac{\partial v_i}{\partial q_s}\left(\frac{dq_s}{dx} + \mu\frac{dq_s}{dy}\right) = 0.$$

When the formal equations satisfied by v_i are used, and the equivalent values of $\frac{\partial v_i}{\partial x} + \mu\frac{\partial v_i}{\partial y}$ and $\frac{\partial v_i}{\partial p_s}, \frac{\partial v_i}{\partial q_s}$, as given by those equations, are substituted in the last relation, it becomes

$$\frac{\partial v_i}{\partial p_1}\left[X + \sum_{s=1}^{n} P_s\left(\frac{dp_s}{dx} + \mu\frac{dp_s}{dy}\right)\right] + \frac{\partial v_i}{\partial q_1}\left[Y + \sum_{s=1}^{n} P_s\left(\frac{dq_s}{dx} + \mu\frac{dq_s}{dy}\right)\right] = 0;$$

and this relation, when regard is paid to the equations

$$\frac{Q_1}{P_1} = \ldots = \frac{Q_n}{P_n} = \mu,$$

can be transformed so that it becomes

$$\frac{\partial v_i}{\partial p_1}\left[X + \sum_{s=1}^{n}\left(P_s\frac{dp_s}{dx} + Q_s\frac{dp_s}{dy}\right)\right] + \frac{\partial v_i}{\partial q_1}\left[Y + \sum_{s=1}^{n}\left(P_s\frac{dq_s}{dx} + Q_s\frac{dq_s}{dy}\right)\right] = 0.$$

Further, each of the equations

$$f_j = 0,$$

for $j=1, \ldots, n$, becomes an identity when the values of $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ are substituted therein. Hence

$$\frac{\partial f_j}{\partial x}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial p_s}\frac{\partial p_s}{\partial x}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial q_s}\frac{\partial q_s}{\partial x}=0,$$

$$\frac{\partial f_j}{\partial y}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial p_s}\frac{\partial p_s}{\partial y}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial q_s}\frac{\partial q_s}{\partial y}=0,$$

$$\frac{\partial f_j}{\partial z_\mu}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial p_s}\frac{\partial p_s}{\partial z_\mu}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial q_s}\frac{\partial q_s}{\partial z_\mu}=0,$$

for $\mu=1, \ldots, n$; and therefore

$$\frac{df_j}{dx}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial p_s}\frac{dp_s}{dx}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial q_s}\frac{dq_s}{dx}=0,$$

$$\frac{df_j}{dy}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial p_s}\frac{dp_s}{dy}+\sum_{s=1}^{n}\frac{\partial f_j}{\partial q_s}\frac{dq_s}{dy}=0.$$

Multiplying these equations by l_j, and adding the respective equations for all values of j, we have

$$X+\sum_{s=1}^{n}P_s\frac{dp_s}{dx}+\sum_{s=1}^{n}Q_s\frac{dq_s}{dx}=0,$$

$$Y+\sum_{s=1}^{n}P_s\frac{dp_s}{dy}+\sum_{s=1}^{n}Q_s\frac{dq_s}{dy}=0.$$

When the values thus given for X and Y are substituted in the earlier equation which is homogeneous and linear in $\frac{\partial v_i}{\partial p_1}$ and $\frac{\partial v_i}{\partial q_1}$, it becomes

$$\frac{\partial v_i}{\partial p_1}\sum_{s=1}^{n}\left\{Q_s\left(\frac{dp_s}{dy}-\frac{dq_s}{dx}\right)\right\}+\frac{\partial v_i}{\partial q_1}\sum_{s=1}^{n}\left\{P_s\left(\frac{dq_s}{dx}-\frac{dp_s}{dy}\right)\right\}=0,$$

and therefore, as

$$Q_s=\mu P_s,$$

for $s=1, \ldots, n$, we have

$$\left(\mu\frac{\partial v_i}{\partial p_1}-\frac{\partial v_i}{\partial q_1}\right)\sum_{s=1}^{n}\left\{P_s\left(\frac{dp_s}{dy}-\frac{dq_s}{dx}\right)\right\}=0.$$

It is impossible, owing to the independence of $v_1, \ldots, v_n$, $f_1, \ldots, f_n$, quà functions of $p_1, \ldots, p_n, q_1, \ldots, q_n$, that the quantity

$$\mu\frac{\partial v_i}{\partial p_1}-\frac{\partial v_i}{\partial q_1}$$

shall be evanescent. For let

$$\frac{1}{P_1}\frac{\partial v_i}{\partial p_1}=M,$$

so that

$$\frac{\partial v_i}{\partial p_1} = MP_1,$$

and therefore, on account of the equations satisfied by v_i,

$$\frac{\partial v_i}{\partial p_s} = MP_s,$$

for $s = 2, \ldots, n$: then, if

$$\frac{\partial v_i}{\partial q_1} - \mu \frac{\partial v_i}{\partial p_1} = 0,$$

we should have

$$\begin{aligned} \frac{\partial v_i}{\partial q_1} &= \mu \frac{\partial v_i}{\partial p_1} \\ &= \mu MP_1 \\ &= MQ_1, \end{aligned}$$

and therefore, on account of the equations satisfied by v_i,

$$\frac{\partial v_i}{\partial q_s} = MQ_s,$$

for $s = 2, \ldots, n$. Now

$$P_s = \sum_{j=1}^{n} l_j \frac{\partial f_j}{\partial p_s}, \quad Q_s = \sum_{j=1}^{n} l_j \frac{\partial f_j}{\partial p_s};$$

consequently, we should have

$$\left\| \begin{array}{c} \dfrac{\partial v_i}{\partial p_1}, \ldots, \dfrac{\partial v_i}{\partial p_n}, \dfrac{\partial v_i}{\partial q_1}, \ldots, \dfrac{\partial v_i}{\partial q_n} \\ \dfrac{\partial f_1}{\partial p_1}, \ldots, \dfrac{\partial f_1}{\partial p_n}, \dfrac{\partial f_1}{\partial q_1}, \ldots, \dfrac{\partial f_1}{\partial q_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial f_n}{\partial p_1}, \ldots, \dfrac{\partial f_n}{\partial p_n}, \dfrac{\partial f_n}{\partial q_1}, \ldots, \dfrac{\partial f_n}{\partial q_n} \end{array} \right\| = 0.$$

Hence v_i, regarded as a function of $p_1, \ldots, p_n, q_1, \ldots, q_n$, would not be functionally distinct from $f_1, \ldots, f_n$, contrary to the hypothesis as to the actual construction of $v_1, \ldots, v_n$. Thus $\mu \frac{\partial v_i}{\partial p_1} - \frac{\partial v_i}{\partial q_1}$ is not evanescent; and therefore

$$\sum_{s=1}^{n} P_s \left(\frac{dp_s}{dy} - \frac{dq_s}{dx} \right) = 0,$$

that is,

$$\sum_{s=1}^{n} \left\{ \left(\frac{dp_s}{dy} - \frac{dq_s}{dx} \right) \sum_{j=1}^{n} \left(l_j \frac{\partial f_j}{\partial p_s} \right) \right\} = 0.$$

The quantities $l_1, \ldots, l_n$ are proportional to the first minors of any row of constituents in Δ, which has n distinct roots; and thus, taking the n sets of quantities $l_1, \ldots, l_n$ associated with the n roots of $\Delta = 0$ in succession, we have n independent sets such that the quantity

$$\begin{vmatrix} l_1^{(1)}, & \ldots, & l_n^{(1)} \\ \cdots & \cdots & \cdots \\ l_1^{(n)}, & \ldots, & l_n^{(n)} \end{vmatrix}$$

does not vanish, where $l_1^{(r)}, \ldots, l_n^{(r)}$ are the set associated with the root μ_r.

We thus have n equations, homogeneous and linear in the quantities

$$\frac{dp_1}{dy} - \frac{dq_1}{dx}, \ \ldots, \ \frac{dp_n}{dy} - \frac{dq_n}{dx}.$$

The determinant of their coefficients is

$$\begin{vmatrix} l_1^{(1)}, & \ldots, & l_n^{(1)} \\ \cdots & \cdots & \cdots \\ l_1^{(n)}, & \ldots, & l_n^{(n)} \end{vmatrix} J\left(\frac{f_1, \ \ldots, \ f_n}{p_1, \ \ldots, \ p_n}\right):$$

neither of the factors in this quantity vanishes: and therefore

$$\frac{dp_s}{dy} - \frac{dq_s}{dx} = 0,$$

for $s = 1, \ldots, n$. Consequently, the n equations

$$\left.\begin{array}{l} dz_1 = p_1 dx + q_1 dy \\ \cdots\cdots\cdots\cdots\cdots \\ dz_n = p_n dx + q_n dy \end{array}\right\},$$

where the values of $p_1, \ldots, p_n, q_1, \ldots, q_n$ are given by

$$f_1 = 0, \ \ldots, \ f_n = 0, \quad v_1 = c_1, \ \ldots, \ v_n = c_n,$$

are a completely integrable system: their integral equivalent contains $2n$ arbitrary constants: and it constitutes a complete integral of the original system

$$f_1 = 0, \ \ldots, \ f_n = 0.$$

Hamburger's theorem is thus established.

Note 1. When the complete integral has thus been obtained, the customary Lagrangian process of varying the parameters,

subject to the conservation of form of the equations, can be used in order to deduce other integrals from the complete integral.

Note 2. If only a number of integrals

$$v_1 = c_1, \ldots, v_m = c_m,$$

where $m < n$, of the various subsidiary systems are known, they can be used to eliminate m of the dependent variables, say $z_1, \ldots, z_m$, and also their derivatives, from $f_1 = 0, \ldots, f_n = 0$. The integration of the surviving equations is a problem of simpler extent than the original problem.

Note 3. The preceding theorem requires an integral

$$v_i = c_i$$

of a subsidiary system. That subsidiary system may have a number of functionally independent integrals

$$v_i^{(1)}, v_i^{(2)}, \ldots,$$

the number not being greater than $n + 3$: if the number is greater than unity, we replace the equation

$$v_i = c_i$$

by the equation

$$\phi_i (v_i^{(1)}, v_i^{(2)}, \ldots) = 0,$$

where ϕ_i is arbitrary. The argument then proceeds as before.

Note 4. The case, when the equations are linear in the derivatives and are of the form

$$f_i = -p_i + \pi_i + \sum_{s=1}^{n} a_{is} q_s = 0,$$

for $i = 1, \ldots, n$, being the case treated in the earlier sections of this chapter, is included simply in the general case. The critical equation $\Delta = 0$, being

$$\left\| \frac{\partial f_j}{\partial q_j} - \mu \frac{\partial f_j}{\partial p_j} \right\| = 0,$$

becomes

$$\begin{vmatrix} a_{11} + \mu, & a_{12} & , \ldots \\ a_{21} & , a_{22} + \mu, & \ldots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

that is, $\Theta = 0$, being the critical equation for the simpler case.

Note 5. When no one of the dependent variables occurs explicitly in the original system, then

$$\frac{df_i}{dx} = \frac{\partial f_i}{\partial x}, \quad \frac{df_i}{dy} = \frac{\partial f_i}{\partial y},$$

for $i = 1, \ldots, n$, so that the equations are simplified. In that case, it may be possible to construct $v_1, \ldots, v_n$, so that no one of them contains any of the dependent variables explicitly: each of the equations

$$dz_r = p_r dx + q_r dy, \qquad (r = 1, \ldots, n),$$

is then completely integrable by itself without reference to the other equations.

175. Sometimes a member of the final integral equivalent can be obtained more directly as follows. Let

$$u(x, y, z_1, \ldots, z_n) = 0$$

be one integral relation in the integral equivalent of a system

$$f_1 = 0, \ldots, f_n = 0;$$

then the equations

$$\frac{\partial u}{\partial x} + \sum_{i=1}^{n} p_i \frac{\partial u}{\partial z_i} = 0,$$

$$\frac{\partial u}{\partial y} + \sum_{i=1}^{n} q_i \frac{\partial u}{\partial z_i} = 0,$$

are satisfied in connection with the system. If then the quantities $p_1, \ldots, p_n$ are eliminated from the equation

$$\frac{\partial u}{\partial x} + \sum_{i=1}^{n} p_i \frac{\partial u}{\partial z_i} = 0,$$

by means of

$$f_1 = 0, \ldots, f_n = 0,$$

the resulting equation must effectively be the same as

$$\frac{\partial u}{\partial y} + \sum_{i=1}^{n} q_i \frac{\partial u}{\partial z_i} = 0.$$

When the latter equation is used to eliminate any of the quantities $q_1, \ldots, q_n$, say q_n (on the supposition that $\frac{\partial u}{\partial z_n}$ is not zero), from the transformed shape of

$$\frac{\partial u}{\partial x} + \sum_{i=1}^{n} p_i \frac{\partial u}{\partial z_i} = 0,$$

the result must be an identity: the necessary conditions that it should reduce to an identity are a number of relations among the derivatives of u, which accordingly are a set of simultaneous partial equations for the determination of u.

It is clear, however, that the process is only of limited application; for instance, if u be a function of x, y, z_1 only, it can be only in the case of equations of exceedingly special form that the transformation of the equation

$$\frac{\partial u}{\partial x}+p_1\frac{\partial u}{\partial z_1}=0$$

will lead to the equation

$$\frac{\partial u}{\partial y}+q_1\frac{\partial u}{\partial z_1}=0,$$

in a way that gives useful relations between the derivatives of u. Moreover, just as in the classical problem of the three bodies*, it is not a fact that any integral leads to an identically satisfied equation: it may only lead to a relation merely compatible with the others.

Should the method fail, then it is necessary to fall back upon the method given in Hamburger's general theory.

Ex. 1. Let it be required to integrate the equations

$$\left.\begin{aligned} p_1(xq_1+yq_2-xy)+q_1(y^2-yq_1-xq_2)=0\\ p_2(xq_1+yq_2-xy)+q_2(x^2-yq_1-xq_2)=0\\ p_3(xq_1+yq_2-xy)+q_3(y^2-yq_1-xq_2)=0\end{aligned}\right\}.$$

To test the method just suggested in § 175, we assume that

$$u=u(x, y, z_1, z_2, z_3)=0$$

is an equation in the integral equivalent: then we have

$$\frac{\partial u}{\partial x}+p_1\frac{\partial u}{\partial z_1}+p_2\frac{\partial u}{\partial z_2}+p_3\frac{\partial u}{\partial z_3}=0,$$

which must be consistent with the given equations, and therefore

$$(xy-xq_1-yq_2)\frac{\partial u}{\partial x}+q_1\frac{\partial u}{\partial z_1}(y^2-yq_1-xq_2)$$
$$+q_2\frac{\partial u}{\partial z_2}(x^2-yq_1-xq_2)+q_3\frac{\partial u}{\partial z_3}(y^2-yq_1-xq_2)=0.$$

* See vol. III of this work, §§ 265, 266.

This effectively must be the same as

$$\frac{\partial u}{\partial y}+q_1\frac{\partial u}{\partial z_1}+q_2\frac{\partial u}{\partial z_2}+q_3\frac{\partial u}{\partial z_3}=0,$$

and therefore the equation

$$(xy-xq_1-yq_2)\frac{\partial u}{\partial x}+q_2\frac{\partial u}{\partial z_2}(x^2-yq_1-xq_2)$$
$$=(y^2-yq_1-xq_2)\left(\frac{\partial u}{\partial y}+q_2\frac{\partial u}{\partial z_2}\right),$$

quà equation in q_1 and q_2, must be an identity. Hence

$$xy\frac{\partial u}{\partial x}=y^2\frac{\partial u}{\partial y},$$

$$x\frac{\partial u}{\partial x}=y\frac{\partial u}{\partial y},$$

$$-y\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial z_2}=y^2\frac{\partial u}{\partial z_2}-x\frac{\partial u}{\partial y},$$

from the terms independent of q_1 and q_2, from the coefficient of q_1, and from the coefficient of q_2, respectively: the other terms in q_1 and q_2 disappear of themselves. These three equations are equivalent to the two

$$x\frac{\partial u}{\partial z_2}+\frac{\partial u}{\partial y}=0,$$

$$y\frac{\partial u}{\partial z_2}+\frac{\partial u}{\partial x}=0,$$

which are a complete Jacobian system: they have three independent integrals, viz.

$$z_1,\quad z_2-xy,\quad z_3,$$

and therefore we should take

$$z_2-xy=g(z_3),$$
$$z_1=f(z_3),$$

where f and g are arbitrary functions.

It is conceivable that there should be an integral in the full integral equivalent which does not involve z_3: let it be

$$v(x,y,z_1,z_2)=0.$$

Then the two equations

$$\frac{\partial v}{\partial x}+p_1\frac{\partial v}{\partial z_1}+p_2\frac{\partial v}{\partial z_2}=0,$$

$$\frac{\partial v}{\partial y}+q_1\frac{\partial v}{\partial z_1}+q_2\frac{\partial v}{\partial z_2}=0,$$

must be treated in a similar manner. We substitute in the first for p_1 and p_2 by means of the original equations and, after substitution, we eliminate q_2

by means of the second: the result should be an identity. The necessary conditions are easily found to be

$$y\frac{\partial v}{\partial y}-x\frac{\partial v}{\partial x}+(y^2-x^2)\frac{\partial v}{\partial z_1}+\frac{1}{\dfrac{\partial v}{\partial z_2}}\left(y\frac{\partial v}{\partial x}-x\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial z_1}=0,$$

$$xy\frac{\partial v}{\partial x}-x^2\frac{\partial v}{\partial y}+\frac{1}{\dfrac{\partial v}{\partial z_2}}\left(y\frac{\partial v}{\partial x}-x\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}=0.$$

The second of these gives either

$$y\frac{\partial v}{\partial x}-x\frac{\partial v}{\partial y}=0,$$

or

$$x\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial y}=0.$$

Using the latter alternative, the first condition can be transformed to

$$\left(y\frac{\partial v}{\partial z_1}-\frac{\partial v}{\partial y}\right)\left(y\frac{\partial v}{\partial y}-x\frac{\partial v}{\partial x}\right)=0.$$

If we could have

$$y\frac{\partial v}{\partial z_1}-\frac{\partial v}{\partial y}=0,$$

together with

$$x\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial y}=0,$$

then completing the system, we should have

$$\frac{\partial v}{\partial z_1}=0,$$

and so $\frac{\partial v}{\partial y}=0$, $\frac{\partial v}{\partial z_2}=0$: no integral would be obtained. When we take

$$y\frac{\partial v}{\partial y}-x\frac{\partial v}{\partial x}=0,$$

together with

$$x\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial y}=0,$$

they become

$$x\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial y}=0,$$

$$y\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial x}=0,$$

the system already used. Thus we obtain no new integral from the alternative

$$x\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial y}=0.$$

Using the prior alternative

$$y\frac{\partial v}{\partial x}-x\frac{\partial v}{\partial y}=0,$$

the first equation becomes

$$y\frac{\partial v}{\partial y}-x\frac{\partial v}{\partial x}+(y^2-x^2)\frac{\partial v}{\partial z_1}=0:$$

the two equations can be replaced by

$$x\frac{\partial v}{\partial z_1}-\frac{\partial v}{\partial x}=0,$$

$$y\frac{\partial v}{\partial z_1}-\frac{\partial v}{\partial y}=0.$$

These are a complete Jacobian system: they have two independent integrals

$$z_1-\tfrac{1}{2}x^2-\tfrac{1}{2}y^2, \quad z_2;$$

accordingly, we take

$$z_1-\tfrac{1}{2}(x^2+y^2)=k(z_2),$$

where k is an arbitrary function.

Thus an integral equivalent of the original equations is

$$z_1-\tfrac{1}{2}(x^2+y^2)=k(z_2), \quad z_2-xy=g(z_3), \quad z_1=f(z_3),$$

where f, g, k are arbitrary functions.

Had it been impossible to obtain a third integral by the preceding process, the known integrals could have been used as follows.

Owing to the relation

$$z_1=f(z_3),$$

we have

$$p_1q_3-p_3q_1=0;$$

so that, when this integral is retained, the third differential equation is a consequence of the first and can therefore be neglected. Again, eliminating z_3 between

$$z_2-xy=g(z_3), \quad z_1=f(z_3),$$

let the result be

$$z_2-xy=\theta(z_1).$$

It is easy to verify that

$$p_2(xq_1+yq_2-xy)+q_2(x^2-yq_1-xq_2)$$
$$=\{p_1(xq_1+yq_2-xy)+q_1(y^2-yq_1-xq_2)\}\,\theta'(z_1),$$

so that the second differential equation is satisfied if the first is satisfied; it need not be retained when the first is retained. Substituting the value of z_2 in the first equation, we have

$$p_1q_1\{x+y\theta'(z_1)\}-q_1^2\{y+x\theta'(z_1)\}+(y^2-x^2)q_1=0$$

as the one equation to be satisfied, or neglecting $q_1=0$, we have

$$p_1\{x+y\theta'(z_1)\}-q_1\{y+x\theta'(z_1)\}=x^2-y^2.$$

This is an equation of Lagrangian form. To integrate it, we construct two integrals of the ordinary equations

$$\frac{dx}{x+y\theta'(z_1)}=\frac{dy}{-y-x\theta'(z_1)}=\frac{dz_1}{x^2-y^2}:$$

these are obtainable in the form

$$z_1 - \tfrac{1}{2}(x^2+y^2) = \text{constant},$$
$$xy + \theta(z_1) = \text{constant};$$

and therefore a general integral is given by

$$z_1 - \tfrac{1}{2}(x^2+y^2) = k\{xy+\theta(z_1)\},$$

where k is an arbitrary functional form. Hence

$$z_1 - \tfrac{1}{2}(x^2+y^2) = k(z_2),$$

agreeing with the former result.

Ex. 2. As an illustration of the general method, we still consider the same system as in the last example. It is easy to see that, as the equation for μ is

$$\begin{vmatrix} p_1x+y^2-2yq_1-xq_2-\mu\theta, & p_2x-yq_2, & p_3x-yq_3 \\ p_1y-xq_1, & p_2y+x^2-yq_1-2xq_2-\mu\theta, & p_3y-xq_3 \\ 0, & 0, & y^2-yq_1-xq_2-\mu\theta \end{vmatrix} = 0,$$

where θ denotes xq_1+yq_2-xy, one root is given by

$$\mu\theta = y^2-yq_1-xq_2.$$

The corresponding sets of quantities l_1, l_2, l_3 are such that

$$l_1(p_1x-yq_1)+l_2(p_2x-yq_2)+l_3(p_3x-yq_3)=0,$$
$$l_1(p_1y-xq_1)+l_2(p_2y-xq_2)+l_3(p_3y-xq_3)=0:$$

hence we may take

$$l_1 = p_2q_3-p_3q_2, \quad l_2=p_3q_1-p_1q_3, \quad l_3=p_1q_2-p_2q_1.$$

But by the given equations

$$p_3q_1-p_1q_3=0,$$

so that

$$l_2=0;$$

and then

$$\frac{l_3}{l_1} = -\frac{p_1}{p_3} = -\frac{q_1}{q_3}.$$

Again

$$X = l_1\frac{\partial f_1}{\partial x}+l_2\frac{\partial f_2}{\partial x}+l_3\frac{\partial f_3}{\partial x}=0,$$

$$Y = l_1\frac{\partial f_1}{\partial y}+l_2\frac{\partial f_2}{\partial y}+l_3\frac{\partial f_3}{\partial y}=0,$$

in the present case. Hence the equations for v are

$$\frac{\partial v}{\partial p_2}=0, \quad \frac{\partial v}{\partial q_2}=0,$$

$$\frac{\partial v}{\partial p_3}+\frac{p_1}{p_3}\frac{\partial v}{\partial p_1}=0,$$

$$\frac{\partial v}{\partial q_3}+\frac{q_1}{q_3}\frac{\partial v}{\partial q_1}=0,$$

$$(xq_1+yq_2-xy)\frac{dv}{dx}+(y^2-yq_1-xq_2)\frac{dv}{dy}=0.$$

We make the system complete: it then is

$$\frac{\partial v}{\partial p_2}=0,\ \frac{\partial v}{\partial q_2}=0,\ \frac{\partial v}{\partial x}=0,\ \frac{\partial v}{\partial y}=0,$$

$$\frac{\partial v}{\partial z_1}=0,\ \frac{\partial v}{\partial z_2}=0,\ \frac{\partial v}{\partial z_3}=0,$$

$$\frac{\partial v}{\partial p_3}+\frac{p_1}{p_3}\frac{\partial v}{\partial p_1}=0,\ \frac{\partial v}{\partial q_3}+\frac{q_1}{q_3}\frac{\partial v}{\partial q_1}=0.$$

The two integrals, independent so far as these equations are concerned, are

$$v=\frac{p_3}{p_1},\quad v=\frac{q_3}{q_1}.$$

But, from the original system of equations, $p_3q_1-p_1q_3=0$: hence we take

$$\frac{p_3}{p_1}=a,\ \frac{q_3}{q_1}=a.$$

Then

$$\begin{aligned}dz_3&=p_3dx+q_3dy\\&=a(p_1dx+q_1dy)\\&=adz_1;\end{aligned}$$

and therefore

$$z_3=az_1+b,$$

an integral of the complete type.

The discussion for the other values of μ is left as an exercise.

Ex. 3. Integrate the equations

$$\left.\begin{aligned}x(q_1+q_2-y)p_1&=yq_1^2+yq_1q_2-y^2q_1+(x^2-y^2)q_2\\x(q_1+q_2-y)p_2&=yq_1q_2+yq_2^2-x^2q_2\end{aligned}\right\}.$$

[The equation for μ is

$$\begin{vmatrix}\mu x(q_1+q_2-y)-xp_1+2yq_1+yq_2-y^2, & -xp_2+yq_2\\ -xp_1+yq_1+x^2-y^2, & \mu x(q_1+q_2-y)-xp_2+yq_1+2yq_2-x^2\end{vmatrix}=0;$$

and the two values of μ are given by

$$\mu x(q_1+q_2-y)=-y(q_1+q_2)+x^2,$$

$$\mu x(q_1+q_2-y)=-2y(q_1+q_2)+x(p_1+p_2)+y^2.$$

Integrals of the complete type are

$$\left.\begin{aligned}z_1+az_2&=\tfrac{1}{2}(x^2+y^2)+b\\z_1+z_2&=a'xy+b'\end{aligned}\right\};$$

and integrals of the general type are

$$\left.\begin{aligned}z_1-\tfrac{1}{2}(x^2+y^2)&=f(z_2)\\z_1+z_2&=g(x,y)\end{aligned}\right\},$$

where f and g are arbitrary functions.]

Ex. 4. Integrate the equations

$$\left.\begin{aligned}\lambda p_1&=q_1q_3(q_1y+q_2)\\ \lambda p_2&=-yq_1q_2q_3+q_2(x-q_2)(1-q_3)\\ \lambda p_3&=-q_3(1-q_3)(q_1y+q_2)\end{aligned}\right\},$$

where λ denotes $q_1(q_2+xq_3-x)$.

Ex. 5. Merely as an indication that the general method is not always effective, consider the comparatively simple pair of equations

$$\left.\begin{aligned}f_1&=xp_1+yq_2-1=0\\ f_2&=yp_2+xq_1-1=0\end{aligned}\right\}.$$

The equation for μ becomes

$$\mu^2-1=0;$$

the relation between the quantities l_1 and l_2 is

$$l_2=l_1\mu.$$

Also

$$P_1=l_1x,\qquad P_2=l_2y,$$

$$X=l_1(p_1+q_1),\quad Y=l_2(p_2+q_2).$$

The differential equations for v are

$$\frac{\partial v}{\partial p_2}=\frac{l_2y}{l_1x}\frac{\partial v}{\partial p_1}=\mu\frac{y}{x}\frac{\partial v}{\partial p_1},$$

$$\frac{\partial v}{\partial q_2}=\mu\frac{y}{x}\frac{\partial v}{\partial q_1},$$

$$\frac{dv}{dx}+\mu\frac{dv}{dy}-\frac{p_1+q_1}{x}\frac{\partial v}{\partial p_1}-\frac{p_2+q_2}{x}\frac{\partial v}{\partial q_1}=0.$$

First, let $\mu=1$, so that the equations are

$$\frac{\partial v}{\partial p_2}-\frac{y}{x}\frac{\partial v}{\partial p_1}=0,\quad \frac{\partial v}{\partial q_2}-\frac{y}{x}\frac{\partial v}{\partial q_1}=0,$$

$$\frac{\partial v}{\partial x}+p_1\frac{\partial v}{\partial z_1}+p_2\frac{\partial v}{\partial z_2}+\frac{\partial v}{\partial y}+q_1\frac{\partial v}{\partial z_1}+q_2\frac{\partial v}{\partial z_2}$$
$$-\frac{p_1+q_1}{x}\frac{\partial v}{\partial p_1}-\frac{p_2+q_2}{x}\frac{\partial v}{\partial q_1}=0.$$

When we complete the system, it becomes

$$\frac{\partial v}{\partial p_1}=0,\ \frac{\partial v}{\partial p_2}=0,\ \frac{\partial v}{\partial q_1}=0,\ \frac{\partial v}{\partial q_2}=0,\ \frac{\partial v}{\partial z_1}=0,\ \frac{\partial v}{\partial z_2}=0,$$

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}=0:$$

the only integral is

$$v=v_1=x-y,$$

and it is useless for the application of the process.

Similarly, when $\mu=-1$, it appears that the only derivable value of v is

$$v=v_2=x+y,$$

and it is useless for the application of the process. In fact, p_1, q_1, p_2, q_2 cannot be deduced from

$$f_1=0, \quad f_2=0, \quad v_1=c_1, \quad v_2=c_2.$$

It may be remarked that, if we write

$$z_1=z+\log(x+y),$$

and eliminate z_2 between the two equations, then z satisfies an equation of the second order which (in the usual notation) is

$$r+\frac{p}{x}=t-\frac{q}{y}.$$

This equation (as may easily be verified) does not possess an intermediate integral. When we take

$$x+y=2u, \quad x-y=2v,$$

the equation becomes

$$\frac{\partial^2 z}{\partial u\,\partial v}-\frac{u}{u^2-v^2}\frac{\partial z}{\partial u}+\frac{v}{u^2-v^2}\frac{\partial z}{\partial v}=0,$$

which is of Laplace's linear type having equal invariants, hereafter to be considered.

Can the Jacobian process be generalised?

176. The preceding investigations of Hamburger shew that, for limited classes of equations*, it is possible to construct auxiliary systems suggested by the analogy of the subsidiary equations constructed in association with a linear equation. And it is precisely this linear form which has made the process effective for the appropriate equations. Moreover, when the original simultaneous equations propounded for integration are not linear, Hamburger's method is to change the set into an amplified set with an amplified aggregate of dependent variables, the new set being linear.

It is natural to enquire whether, as the Lagrangian subsidiary equations for a single linear equation in a single dependent variable have thus been generalised so as to be associable with a number of linear equations in the same number of dependent variables, there is any corresponding possibility of generalising the Jacobian method of proceeding with a single equation. It is, however, possible to see, from even the simplest case, that the

* The limitations are imposed by the hypothesis that the equations in the auxiliary systems are so far consistent with one another as to possess one or more integrals: also, there are only two independent variables.

generalisation of the Jacobian method requires, in order to be effective, a process which is of too high an order for the applicability of analysis in its present stage of attainment.

Thus let

$$f = f(z_1, z_2, p_1, q_1, p_2, q_2, x, y) = 0,$$
$$g = g(z_1, z_2, p_1, q_1, p_2, q_2, x, y) = 0,$$

be propounded as a couple of equations, algebraically independent of one another; and let

$$h = h(z_1, z_2, p_1, q_1, p_2, q_2, x, y) = \text{constant}$$

be an equation compatible with them. Then, denoting the second derivatives of z_1 and z_2 by r_1, s_1, t_1, and r_2, s_2, t_2 respectively, we have

$$\frac{\partial f}{\partial p_1} r_1 + \frac{\partial f}{\partial q_1} s_1 + \frac{\partial f}{\partial p_2} r_2 + \frac{\partial f}{\partial q_2} s_2 + \frac{df}{dx} = 0,$$
$$\frac{\partial g}{\partial p_1} r_1 + \frac{\partial g}{\partial q_1} s_1 + \frac{\partial g}{\partial p_2} r_2 + \frac{\partial g}{\partial q_2} s_2 + \frac{dg}{dy} = 0,$$
$$\frac{\partial h}{\partial p_1} r_1 + \frac{\partial h}{\partial q_1} s_1 + \frac{\partial h}{\partial p_2} r_2 + \frac{\partial h}{\partial q_2} s_2 + \frac{dh}{dy} = 0;$$

the elimination of r_1 and r_2 leads to the equation

$$\frac{\partial(f, g, h)}{\partial(p_1, p_2, q_1)} s_1 + \frac{\partial(f, g, h)}{\partial(p_1, p_2, q_2)} s_2 + \frac{\partial(f, g, h)}{\partial(p_1, p_2, x)} = 0.$$

Similarly, we have the equation

$$\frac{\partial(f, g, h)}{\partial(q_1, q_2, p_1)} s_1 + \frac{\partial(f, g, h)}{\partial(q_1, q_2, p_2)} s_2 + \frac{\partial(f, g, h)}{\partial(p_1, p_2, x)} = 0.$$

It is generally impossible to eliminate s_1 and s_2 between these two equations. Thus, by associating only a single equation with a given pair, it is not possible to generalise Jacobi's process.

If, however, a second equation, say

$$k = k(z_1, z_2, p_1, q_1, p_2, q_2, x, y) = \text{constant},$$

can be associated with the first pair and with the equation $g = \text{constant}$, so that

$$\frac{\partial k}{\partial p_1} r_1 + \frac{\partial k}{\partial q_1} s_1 + \frac{\partial k}{\partial p_2} r_2 + \frac{\partial k}{\partial q_2} s_2 + \frac{dk}{dx} = 0,$$

we find

$$\frac{\partial(f, g, h, k)}{\partial(p_1, q_1, p_2, q_2)} s_1 + \frac{\partial(f, g, h, k)}{\partial(p_1, x, p_2, q_2)} = 0,$$

$$\frac{\partial(f, g, h, k)}{\partial(p_1, q_1, p_2, q_2)} s_2 + \frac{\partial(f, g, h, k)}{\partial(p_1, q_1, p_2, x)} = 0.$$

Similarly, we should have

$$\frac{\partial(f, g, h, k)}{\partial(p_1, q_1, p_2, q_2)} s_1 + \frac{\partial(f, g, h, k)}{\partial(y, q_1, p_2, q_2)} = 0,$$

$$\frac{\partial(f, g, h, k)}{\partial(p_1, q_1, p_2, q_2)} s_2 + \frac{\partial(f, g, h, k)}{\partial(p_1, q_1, y, q_2)} = 0.$$

Consequently,

$$\frac{\partial(f, g, h, k)}{\partial(x, p_1, p_2, q_2)} + \frac{\partial(f, g, h, k)}{\partial(y, q_1, p_2, q_2)} = 0,$$

$$\frac{\partial(f, g, h, k)}{\partial(x, p_2, p_1, q_1)} + \frac{\partial(f, g, h, k)}{\partial(y, q_2, p_1, p_2)} = 0,$$

which may be regarded as two equations for the determination of h and k. The first of them secures the relation

$$\frac{\partial p_1}{\partial y} = \frac{\partial q_1}{\partial x};$$

the second secures the relation

$$\frac{\partial p_2}{\partial y} = \frac{\partial q_2}{\partial x};$$

and the two equations are thus the necessary and sufficient conditions for integrability.

The two equations are also two equations for the determination of h and k, being lineo-linear in the derivatives of these quantities. They are simpler in form than the original equations $f = 0$, $g = 0$: yet, even so, the integration of the two equations appears to be an operation of the second order, which is not resoluble into operations of the first order.

Thus the Jacobian process cannot be generalised when there are two, or more than two, dependent variables without requiring for its completion operations of higher order than are required for Hamburger's generalisation of Lagrange's process.

Note. It may happen that three equations

$$f = 0, \quad g = 0, \quad h = \text{constant},$$

are given: conditions as to coexistence must be satisfied. The two preceding equations can be regarded as simultaneous equations for the determination of one dependent variable: the conditions that they possess a common integral will be the conditions for the coexistence of the three equations. When these are satisfied, k can be determined by the usual processes: and then the equations

$$f=0, \quad g=0, \quad h=\text{constant}, \quad k=\text{constant},$$

give values of p_1, q_1, p_2, q_2 which make

$$dz_1 = p_1 dx + q_1 dy, \quad dz_2 = p_2 dx + q_2 dy,$$

a completely integrable system.

Ex. Shew that, if n dependent variables $z_1, \ldots, z_n$ are to be determined in terms of two independent variables x and y by means of n partial differential equations of the first order

$$f_1=0, \ \ldots, \ f_n=0,$$

and if other n equations

$$v_1=c_1, \ \ldots, \ v_n=c_n,$$

where $c_1, \ldots, c_n$ are constants, can be associated with them, then the equations

$$\frac{\partial(f_1, \ldots, f_n, v_1, \ldots, v_n)}{\partial(x, p_r, p_1, q_1, \ldots, p_n, q_n)} + \frac{\partial(f_1, \ldots, f_n, v_1, \ldots, v_n)}{\partial(y, q_r, p_1, q_1, \ldots, p_n, q_n)} = 0,$$

for $r=1, \ldots, n$, derivatives with regard to q_r not appearing in the first expression and those with regard to p_r not appearing in the second expression, are satisfied.

177. In a preceding example of very simple type (§ 175, Ex. 5), it was seen that the elimination of one of the dependent variables and its derivatives led to an equation of the second order. This result is partly due to the special form of the equations there given: that it is not true in general for two equations of the form

$$f(z_1, z_2, p_1, p_2, q_1, q_2, x, y)=0,$$

$$g(z_1, z_2, p_1, p_2, q_1, q_2, x, y)=0,$$

can easily be seen. Taking the derivatives of the first order of both equations, we have

$$\frac{df}{dx}=0, \quad \frac{df}{dy}=0, \quad \frac{dg}{dx}=0, \quad \frac{dg}{dy}=0,$$

which, with $f=0$ and $g=0$, are six equations: and as regards z_2 and its derivatives, the quantities that occur in them are z_2, p_2,

q_2, r_2, s_2, t_2, in the usual notation. Six quantities cannot usually be eliminated between six equations; and therefore we cannot infer that z_1 is usually determined by an equation of the second order. But if $f=0$ and $g=0$ do not explicitly involve the dependent variables, then only five quantities occur for elimination: in that case, there survives a single equation of the second order.

In the more general case, when the dependent variables do occur explicitly, we associate with the preceding six equations the set

$$\frac{d^2f}{dx^2}=0,\quad \frac{d^2f}{dxdy}=0,\quad \frac{d^2f}{dy^2}=0,\quad \frac{d^2g}{dx^2}=0,\quad \frac{d^2g}{dxdy}=0,\quad \frac{d^2g}{dy^2}=0,$$

thus making twelve in all. There are ten quantities to be eliminated; and therefore two equations will survive after the elimination, being two equations of the third order satisfied by z_1.

Ex. Prove that, if there be m dependent variables and two independent variables, and if there be m partial differential equations of the first order involving the dependent variables explicitly, then usually the lowest order of differential equation satisfied by a single variable is $2m-1$, and that usually the number of equations of that order satisfied by the single variable is m.

We shall return to the discussion of this matter in Chapter XXI. Meanwhile, these results, as well as other considerations, indicate that we require the theory of equations of order higher than the first; accordingly, we proceed to the consideration of that theory.

THEORY

OF

DIFFERENTIAL EQUATIONS.

THEORY

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PART IV.

PARTIAL DIFFERENTIAL EQUATIONS.

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CHAPTER XII.

General Integrals of Equations of Orders Higher than the First.

The present chapter is devoted to general explanations, connected with the existence-theorem and with the kinds of integrals that are possessed by equations of order higher than the first, particularly those equations having general integrals without partial quadratures. For the most part, though not entirely, the equations considered are of the second order in two independent variables. The discussion is based mainly upon the memoir of Ampère, quoted in § 179, and upon the first chapter of the memoir by Imschenetsky, quoted in § 180.

178. After the discussion of equations of the first order which involve only a single dependent variable, and a discussion of sets of equations of the first order which involve several dependent variables and are integrable by any generalisation of any process that is effective for equations involving only one dependent variable, the next subject for consideration is manifestly the theory of partial differential equations of the second order. We shall begin with the simplest aggregate of such equations; and, for that aggregate, we shall assume that there is only a single dependent variable z, and that there are only two independent variables x and y. Denoting the first derivatives of z by p and q, and the second derivatives by r, s and t, as usual, we may take an equation of the second order in the form

$$f(x, y, z, p, q, r, s, t)=0.$$

Such an equation certainly possesses integrals. We shall assume that f either is in a form or can be brought into a form which makes it a regular function of its arguments: in that case, we have seen that Cauchy's existence-theorem applies and that

integrals, characterised by certain properties, do exist. Thus there is an integral z determined by the characteristic properties:—

(i) it is a regular function of x and y within fields of variation round a and b, given by

$$|x-a| \leqslant \rho, \quad |y-b| \leqslant \rho,$$

where ρ is not infinitesimal;

(ii) when $x=a$, the integral z reduces to $\phi_0(y)$ and the derivative $\frac{\partial z}{\partial x}$ reduces to $\phi_1(y)$, where $\phi_0(y)$ and $\phi_1(y)$ are regular functions of y within the domain of b and are otherwise arbitrary.

There is a single condition, of a formal type, which must be satisfied, or the existence of the foregoing integral cannot be established: it is that, if

$$\phi_0(b)=c, \quad \phi_1(b)=\lambda, \quad \phi_0'(b)=\mu, \quad \phi_1'(b)=\beta, \quad \phi_0''(b)=\gamma,$$

the equation

$$f(a, b, c, \lambda, \mu, \theta, \beta, \gamma)=0,$$

regarded as an equation in θ, should have at least one simple root. When this condition is satisfied, each simple root θ determines an integral z as above: and the integral thus associated with that simple root is unique.

When the condition is not satisfied so that $f=0$, regarded as an equation in θ, has no simple root, then the existence of such an integral is not established. But it may then happen that $f=0$, regarded as an equation in γ, has simple roots. In that case, the theorem establishes the existence of an integral z, regular as before in the domain of a and b, but now such that, when $y=b$, the integral z reduces to $\psi_0(x)$ and the derivative $\frac{\partial z}{\partial y}$ reduces to $\psi_1(x)$, where $\psi_0(x)$ and $\psi_1(x)$ are regular functions of x within the domain of a and are otherwise arbitrary.

If, however, $f=0$, regarded as an equation in γ, has no simple roots and, as before, has no simple roots when regarded as an equation in θ, it may have simple roots when regarded as an equation in β. In that case, there is an integral of a similar type, obtainable most easily through a transformation of the independent variables.

Thus the equation is proved to possess an integral characterised by definite properties except only in the case where $f=0$, regarded as an equation in θ, β, γ in turn, possesses no simple roots, so that we should have

$$\frac{\partial f}{\partial \theta}=0, \quad \frac{\partial f}{\partial \beta}=0, \quad \frac{\partial f}{\partial \gamma}=0,$$

concurrently with $f=0$. Returning now to the differential equation

$$f(x, y, z, p, q, r, s, t)=0,$$

the quantities

$$\frac{\partial f}{\partial r}, \quad \frac{\partial f}{\partial s}, \quad \frac{\partial f}{\partial t}$$

will usually be variable quantities; and then values can usually be given to their arguments such that, while f vanishes for those values, not all the three quantities $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ vanish. It may, however, happen that there are values of the arguments which satisfy the four equations

$$f=0, \quad \frac{\partial f}{\partial r}=0, \quad \frac{\partial f}{\partial s}=0, \quad \frac{\partial f}{\partial t}=0;$$

and then the Cauchy existence-theorem does not apply. In that case, there may be variable values of z (and of x and y) which satisfy all the four equations: such values of z, if any, will be called *singular integrals*. In all other cases, the existence-theorem establishes the existence of an integral with the specified properties: as its existence was first established by Cauchy, it frequently is called the *Cauchy integral*.

It will be noticed that two arbitrary functions enter into the expression of the specified properties.

The form thus stated is the simplest form of the Cauchy integral, in so far that the initial conditions are in their simplest form. As indicated (§ 24) in the discussion of the existence theorem, the initial conditions can be taken in an ampler form as follows:

> For all the values of x and y satisfying a given relation that is not critical with regard to the form of the differential equation, that is, for all points of a given analytical plane

curve, the variable z and its derivative in any direction, that is not tangential to the curve, acquire values represented by arbitrarily assigned functions of x and y.

This undoubtedly is more general. However, as it arises through a transformation of the variables from the simpler case, and does not otherwise add any element of generality to the solution, we shall usually be content to take the initial conditions in their simpler form.

It will be convenient to assume, for the purpose of immediate discussion as well as for simplicity of statement, that the equation can be resolved with regard to r, so that it takes the form

$$r = g(x, y, z, p, q, s, t):$$

the original equation can be regarded as the aggregate of all these resolved equations. The Cauchy integral is then a regular function of the variables in the domain of a and b: two arbitrary functions $\phi_0(y)$ and $\phi_1(y)$, subject solely to the condition of being regular in the domain of b, affect its form: and it is a unique integral as satisfying these conditions.

But while it thus possesses arbitrarily assigned elements, which frequently can be specialised so as to include integrals otherwise obtained, there is no certainty that specialisation or definition of these elements will secure that the integral shall include every integral; and therefore there is no certainty that the Cauchy integral is completely comprehensive. A question thus arises as to whether the equation possesses any integral that is more comprehensive; a further question is stirred as to the different kinds of integral that the equation may possess. Even so, limiting assumptions have been made: all singularities and other deviations from regularity in the form of the original equation have been avoided.

Two Definitions of the General Integral.

179. It is usual to assign, to the most comprehensive integral known, the name of the *general integral*, for partial equations of order higher than the first; but there are two definitions of the general integral. One of these definitions is due* to Ampère;

* *Journ. de l'Éc. Polytechnique, cah.* XVII (1815), p. 550.

the other, given* by Darboux, is based upon the researches of Cauchy.

According to Ampère, an integral (no matter how obtained) is general when the only relations, which are free from the arbitrary elements and to which the integral leads among the variables and the derivatives of the dependent variable up to any order whatever, are those expressed by the differential equation and by equations deduced from the equation by differentiation. Thus, in this sense,

$$z = \phi(x+y) + \psi(x-y)$$

is a general integral of the equation

$$r = t;$$

for the relations to which the integral equation gives rise are

$$\frac{\partial^{2+m+n} z}{\partial x^{2+m} \partial y^n} = \frac{\partial^{2+m+n} z}{\partial x^m \partial y^{2+n}},$$

for all positive integer values of m and n, and all of these relations are derivatives of the differential equation. But

$$z = a(x^2+y^2) + 2hxy + bx + cy + d,$$

where a, b, c, d, h are arbitrary constants, is not a general integral of the same equation: for it satisfies relations

$$\frac{\partial^3 z}{\partial x^3} = 0, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 z}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 z}{\partial y^3} = 0,$$

no one of which can be derived from the differential equation, though derivatives of the differential equation are not inconsistent with the relations.

According to Darboux, an integral (no matter how obtained) is general if the arbitrary elements which it contains can be determined so as to give the Cauchy integral, involving assigned functional values to z and to one of its derivatives in specified circumstances. Thus, in this sense also,

$$z = \phi(x+y) + \psi(x-y)$$

is a general integral of the equation

$$r = t:$$

* *Théorie générale des surfaces*, t. II, pp. 97, 98.

for, if the initial conditions of the Cauchy integral are that $z = f(y)$ and $\frac{\partial z}{\partial x} = g(y)$, when $x = a$, then if

$$\sigma(u) = \tfrac{1}{2} f(u) + \tfrac{1}{2}\int_0^u g(u)\,du,$$

$$\rho(v) = \tfrac{1}{2} f(v) - \tfrac{1}{2}\int_0^v g(v)\,dv,$$

the required Cauchy integral is given by

$$z = \sigma(x + y - a) + \rho(a - x + y),$$

the functions ϕ and ψ thus having been appropriately determined.

The tenour of the Ampère definition of a general integral is different from that of the Cauchy general integral. Though the difference between the integrals is often of no account, yet for particular equations it can be significant: and therefore it is worth while to estimate which of the two integrals is the more comprehensive.

It seems clear that, in the matter of comprehensiveness, the Cauchy general integral has some advantage over the Ampère general integral.

The limitations, which are imposed in the course of establishing the Cauchy integral, are of a qualitative kind: they are restrictions as to the position and the extent of the domains within which the various functions that occur are regular, or they are hypotheses as to the resolubility of the differential equation: but no positive relations (other than derivatives of the differential equation) are used or are required in order to secure the convergence of the power-series obtained, or the continuity of the functions, or the freedom of the initial conditions. Consequently, an integral that is general in the sense of the Darboux-Cauchy definition is general also in the sense of Ampère's definition. As against this inference, it must be borne in mind that, however arbitrarily the initial conditions are chosen either as regards the position of the domain or the forms of the assigned functions, the Cauchy integral is always a regular function of the variables and that deviations from regularity have been excluded from consideration: there is no such restriction on the Ampère integral.

The restriction can often be partly removed by considering a part of a domain and by taking a regular expression for a branch of a non-regular integral in that region: but this modification is not always possible, and there are deviations from regularity such that the removal of the restriction cannot be made complete.

But on the other hand, classes of equations can be constructed which have integrals that are general in the Ampère sense and certainly are not general in the Darboux-Cauchy sense. It is true that such classes of equations are special in type, and that a similar difference need not exist for equations that are not of any special type: but the mere existence of such equations is a limitation upon the comparative comprehensiveness of the Ampère integral.

An instance is adduced* by Goursat in the example

$$s = yq:$$

simple quadratures lead to an integral

$$z = \theta(x) + \int_a^y e^{xu} \phi(u)\, du,$$

where θ and ϕ are arbitrary functions, and a is any constant. Now the quantity z', where

$$z' = \int_a^y e^{xu} \phi(u)\, du,$$

satisfies the differential equation: all the relations between the variables and the derivatives of z', which are free from arbitrary elements, are constituted by the differential equation and by derivatives of the differential equation. Thus z' is an integral of the differential equation which is general in the sense of Ampère's definition: it clearly is less comprehensive than the integral

$$z = \theta(x) + \int_a^y e^{xu} \phi(u)\, du,$$

which is easily seen to be general in the Darboux-Cauchy sense.

We shall return to a further discussion of the matter when dealing with linear equations of the second order. It is manifest that the foregoing explanations can be applied to equations of

* *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. II, p. 212. This treatise, when quoted hereafter, will be referred to as Goursat, *S. O.*

order higher than the second: and meanwhile, we may regard the general integral as one that is largely (though not universally) comprehensive.

CLASSES OF INTEGRALS.

180. Before proceeding to the discussion of certain properties of general integrals, taken according to either of the definitions just indicated, we shall mention some other classes of integrals and briefly outline some of their relations with one another*.

Speaking broadly, we may define an integral of a partial differential equation of the second (or of higher) order as a relation between the variables such that, in virtue of the relation itself and of derivatives from it, the differential equation is satisfied. When the integral relation does not involve derivatives of the dependent variable, it is sometimes called a *primitive*: the more frequent practice is not to give any special title to such a relation. When the integral relation does involve derivatives of the dependent variable, these derivatives being of order lower than that of the equation, the integral relation is usually called an *intermediate integral*: it has not been proved, and it is not a fact, that intermediate integrals are always possessed by differential equations of order higher than the first.

We have already referred to *general integrals*: after the provisional explanations and for the sake of simplicity, we shall regard the Ampère definition as giving a necessary qualification (though not a complete qualification) for a general integral. A *particular integral* is a special case of the general integral: it is distinguished by the property that while, in conjunction with its derivatives, it leads to the differential equation and to derivatives of the differential equation, it leads also to other differential equations not obtainable as derivatives of the differential equation.

It has been customary with writers, following Lagrange, to refer to *complete integrals* or *complete primitives*: Ampère however considered, and gave† reasons for considering, that such

* In this connection, reference may be made to the first chapter of Imschenetsky's memoir on equations of the second order with two independent variables, *Grunert's Archiv*, t. LIV (1872), pp. 209—360.

† See the memoir (p. 554) cited in § 179.

integrals are always particular integrals. Their special occurrence is due to the fact that Lagrange proceeded to a differential equation from an integral relation, by eliminating from the latter and from its derivatives, as many arbitrary elements as possible. Thus, let an integral equation

$$g(x, y, z, a_1, a_2, a_3, a_4, a_5) = 0$$

be given, involving five arbitrary constants. When the two first derivatives and the three second derivatives are formed, there are six equations in all: from these, the five constants can be eliminated and the eliminant may be a single equation of the second order. Also, if the constants be independent of one another, not more than a single equation will result, unless some functional combinations occur: and similarly, subject to the same exceptional occurrence of functional combinations, not more than five independent constants could be eliminated. Thus five is the greatest number of arbitrary constants which could be expected in an integral, when its generality depends on arbitrary constants alone; hence the name assigned* to such an integral by Lagrange.

There are other integrals of various types. We have seen that the equations

$$f = 0, \quad \frac{\partial f}{\partial r} = 0, \quad \frac{\partial f}{\partial s} = 0, \quad \frac{\partial f}{\partial t} = 0,$$

could be satisfied simultaneously; if they lead to an integral, on the analogy of the corresponding case for equations of the first order, it is called *singular*.

It is sufficiently clear that, just as in the case of equations of the first order where the more obvious classes of integrals are not sufficiently comprehensive to include all types of special integrals, so in the case of equations of higher order there will be integrals (which may be called *special*, for convenience), possessed by particular equations and not included in the preceding classes.

* A similar argument, applied to an equation

$$f(x, y, z, p, q, r, s, t) = 0,$$

shews that, if initial values chosen for x and y be regarded as pure constants, the values of the six quantities z, p, q, r, s, t for those initial values are connected by a single relation, so that it might be considered that there are five independent arbitrary constants at our disposal.

Frequently they will be peculiarly associated with the form of the equation when it is quite regular: when the equation is not regular, special integrals will frequently occur, particularly associated with singularities of the form or with other deviations from regularity*.

It is not unusual to attempt some classification of intermediate integrals, though this is the less important because such integrals do not always exist. Still, when they do exist, two classes are selected as being of wider range than others. If an intermediate integral involves one arbitrary function in its expression, it is usually called an *intermediate general integral.* If it involves two arbitrary independent constants (this being usually the greatest number of constants that can be eliminated from an integral and its two derivatives leading to an equation of the second order), it is sometimes called an *intermediate complete integral.* Well-known instances of equations possessing an intermediate general integral are provided by the equations

$$U(rt - s^2) + Rr + 2Ss + Tt = V,$$

when certain conditions are satisfied, the quantities R, S, T, U, V being functions of x, y, z, p, q. An instance of an equation possessing a complete intermediate integral is given by

$$(rt - s^2)^2 + (pt - qs)(ps - qr) = 0:$$

the intermediate integral is

$$c^2 q - cz + p = a,$$

where a and c are arbitrary constants.

Instances hereafter will occur freely in which it appears that a differential equation of the second order does not possess any intermediate integral. Thus the equation†

$$s = z$$

cannot possess an intermediate integral. Such an integral, if possessed, would have one of the forms

$$p = f(x, y, z, q), \quad q = g(x, y, z), \quad p = h(x, y, z).$$

* See the Supplementary Note, at the end of Chapter XVI.

† The example is quoted by Imschenetsky, (*l.c.*) p. 222, from Raabe.

If the first exists, the equation $s=z$ should be obtainable from

$$\left.\begin{aligned} p &= f \\ s &= \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + t\frac{\partial f}{\partial q} \\ r &= \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial q} \end{aligned}\right\}.$$

Obviously, the last of these relations cannot be used to obtain the equation $s=z$. It is clear that t must not occur: hence the second relation gives $\frac{\partial f}{\partial q}=0$, so that f does not involve q. Unless $\frac{\partial f}{\partial z}=0$, the second relation will reintroduce q, which ought not to remain if the equation $s=z$ is to be obtained. Hence f must not involve z: and then it is obvious that, as f does not involve z, no combination of

$$p=f, \quad s=\frac{\partial f}{\partial y},$$

can lead to $s=z$. Thus no relation of the form $p=f$ is an intermediate integral of the original equation. Similarly, no relation of the form $q=g(x, y, z)$ can be an intermediate integral of the original equation.

It must not however be assumed that, when an equation possesses a complete primitive, it necessarily possesses an intermediate complete integral. Let the primitive be

$$f(x, y, z, a_1, a_2, a_3, a_4, a_5)=0;$$

it is not generally possible to eliminate more than two of the arbitrary constants between $f=0$ and its two first derivatives: the eliminant generally contains three arbitrary constants, and thus it is not an intermediate complete integral. Indeed, the differential equation of the second order could not generally be deduced from the eliminant: for the three included constants could not generally be eliminated between the eliminant-equation and its two first derivatives. While this is the case in general, it is not the case universally: for constants may coalesce at either of the stages in such a way as to make the elimination possible.

The real importance of the intermediate integral, when it occurs, lies in the fact that it enables us to construct the primitive

of an equation of the second order by two operations of what may be called grade unity: ordinarily the construction of the primitive is an irresoluble operation of grade two.

Ex. The relations of the various integrals to one another can be illustrated simply by reference to the equation

$$r=t,$$

of which a general integral (as has already been seen) is given by

$$z=\phi(x+y)+\psi(x-y),$$

where ϕ and ψ are arbitrary functions.

A complete integral is given by

$$z=a+\beta x+\gamma y+\delta xy+\epsilon(x^2+y^2);$$

as it can be expressed in the form

$$\begin{aligned}z=a&+\tfrac{1}{2}(\beta+\gamma)u+\tfrac{1}{4}(2\epsilon+\delta)u^2\\&+\tfrac{1}{2}(\beta-\gamma)v+\tfrac{1}{4}(2\epsilon-\delta)v^2,\end{aligned}$$

where $u=x+y$, $v=x-y$, it is a particular form of the general integral. Also, for this complete integral,

$$p=\beta+\delta y+2\epsilon x,$$
$$q=\gamma+\delta x+2\epsilon y,$$

so that

$$p+q=\beta+\gamma+(2\epsilon+\delta)(x+y),$$
$$p-q=\beta-\gamma+(2\epsilon-\delta)(x-y),$$

both of which are intermediate integrals: for the former leads to

$$r+s=s+t,$$

and the latter leads to

$$r-s=-(s-t),$$

both of which are the original equation.

Another complete integral is given by

$$z=a+\beta x+\gamma y+\delta xy+\epsilon(x^3+3xy^2);$$

it also is a particular form of the general integral, because it can be expressed in the form

$$\begin{aligned}z=a&+\tfrac{1}{2}(\beta+\gamma)u+\tfrac{1}{4}\delta u^2+\tfrac{1}{2}\epsilon u^3\\&+\tfrac{1}{2}(\beta-\gamma)v-\tfrac{1}{4}\delta v^2+\tfrac{1}{2}\epsilon v^3,\end{aligned}$$

where $u=x+y$, $v=x-y$, as before. For this complete integral,

$$p=\beta+\delta y+3\epsilon(x^2+y^2),$$
$$q=\gamma+\delta x+6\epsilon xy,$$

so that

$$p+q=\beta+\gamma+\delta(x+y)+3\epsilon(x+y)^2,$$
$$p-q=\beta-\gamma-\delta(x-y)+3\epsilon(x-y)^2,$$

which are not in the form of intermediate integrals, as both of them involve effectively three arbitrary constants. Elimination, however, is possible from their derived equations: thus, from the first,

$$r+s=\delta+6\epsilon(x+y)=s+t,$$

there being a combination of quantities in the elimination.

Similarly,

$$p+q=\theta(x+y), \quad p-q=\chi(x-y),$$

are intermediate general integrals, θ and χ denoting arbitrary functions.

Character of the General Integral.

181. We have seen that the general integral of an equation of the second order involves two arbitrary functions in its expression: for when a position is selected, with only the limitation that it shall be on an assigned curve, the values of the dependent variable and one of its derivatives at any such position become assigned functions of the independent variables. Now the general integral, whether obtained in the Ampère sense or in the Darboux-Cauchy sense, must satisfy the tests implied in the Ampère definition: for we have seen that an integral, which is general in the Darboux-Cauchy sense, is general also in the Ampère sense.

Suppose, then, that the differential equation of the second order is of the customary form

$$f(x, y, z, p, q, r, s, t)=0:$$

and let derivatives of f, of all orders up to and including those of $n-2$ be formed, where $n \geqslant 2$: the total number of equations thus possessed is

$$\begin{aligned} &1+2+\ldots+(n-1) \\ &=\tfrac{1}{2}n(n-1). \end{aligned}$$

Each of these equations is free from all the arbitrary elements that occur in the integral: and the derivatives of the dependent variable that occur are of all orders, up to and including those of order n.

Next, consider the integral equation; and suppose that it is given in finite form, whether it furnishes the value of z explicitly or implicitly. When all the derivatives of the integral equation, of orders up to n inclusive, are formed, there are

$$\begin{aligned} &1+2+\ldots+(n+1), \\ &=\tfrac{1}{2}(n+1)(n+2), \end{aligned}$$

equations in all. When all the arbitrary elements are eliminated from this tale of $\frac{1}{2}(n+1)(n+2)$ equations, it is required (from the property that the integral is general, in Ampère's sense) that the resulting equations should be exactly equivalent to the preceding tale of $\frac{1}{2}n(n-1)$ equations derived from the differential equation. Hence the number of such elements to be eliminated is

$$\begin{aligned} &\tfrac{1}{2}(n+1)(n+2) - \tfrac{1}{2}n(n-1) \\ &= 2n+1, \end{aligned}$$

being therefore the number of arbitrary elements that occur in the integral equation and in derivatives from the integral equation, when derivatives of the dependent variable up to order n are formed.

It therefore appears that *a general integral in finite form contains arbitrary elements in such a manner that their number increases with successive differentiations of the integral.*

As a matter of fact it can be verified that, in even the simplest instances such as in the integral of $s=0$, the increase in the number of arbitrary elements arises through the introduction of new derivatives of arbitrary functions. The arguments of the two arbitrary functions, which occur in the general integral of an equation of the second order, may be different or they may be the same: it is to be noted that, when we proceed from derivatives of the integral equation in successive orders, two arbitrary elements (being the next higher derivatives of the arbitrary functions of specific arguments) are introduced at each successive stage.

Thus the general integral of the equation

$$r=t$$

is

$$z=\phi(x+y)+\psi(x-y):$$

the general integral (as will be proved later) of the equation

$$r-t=\frac{2p}{x}$$

is

$$z=\phi(x+y)+\psi(y-x)-x\{\phi'(x+y)-\psi'(y-x)\}:$$

where, in each case, ϕ and ψ are arbitrary functions. It is easy to verify the foregoing theorem for each of these equations.

Among the methods of analysis applied to partial equations of order higher than the first, there are two modes of occurrence of

arbitrary functions in an equation giving a general integral which arise more frequently than others.

In one of these modes, the arbitrary functions present themselves as possessed of determinate arguments which are expressed, explicitly or implicitly, in terms of the independent variables.

Thus, for the equation

$$r=t,$$

already quoted, the general integral is

$$z=\phi(x+y)+\psi(y-x),$$

where the arguments of the arbitrary functions ϕ and ψ are explicit functions of x and y.

The general integral of the equation

$$pqr=s(1+p^2)$$

is given by the elimination of u between the two equations

$$\left.\begin{array}{r} z-\phi(u)-ux-(1+u^2)^{\frac{1}{2}}\,f(y)=0 \\ \phi'(u)+x+u(1+u^2)^{-\frac{1}{2}}f(y)=0 \end{array}\right\};$$

the argument of the arbitrary function ϕ is an implicit function of x and y, affected also by the occurrence and the form of the arbitrary function f.

In the other of the modes referred to, the arbitrary functions present themselves as possessed of arguments involving parameters, which are subject to quadratures of a multiplicity dependent upon the number of independent variables. This mode of occurrence is frequent in the integrals of many of the partial differential equations of mathematical physics: and there are two distinct forms of this mode of occurrence, according as the parameter is independent, or is not independent, of the variables.

Thus the equation

$$r=q,$$

which effectively is an equation in Fourier's theory of the conduction of heat, is satisfied by

$$z=\int_{-\infty}^{\infty}\phi(x+2uy^{\frac{1}{2}})\,e^{-u^2}du:$$

the parameter u of integration is independent of x and y. Again, the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which is the equation obeyed by gravitational potential in free space, is satisfied* by

$$V=\int_0^{2\pi} f(z+ix\cos u+iy\sin u,\ u)\,du,$$

* Whittaker, *Math. Ann.*, t. LVII (1903), p. 337.

where f is an arbitrary function of its two arguments, and the parameter u of integration is independent of x and y. The equation

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0$$

is satisfied by*

$$V=\int_0^{2\pi}\int_0^{\pi} f(x_1 \sin u \cos v+x_2 \sin u \sin v+x_3 \cos u+ix_4,\ u,\ v)\, du\, dv,$$

and by†

$$V=\int_0^{2\pi} f(x_1 \cos t+x_2 \sin t+ix_3,\ x_1 \sin t-x_2 \cos t+ix_4,\ t)\, dt.$$

In all of these forms the parameters of integration are independent of the variables.

On the other hand, equations occur possessing integrals in which there are quadratures with regard to variables while all other magnitudes in the subject of quadrature are kept constant. Thus a primitive of the equation

$$r-t+\frac{4p}{x+y}=0$$

is

$$(x+y)\, z+e^{\frac{2y}{x+y}} F(x+y)+e^{\frac{2y}{x+y}}\int e^{-\frac{2y}{a}} f(2y-a)\, dy=0,$$

where, after the quadrature, a is to be replaced by $x+y$, and F, f are arbitrary functions: but it is to be noted that an equivalent form is

$$z+e^{\frac{2y}{x+y}} G(x+y)=g(y-x)+(x+y)\, g'(y-x)+(x+y)^2 g''(y-x)+\ldots,$$

provided the series on the right-hand side converges, the functions G and g being arbitrary. In the latter expression, the integral is in finite form as regards the function G but not as regards the function g.

PARTIAL QUADRATURES.

182. There is a fundamental difference between the two classes of equations thus constituted. In the latter class, there is a quadrature, either definite or indefinite, with regard to a parameter or a variable, while all other magnitudes occurring in the subject of quadrature remain constant: it is usual to describe these as *integrals with partial quadratures*. In the former class, there are no such quadratures; and it is usual to describe‡ such general integrals as *integrals without partial quadratures*.

* **Whittaker, *Math. Ann.*, t. LVII (1903), p. 345.**

† **Bateman, *Proc. Lond. M. S.*, Ser. 2, Vol. I (1904), p. 457.**

‡ **Ampère called them the *first class* (*l.c.*, p. 558): but he did not develope the classification, and so a more definite description is preferable.**

The difference between the two classes of equations is not solely formal, as regards the absence or the presence of partial quadratures: it affects the character of the dependent variable and its derivatives.

In the case of equations, the integrals of which do not involve partial quadratures, derivatives of the integral equations with regard to the independent variables introduce no new arguments but only direct derivatives of the functional forms that occur in the integral. But in the case of equations the integrals of which do involve partial quadratures, such derivation often leads to subjects of quadrature quite distinct from those that occur in the original integral; and so it can lead to new arguments for the functional forms.

Accordingly, the class of equations whose integrals do not involve partial quadratures is simpler than the class of equations whose integrals are affected by partial quadratures: and the properties of the simpler class have been more fully developed than those of the other.

As already remarked, the two types indicated are the two more usual modes of occurrence: but they are not completely comprehensive. For example, the arbitrary elements might be defined in connection with some other differential equation, either of lower order than the given equation and not involving more independent variables, or involving fewer independent variables than the given equation and not of higher order. It is at least conceivable that precise selection from such a mode of determining the arbitrary elements might lead to new classes of integrable equations. Even so, it is clear that the methods of occurrence of arbitrary functions are not completely exhausted.

183. As the class of equations, whose general integrals (in finite form as regards one or other of the arbitrary functions) do not involve partial quadratures, is thus marked off from the others and is the simplest of all, it is convenient to have a means of testing whether a given equation does or does not belong to the class. Such a test (not, however, absolute) was devised by Ampère, as follows: the explanation is associated with an equation of the second order in two independent variables, but the test is easily seen to be applicable to equations of higher orders also and to equations in a greater number of independent variables.

Let the equation be

$$f(x, y, z, p, q, r, s, t) = 0,$$

and suppose the equation so taken that f is a polynomial function of the arguments r, s, t. Let α be an argument of an arbitrary function, that occurs in the general integral in finite form; and transform the independent variables so that α becomes one of them, say to x and α, so that y is a function of x and α. Now

$$dq = s\,dx + t\,dy$$

$$= \left(s + t\frac{\partial y}{\partial x}\right) dx + t\frac{\partial y}{\partial \alpha} d\alpha,$$

so that

$$\frac{dq}{dx} = s + t\frac{\partial y}{\partial x},$$

$$\frac{dq}{d\alpha} = t\frac{\partial y}{\partial \alpha}.$$

Similarly,

$$\frac{dp}{dx} = r + s\frac{\partial y}{\partial x};$$

and therefore

$$s = \frac{dq}{dx} - t\frac{\partial y}{\partial x},$$

$$r = \frac{dp}{dx} - \frac{dq}{dx}\frac{\partial y}{\partial x} + t\left(\frac{\partial y}{\partial x}\right)^2,$$

and the value of t is given by

$$t = \frac{\dfrac{dq}{d\alpha}}{\dfrac{\partial y}{\partial \alpha}}.$$

Let substitution for r and s in terms of t be made in $f = 0$: and let the result, arranged in powers of t, be

$$P + Qt + Rt^2 + \dots + Xt^n = 0,$$

where $P, Q, \dots, X$ involve derivatives of p and q with regard to x but not derivatives with regard to α. The differential equation is to be satisfied identically when the integral is substituted in it. Also, as t has the value

$$\frac{dq}{d\alpha} \div \frac{\partial y}{\partial \alpha},$$

it contains a derivative of the arbitrary function of α in the general integral which is of at least one order higher than derivatives of that function occurring in derivatives of p and q with regard to x: that is, t contains a derivative of an arbitrary function that does not occur in $P, Q, \ldots, X$. In order therefore that the equation may be satisfied identically, we must have

$$P=0, \quad Q=0, \quad R=0, \ldots, X=0,$$

in virtue of the given equation. When these relations are consistent with one another, in virtue of the given equation and of a value of $\frac{\partial y}{\partial x}$, all the necessary conditions are satisfied. Unless they are satisfied, the equation cannot possess a general integral, which can be expressed in a finite form without partial quadratures as regards at least one of the arbitrary functions. But, though the conditions are necessary, we are not in a position to declare them sufficient to secure an integral of the character indicated; they only provide a qualifying test.

Moreover, as $\frac{\partial y}{\partial x}$ is the derivative of y with regard to x when α is constant, the value of $\frac{\partial y}{\partial x}$ determines α. We may take

$$\alpha = y - \int \frac{\partial y}{\partial x}\, dx,$$

or we may take α any function of the right-hand side: or if $\theta(x, y) = \text{constant}$ be an integral of the relation

$$dy - \frac{\partial y}{\partial x}\, dx = 0,$$

we may take

$$\alpha = \theta(x, y).$$

In any case, the value of $\frac{\partial y}{\partial x}$ determines the argument α: but in order to obtain α explicitly, an explicit value of $\frac{\partial y}{\partial x}$ in terms of y and x would require to be known.

We shall recur to this analysis in a later chapter (Chapter XVII).

Ex. 1. Consider the equation (§ 181)

$$r - t + \frac{4p}{x+y} = 0,$$

the general integral of which is of finite form as regards one arbitrary function, and may be free from partial quadratures though not then of finite form as regards the other arbitrary function.

Making the same assumptions as in the text, and writing

$$\theta = \frac{\partial y}{\partial x},$$

we find the Ampère conditions to be

$$\frac{dp}{dx} - \frac{dq}{dx}\theta + \frac{4p}{x+y} = 0,$$
$$\theta^2 - 1 = 0.$$

These are consistent with one another : and, as

$$\frac{dp}{dx} = r + s\theta, \quad \frac{dq}{dx} = s + t\theta,$$

they are consistent with the original equation. Hence the original equation may possess a general integral which is free from partial quadratures.

The two values of θ are 1, -1 : the two values of a for this equation can be taken

$$a = y + x, \quad a = y - x:$$

that is, the arbitrary functions which occur are functions of $y+x$ and $y-x$ respectively.

Ex. 2. Consider the equation *

$$st + x(rt - s^2)^2 = 0.$$

Making the assumptions in the text, and writing

$$\theta = \frac{\partial y}{\partial x},$$

we have

$$rt - s^2 = t\left(\frac{dp}{dx} + \frac{dq}{dx}\theta\right) - \left(\frac{dq}{dx}\right)^2,$$

so that, when substitution in the differential equation is effected, the equation becomes

$$x\left(\frac{dq}{dx}\right)^4 + t\left\{-2x\left(\frac{dq}{dx}\right)^2\left(\frac{dp}{dx} + \frac{dq}{dx}\theta\right) + \frac{dq}{dx}\right\} + t^2\left\{x\left(\frac{dp}{dx} + \frac{dq}{dx}\theta\right)^2 - \theta\right\} = 0.$$

If then the equation can have a general integral in finite form free from partial quadratures, we must have

$$x\left(\frac{dq}{dx}\right)^4 = 0,$$

$$\frac{dq}{dx}\left\{1 - 2x\frac{dq}{dx}\left(\frac{dp}{dx} + \frac{dq}{dx}\right)\theta\right\} = 0,$$

$$x\left(\frac{dp}{dx} + \frac{dq}{dx}\theta\right)^2 - \theta = 0.$$

* Ampère, *l.c.*, p. 608.

In order that these equations may be satisfied, we must have

$$\frac{dq}{dx}=0,$$

$$x\left(\frac{dp}{dx}\right)^2-\theta=0\,;$$

the three equations are consistent with one another in virtue of these two equations. These two equations are also consistent with the original equation: the first of them implies

$$s+t\theta=0,$$

and the second of them implies

$$x\,(r+s\theta)^2-\theta=0\,;$$

the elimination of θ between the two gives

$$x\,(rt-s^2)^2+st=0,$$

which is the original equation. Thus the two equations are consistent with one another and with the original equation: so far there is nothing to prevent the general integral from being expressible in a finite form without partial quadratures.

The equation

$$-\frac{s}{t}=\theta=\frac{\partial y}{\partial x}$$

corresponds to the determination of the argument a. The construction of the argument is a matter for later investigation; the immediate purpose is to test whether there is any reason to prevent the equation from possessing an integral of the required type.

Ex. 3. Prove that the equation

$$rt-a^2x^2t^2+py-qs=0,$$

where a is a constant, cannot have a general integral expressible in finite form without partial quadratures unless the integral also satisfies the equation

$$(py-qs)^2=a^2x^2q^2t^2. \qquad \text{(Ampère.)}$$

The Arbitrary Elements in General Integrals.

184. There are two properties of general integrals without partial quadratures that can be established: one of them relates to the number of arbitrary functions which can occur in a general integral of this type*: the other of them relates to the number of arguments occurring in an arbitrary function. It will now be proved *that the number of independent arbitrary functions in the*

* Ampère, (*l.c.*), p. 583; Imschenetsky, (in the memoir quoted in § 180), p. 236. See also a memoir by the author, *Proc. L. M. S.*, t. XXIX (1898), p. 5.

general integral of a differential equation of any order, when that general integral is of finite form and without partial quadratures, is equal to the order of the equation; also that, usually though not universally, the number of arguments in each such arbitrary function is less by unity than the number of independent variables.

Most of the preceding explanations have been concerned with equations of the second order in two independent variables. As the propositions just stated are of wider application, we shall assume that the differential equation is of order m in n independent variables.

Accordingly, we take an equation

$$F=0$$

of order m in n independent variables $x_1, \ldots, x_n$; as usual, the dependent variable is denoted by z.

The integral system will, of course, contain these variables. Suppose that, in addition to them, it involves a number k of variable quantities $\alpha, \beta, \gamma, \ldots$, these quantities being not necessarily independent of one another. As the integral system is to be equivalent to a single relation, which shall express z explicitly or implicitly in terms of the independent variables, that integral system must contain $k+1$ equations from which the k variable quantities $\alpha, \beta, \gamma, \ldots$ can be conceived as eliminable.

Further suppose, firstly, that g independent arbitrary functions $\phi, \psi, \ldots$ occur in the integral system and, secondly, that, as a rule, each such function has r arguments though in particular cases the number of arguments may be less than r. These arguments may be considered as connected with the k variable quantities $\alpha, \beta, \gamma, \ldots$. Also it may happen that derivatives (some or all up to a specified order) of the arbitrary functions with regard to their arguments occur in the integral system. Let the highest derivatives of ϕ, which thus occur, be of order p_ϕ; and similarly for the derivatives of the other arbitrary functions.

Using Ampère's test as to whether an integral of a differential equation is general, we construct all the derivatives of the integral system of all orders up to s inclusive, where s is any integer equal to or greater than m: from the aggregate of equations thus obtained, all the arbitrary elements are to be eliminated: the surviving equations are to be equivalent to the equation $F=0$ and to the

equations deduced from $F=0$ by forming its derivatives of all orders up to $s-m$ inclusive.

The quantities to be eliminated from the equations deduced from the integral system are of two groups: one group is constituted by the quantities α, β, γ, ... and their derivatives, the other by the arbitrary functions ϕ, ψ, ... and their derivatives.

As regards the total number in the former group, it is made up of α, β, γ, ... and their derivatives of all orders of all orders up to s inclusive: hence this number is

$$\frac{(s+n)!}{s!\,n!}k$$
$$=\frac{(s+1)(s+2)\dots(s+n)}{n!}k.$$

Some of the derivatives may vanish identically, and then corresponding quantities to be eliminated would not occur: the offset, to be allowed on this account, will be considered later.

Next, we require the total number of quantities connected with the arbitrary functions that have to be eliminated. In the case of any function ϕ, the highest derivative which occurs in the integral system is p_ϕ: if differentiation is being effected with regard to a variable not involved in any of the r arguments of ϕ, no new derivative is then introduced: but when the variable of differentiation is involved in one (or in more than one) of the r arguments of ϕ, then a new derivative is introduced. Now all the r arguments are variable magnitudes; hence the first differentiations of the integral system will introduce the various derivatives of ϕ of order $p_\phi+1$, the second differentiations will introduce those of order $p_\phi+2$, and so on up to the sth differentiations which will introduce those of order $p_\phi+s$; and each of these is a derivative with regard to some combination of the r arguments. Hence the total number of derivatives of ϕ in all (including ϕ itself) is

$$1+r+\frac{r(r+1)}{2!}+\dots+\frac{r(r+1)\dots(r+s+p_\phi-1)}{(s+p_\phi)!}$$
$$=\frac{(r+1)(r+2)\dots(r+s+p_\phi)}{(s+p_\phi)!}$$
$$=\frac{(s+p_\phi+1)(s+p_\phi+2)\dots(s+p_\phi+r)}{r!}.$$

Similarly for the derivatives of the other arbitrary functions: and therefore the total number of the derivatives of the arbitrary functions (including the arbitrary functions themselves) is

$$\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\},$$

where the summation is to be taken for the g arbitrary functions, and the number p may vary from term to term in this sum. This number really is an upper limit for any value of s. It may happen that, owing to the form of the integral system, not all these derivatives actually occur; there then would not be the corresponding quantities to be eliminated.

Consequently, the total number of quantities in the two groups, which have to be eliminated from the integral system and the equations deduced from the integral system, can be as great as

$$\frac{1}{n!}k(s+1)(s+2)\dots(s+n)$$
$$+\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\}.$$

We have seen, however, that there may be an offset, on account of possibly vanishing derivatives of arguments on the one hand, and of possibly non-occurrent derivatives of arbitrary functions on the other hand: let N denote the aggregate number of quantities of this kind within the range considered, which otherwise would be included in the preceding aggregate. Hence, if I is the total number of quantities to be eliminated from the integral system and the equations derived from it, then

$$I=\frac{1}{n!}k(s+1)(s+2)\dots(s+n)$$
$$+\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\}-N.$$

Next, when the integral system of $k+1$ equations is differentiated with regard to all possible combinations of the independent variables so as to give derivatives of z of all orders up to s inclusive, the complete tale of equations (including the original integral system) is J, where

$$J=\frac{(s+1)(s+2)\dots(s+n)}{n!}(k+1).$$

It is from these J equations that the foregoing number I of quantities must be eliminated, in order to give partial differential equations satisfied by z. Usually, the various eliminable quantities disappear singly during the elimination: in that case, the number of eliminant equations is $J-I$. But it may happen that some of the quantities disappear in a combination of several together, and also that this simultaneous disappearance may occur for several combinations: in that case, the number of eliminant equations will be increased say by S. Accordingly, we may say that the number of eliminant equations is

$$J-I+S:$$

each of them is a differential equation satisfied in virtue of the integral system, and s is the highest order of derivative that occurs.

Now by Ampère's test of a general integral, this aggregate of $J-I+S$ equations is to be an exact algebraic equivalent of the partial differential equation $F=0$ of order m and of the equations deduced from $F=0$ by effecting upon it all differentiations with respect to the independent variables of all orders up to $s-m$ inclusive, so that the deduced equations combined will involve derivatives of z of all orders up to s. The total number of equations in this set (including $F=0$) is

$$\frac{1}{n!}(s-m+1)(s-m+2)\ldots(s-m+n),$$

which number therefore must be equal to $J-I+S$. Thus we have the relation

$$\begin{aligned}&\frac{1}{n!}(s+1)(s+2)\ldots(s+n)+N+S\\ &-\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\ldots(s+p+r)\}\\ &=\frac{1}{n!}(s-m+1)(s-m+2)\ldots(s-m+n),\end{aligned}$$

which must hold for all integer values of s such that $s \geqslant m$.

Of the integers that occur in this equation, the various numbers $p_\phi, p_\psi, \ldots$ are given: no one of them depends upon s. Also m, the order of the original differential equation, and n, the number of independent variables, are known and do not depend upon s. The

number r, being the number of arguments in the arbitrary function, and the number g, being the number of arbitrary functions and also the number of different products in the summation typified by Σ, are not yet known: they do not, however, depend upon s. On the other hand, N and S may depend upon s, and, if they are different from zero, they usually do depend upon s; but for comparatively large values of s, both N and S are integers that are small compared with the number of quantities and of equations respectively in question.

With these explanations, let the preceding numerical relation be transformed so as to be of the form

$$\frac{1}{r!}\Sigma\{(s+p+1)(s+p+2)\dots(s+p+r)\}-N-S$$

$$=\frac{1}{n!}\{(s+1)(s+2)\dots(s+n)-(s-m+1)(s-m+2)\dots(s-m+n)\},$$

and let both sides of this form of the relation be expanded in descending powers of s. On the left-hand side, the term containing the highest power of s is

$$\frac{1}{r!}gs^r,$$

on the assumption (which will be made, after the statement made concerning N and S) that neither N nor S contains so high a power of s. On the right-hand side, the term in s^n disappears, and the coefficient of s^{n-1} is

$$\frac{1}{n!}\{1+2+\dots+n+(m-1)+(m-2)+\dots+(m-n)\}$$

$$=\frac{m}{(n-1)!}.$$

Hence

$$\frac{1}{r!}gs^r=\frac{m}{(n-1)!}s^{n-1},$$

which is to hold for all values of s. Consequently,

$$g=m,$$

that is, the number of arbitrary functions in a general integral without partial quadratures is equal to the order of the equation: and

$$r=n-1,$$

that is, the number of arguments in an arbitrary function can be one less than the number of independent variables.

These are the two propositions which were to be established.

Note 1. These propositions are of wide range: some special cases are worthy of special mention.

Let $m=1$: we infer that the general integral of a partial differential equation of the first order in n independent variables contains one arbitrary function which can have $n-1$ arguments.

Let $m=2$, $n=2$: we infer that the general integral of a partial differential equation of the second order in two independent variables, when it is free from partial quadratures, contains two arbitrary functions each of a single argument. But there is nothing in the preceding discussion to shew whether the two arguments are different or are the same.

Let $m=2$, $n=3$: we infer that the general integral of a partial differential equation of the second order in three independent variables, when it is free from partial quadratures, contains two arbitrary functions, each of two arguments. But there is nothing in the preceding discussion to shew what relation, if any, exists between the arguments.

Note 2. The preceding discussion has taken no account of the precise form of the equation $F=0$; and therefore it may be found not to apply to equations of special types. In such cases, all that can be inferred is that, if arbitrary functions contain $n-1$ arguments, the number of them in the general integral is not greater than the order of the equation: while this last property is not necessarily maintained if the functions contain fewer than $n-1$ arguments.

Ex. 1. Prove that the equations

$$x=F_1\{a, \beta, \phi(a), \phi'(a), \ldots, \phi^{(m)}(a)\},$$
$$y=F_2\{a, \beta, \phi(a), \phi'(a), \ldots, \phi^{(m)}(a)\},$$
$$z=F_3\{a, \beta, \phi(a), \phi'(a), \ldots, \phi^{(m)}(a)\},$$

cannot represent the general integral of an equation of the second order.

(Goursat.)

Ex. 2. Prove that each of the quantities

$$z=X_1+X_2+X_3+(x_2-x_3)X_1'+(x_3-x_1)X_2'+(x_1-x_2)X_3',$$
$$z=X_1+X_2+X_3-(2x_1+x_3)X_1'+(x_1+x_3)X_2'+(x_2-x_3)X_3',$$

where X_1, X_2, X_3 are arbitrary functions of x_1, x_2, x_3 respectively and X_1', X_2', X_3' are their derivatives, satisfies a partial differential equation of the second order: and apply Ampère's test to prove that, in neither case, is the integral a general integral.

Ex. 3. The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is satisfied by

$$\begin{aligned} u = f\,(ix + y\cos a + z\sin a) \\ + g\,(x\cos\beta + iy + z\sin\beta) \\ + h\,(x\cos\gamma + y\sin\gamma + iz), \end{aligned}$$

where a, β, γ are arbitrary constants, and where f, g, h are arbitrary functions: discuss the character of the integral thus given.

185. The two results, as regards the number of arbitrary functions and the number of arguments in an arbitrary function contained in a general integral without partial quadratures, can be brought into relation with Cauchy's existence-theorem for an equation of general order in any number of independent variables, as has already been done for an equation of the second order in two variables. It was proved (§ 25) that, for an equation

$$\frac{\partial^m z}{\partial x_1^{\,m}} = Z,$$

where Z is a regular function of all the variables and of all the derivatives (save only the derivative on the left-hand side) of all orders up to m inclusive, then an integral exists, which is a regular function of the variables in non-infinitesimal domains and is such that, when $x_1 = a_1$, the quantities

$$z,\quad \frac{\partial z}{\partial x_1},\ \ldots,\ \frac{\partial^{m-1} z}{\partial x_1^{\,m-1}}$$

become functions of $x_2, \ldots, x_n$, which are regular in the specified domains and otherwise are quite arbitrary. Thus m arbitrary functions occur: and each of them involves $n - 1$ arguments which, when $x_1 = a_1$, are algebraically equivalent to $x_2, \ldots, x_n$.

A more general form of the theorem is obtained merely by transformation of the variables, as follows: an integral exists of the same regular character as before which is such that, when any relation

$$u\,(x_1, \ldots, x_n) = 0$$

exists among the variables, the variable z and the $m-1$ derivatives of the successive orders $1, \ldots, m-1$ become arbitrarily assigned functions of $x_1, \ldots, x_n$. The number of arbitrary functions is the same as before, viz. it is m: the number of arguments is $n-1$, for the arbitrary functions in the initial conditions involve all the n variables subject to the single relation $u=0$.

It thus appears, whether from Cauchy's existence-theorem or from Ampère's investigation on general integrals without partial quadratures, that an equation of any order m contains m arbitrary functions in an integral. Two examples have been given (§ 184, Ex. 2) in which integrals occurred having three arbitrary functions in finite form without quadratures and yet satisfied equations of the second order in three variables: in those instances, the arbitrary functions each involved only a single argument: whereas Ampère's investigation and Cauchy's theorem alike insist on two arguments in the arbitrary functions which occur in the general integral of the equations in question.

It should however be remarked that the integral, as given in the establishment of the existence-theorem, is found as a converging series and usually cannot be changed so as to have a finite form: the Ampère investigation only deals with integrals that are in a finite form and are without partial quadratures.

No inference, however, can be deduced as to the number of arbitrary elements occurring in the explicit expression of an integral involving partial quadratures. As will be seen hereafter (§ 209), it is possible to express the general integral of a linear equation of the second order in terms of only one arbitrary function: the matter will be considered during the discussion of those linear equations.

Equation characteristic of the Argument of an Arbitrary Function in a General Integral.

186. There is still another result which can be obtained when the general integral of the equation is in finite form without partial quadratures.

Let the equation be

$$f(x, y, z, p, q, r, s, t)=0,$$

and suppose that its general integral is of the type specified. Differentiating m times with respect to y, and writing

$$z_{\mu,\nu} = \frac{\partial^{\mu+\nu} z}{\partial x^{\mu} \partial y^{\nu}},$$

we have

$$\frac{\partial f}{\partial r} z_{2,m} + \frac{\partial f}{\partial s} z_{1,m+1} + \frac{\partial f}{\partial t} z_{0,m+2} + U = 0,$$

where the derivatives in U are of order not higher than $m+1$.

Let α be the argument of an arbitrary function in the general integral and, assuming that α involves y, change the independent variables from x and y to x and α: then y is a function of x and α. Also let θ denote the value of $\frac{\partial y}{\partial x}$ when α is constant. Thus

$$\frac{dz_{0,m+1}}{dx} = z_{1,m+1} + z_{0,m+2}\,\theta,$$

$$\frac{dz_{1,m}}{dx} = z_{2,m} + z_{1,m+1}\,\theta,$$

using the same notation as in § 183 for new derivations with regard to x: whence, substituting for $z_{1,m+1}$ and $z_{2,m}$, we have the above equation in the form

$$\left(\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r}\right) z_{0,m+2} + U + \frac{\partial f}{\partial s}\frac{dz_{0,m+1}}{dx} + \frac{\partial f}{\partial r}\left(\frac{dz_{1,m}}{dx} - \theta \frac{dz_{0,m+1}}{dx}\right) = 0.$$

Now, for all values of n, we have

$$\frac{d^n z}{d\alpha^n} = z_{0,n}\left(\frac{\partial y}{\partial \alpha}\right)^n + \ldots,$$

the unspecified derivatives of z being of order less than n. Substituting for $z_{0,m+2}$ and for the other derivatives according to similar formulæ of transformation, we have a term in $\frac{d^{m+2} z}{d\alpha^{m+2}}$ of which

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r}$$

is the coefficient. This quantity $\frac{d^{m+2} z}{d\alpha^{m+2}}$ does not occur elsewhere in the equation, the other derivatives with regard to α being of order

not higher than $m+1$; when the value of z is substituted, the quantity $\dfrac{d^{m+2}z}{d\alpha^{m+2}}$ introduces a derivative of the arbitrary function of α, which is of order higher than any other derivative that occurs in the equation. But the equation must be identically satisfied when this value of z, given by the general integral, is substituted; and therefore the term involving the highest derivative of the arbitrary function of α in the general integral must vanish, that is,

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0.$$

Also, θ is the value of $\dfrac{\partial y}{\partial x}$ when α is constant, so that

$$\frac{\partial \alpha'}{\partial x} + \frac{\partial \alpha}{\partial y} \theta = 0:$$

hence

$$\frac{\partial f}{\partial r}\left(\frac{\partial \alpha}{\partial x}\right)^2 + \frac{\partial f}{\partial s}\frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y} + \frac{\partial f}{\partial t}\left(\frac{\partial \alpha}{\partial y}\right)^2 = 0,$$

being an equation satisfied by α.

The same equation would be satisfied by the argument, say β, of the other arbitrary function: and it may happen that α and β are the same.

We have said that the equation is satisfied by α and by β. In general, derivatives of z would occur in this equation; and it could not be used for the immediate determination of α and β. But if f is linear in r, s, t, and has functions of x and y only for the coefficients of r, s, t, then the determination of α and β is effected by integrating

$$\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial f}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial f}{\partial s} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial f}{\partial t} = 0,$$

a partial differential equation of the first order.

Short of this actuality, however, which belongs to only a restricted class of equations, we can make other inferences from the equation

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0,$$

which is found to recur continually in the investigations on $f = 0$.

Let m and n be its roots, when it is regarded as a quadratic in θ: then we may take

$$\frac{\partial \alpha}{\partial x} + m \frac{\partial \alpha}{\partial y} = 0, \qquad \frac{\partial \beta}{\partial x} + n \frac{\partial \beta}{\partial y} = 0.$$

Various cases occur.

(i) Let m and n be distinct from one another, both being finite. There are two distinct arguments: and the two arbitrary functions in the general integral have these for their respective arguments. The only condition is one of inequality, viz. that

$$\left(\frac{\partial f}{\partial s}\right)^2 - 4 \frac{\partial f}{\partial r} \frac{\partial f}{\partial t}$$

does not vanish.

(ii) Let m and n be the same, and be finite. There is only one quantity: it is the common argument of the two arbitrary functions in the general integral. The condition

$$\left(\frac{\partial f}{\partial s}\right)^2 - 4 \frac{\partial f}{\partial r} \frac{\partial f}{\partial t} = 0$$

must be satisfied.

(iii) If m is zero and n finite though not zero, then α is a function of y alone, and β is not thus restricted. In this case,

$$\frac{\partial f}{\partial t} = 0:$$

hence, when the equation does not involve t explicitly, one of the arbitrary functions in the general integral involves y alone.

(iv) If m is infinite and n is finite though not zero, then α is a function of x alone and β is not thus restricted. In this case,

$$\frac{\partial f}{\partial r} = 0:$$

so that, when the equation does not involve r explicitly, one of the arbitrary functions in the general integral involves x alone.

(v) If m is infinite and n is zero, then α is a function of x only, and β is a function of y only. In this case,

$$\frac{\partial f}{\partial r} = 0, \qquad \frac{\partial f}{\partial t} = 0,$$

so that r and t do not occur explicitly. Resolving the equation with regard to s, we have it in the form

$$s = g(x, y, z, p, q):$$

when it possesses a general integral without partial quadratures, the arguments of the two arbitrary functions in that general integral are respectively x and y. In particular, when the equation

$$s + ap + bq + cz = 0,$$

a, b, c being functions of x and y only, has a general integral without partial quadratures, the arbitrary functions in that general integral are X and Y, arbitrary functions of x and of y respectively.

(vi) If both m and n are zero, then α and β are functions of y alone. In this case,

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial s} = 0,$$

so that s and t do not occur explicitly. Moreover, q cannot then occur explicitly: for, as

$$z = F(x, y, \alpha, \alpha', \ldots, \beta, \beta', \ldots),$$

the occurrence of q in the differential equation would give rise to derivatives of α and of β which (on the assumption that F is in finite form) are of order higher than the derivatives of those quantities found in z, p, or r. Thus the equation can only be of the form

$$f(x, y, z, p, r) = 0:$$

which effectively is an ordinary equation of the second order, having z for its dependent variable, x for its independent variable, and having two arbitrary functions of the parametric variable y for the two arbitrary elements in its integral.

(vii) If both m and n are infinite, then α and β are functions of x alone. The case is similar to the last case with the interchange of x and y, with the corresponding interchanges: in particular, the original differential equation is

$$f(x, y, z, q, t) = 0.$$

Ex. 1. When the equation is

$$r=t,$$

the equation for the arguments is

$$\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial y}\right)^2=0:$$

that is, the arguments of the arbitrary functions in the general integral are

$$\alpha=x+y, \quad \beta=x-y.$$

Ex. 2. The equation

$$(b+cq)^2 r-2(b+cq)(a+cp)s+(a+cp)^2 t=0$$

satisfies the Ampère tests (§ 183): it may therefore have a general integral in finite form without partial quadratures. The arguments of the arbitrary functions in this general integral are given by

$$(b+cq)^2\left(\frac{\partial u}{\partial x}\right)^2-2(b+cq)(a+cp)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+(a+cp)^2\left(\frac{\partial u}{\partial y}\right)^2=0:$$

the two arguments are one and the same, and may be taken as the simplest integral of

$$(b+cq)\frac{\partial u}{\partial x}-(a+cp)\frac{\partial u}{\partial y}=0.$$

This simplest integral is an integral of the equation

$$\frac{dx}{b+cq}=\frac{dy}{-(a+cp)},$$

that is, of

$$a\,dx+b\,dy+c\,(p\,dx+q\,dy)=0,$$

that is, of

$$adx+bdy+c\,dz=0:$$

hence the common argument of the two arbitrary functions in the general integral is $ax+by+cz$.

Ex. 3. Find the arguments of the arbitrary functions in the general integrals of the equations:

(i) $x^2r+2xys+y^2t=0$;

(ii) $x^2r-2xys+y^2t=0$;

(iii) $q^2r-2pqs+p^2t=0$;

(iv) $q^2r+2pqs+p^2t=0$;

(v) $x^2r-y^2t=0$;

(vi) $q(1+q)r-(p+q+2pq)s+p(1+p)t=0$;

proving that, in the case of each equation, the Ampère tests which allow it to possess a general integral expressible in finite form without partial quadratures are satisfied.

Ex. 4. Prove that an integral of the equation

$$x(r+t)+p=0$$

is given by

$$z=\int_0^\pi f(x\cos\phi+iy)\,d\phi+\int_0^\pi g(x\cos\psi-iy)\,d\psi.$$

Does this integral involve two arbitrary functions that are independent of one another? Does the equation possess a general integral in finite form without partial quadratures?

187. A similar discussion, in connection with an equation of order n in two independent variables, leads to a similar result as regards the arguments of the arbitrary functions. If the equation be

$$F(z_{n,0},\ z_{n-1,1},\ \ldots,\ z_{0,n},\ \ldots)=0,$$

where

$$z_{p,q}=\frac{\partial^{p+q}z}{\partial x^p\,\partial y^q},$$

and if u be any one of those arguments, then the equation

$$\frac{\partial F}{\partial z_{n,0}}\left(\frac{\partial u}{\partial x}\right)^n+\frac{\partial F}{\partial z_{n-1,1}}\left(\frac{\partial u}{\partial x}\right)^{n-1}\frac{\partial u}{\partial y}+\ldots+\frac{\partial F}{\partial z_{0,n}}\left(\frac{\partial u}{\partial y}\right)^n=0$$

is satisfied: it being always remembered that the general integral of $F=0$ is assumed to be of finite form and without partial quadratures.

Usually this equation would be only one of a set of equations satisfied in connection with the general integral of $F=0$. For the very restricted class of equations, in which F is linear in the derivatives of order n and has functions of x and y only for the coefficients of these derivatives, the foregoing equation becomes a partial differential equation of the first order for the actual determination of the arguments u.

188. Corresponding results can similarly be obtained for equations of any order in any number of independent variables; it will be sufficient to state them for an equation of the second order in three independent variables. A change in the notation will be made: we denote the dependent variable by v, the independent variables by x, y, z, and we write

$$l,\ m,\ n=\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial z},$$

$$a,\ b,\ c,\ f,\ g,\ h=\frac{\partial^2 v}{\partial x^2},\ \frac{\partial^2 v}{\partial y^2},\ \frac{\partial^2 v}{\partial z^2},\ \frac{\partial^2 v}{\partial y\partial z},\ \frac{\partial^2 v}{\partial z\partial x},\ \frac{\partial^2 v}{\partial x\partial y},$$

respectively. Then an equation of the second order is of the form

$$F(x, y, z, v, l, m, n, a, b, c, f, g, h) = 0:$$

we may suppose that F is a regular function of all its arguments, and we shall assume that F is polynomial in the derivatives a, b, c, f, g, h.

The Ampère test as to the possession of a general integral in finite form without partial quadratures can be applied as before. Let u be an argument of an arbitrary function in that integral, and let the variables be changed from x, y, z to x, y, u, so that z becomes a function of x, y, u: forming the derivatives of z, let

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q.$$

We substitute

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + cp^2,$$

$$b = \frac{dm}{dy} - q\frac{dn}{dy} + cq^2,$$

$$f = \frac{dn}{dy} - cq,$$

$$g = \frac{dn}{dx} - cp,$$

$$h = \frac{dl}{dy} - q\frac{dn}{dx} + cpq = \frac{dm}{dx} - p\frac{dn}{dy} + cpq,$$

in $F = 0$, and arrange it in powers of c in the form

$$F_0 + cF_1 + c^2F_2 + \ldots + c^mF_m = 0:$$

in order that a general integral of the specified type may be possessed, the equations

$$F_0 = 0, \quad F_1 = 0, \ldots, F_m = 0,$$

must be consistent with one another, with

$$F = 0,$$

and with

$$\frac{dl}{dy} - q\frac{dn}{dx} = \frac{dm}{dx} - p\frac{dn}{dy}.$$

But, as before, these conditions are necessary, though not universally sufficient: they provide a qualifying test.

In particular, the equation $F_m = 0$ is

$$p^2 \frac{\partial F}{\partial a} + q^2 \frac{\partial F}{\partial b} + \frac{\partial F}{\partial c} - q \frac{\partial F}{\partial f} - p \frac{\partial F}{\partial g} + pq \frac{\partial F}{\partial h} = 0.$$

Now p and q are the derivatives of z with regard to x and y when u is constant: thus

$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} = 0,$$

$$\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} = 0.$$

Hence u, the argument of an arbitrary function in a general integral of $F=0$, supposed in finite form and without partial quadratures, satisfies the equation

$$\frac{\partial F}{\partial a}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial F}{\partial b}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial F}{\partial c}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial F}{\partial f}\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial g}\frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial h}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0,$$

in connection with an integral of the partial differential equation.

We shall return to the subject during the discussion of the methods of integration of equations of the second order involving more than two independent variables (Chap. XXIV).

Ex. 1. One of the most important of these equations is

$$a+b+c=0,$$

which recurs continually in mathematical physics. A general integral has already (§ 181) been given in a form which requires partial quadratures.

The qualifying conditions that it should possess a general integral without partial quadratures are easily found to be that the relations

$$\frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} = 0,$$

$$\frac{dl}{dy} + p\frac{dn}{dy} - \frac{dm}{dx} - q\frac{dn}{dx} = 0,$$

$$p^2 + q^2 + 1 = 0,$$

shall be consistent with one another and with the original equation. It is easy to see that these conditions are satisfied: and thus the equation possesses a general integral without partial quadratures.

Also, the last relation leads to

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

as the equation satisfied by an argument of an arbitrary function in the integral. A general integral of this partial differential equation of the first order is given by the equations

$$\left.\begin{aligned} u &= \gamma(z + ix\cos a + iy\sin a) + \phi(\gamma, a) \\ 0 &= \quad z + ix\cos a + iy\sin a + \frac{\partial\phi}{\partial\gamma} \\ 0 &= i\gamma\ (-x\sin a + y\cos a) + \frac{\partial\phi}{\partial a} \end{aligned}\right\}.$$

Ex. 2. In connection with the last result, verify that, if

$$\phi(\gamma, a) = \gamma\theta(a) + \psi(a),$$

where θ and ψ are arbitrary functions of a, then

$$v = F(u),$$

where F denotes an arbitrary function, satisfies the equation

$$a + b + c = 0.$$

CHAPTER XIII.

LINEAR EQUATIONS OF THE SECOND ORDER IN TWO INDEPENDENT VARIABLES: THE LAPLACE-TRANSFORMATIONS.

THE greater part of the present chapter is devoted to what is usually called *Laplace's method* for the integration of an equation of the second order that is linear in the dependent variable and its derivatives. The method* given by Laplace was developed and extended by Darboux† : and it is upon his exposition that the present account of the method is based. Some detailed references are given in the course of the chapter: and general mention may here be made of Goursat's discussion of the method‡.

189. We proceed now to consider, in detail, equations of the second order; and we begin with those equations which involve two independent variables. Among them, two classes are marked out from the rest by their simplicity of form: one of these classes is constituted by the equations which are linear in the dependent variable and its derivatives: the other is constituted by the equations which possess an intermediate integral involving the first derivatives of the dependent variable. The two classes are not completely exclusive of one another; but the main methods of dealing with them are quite distinct. We discuss first the equations which are linear.

The most general form of equation, which is linear in the dependent variable and its derivatives, and which is of the second order, is (in the ordinary notation)

$$Rr + 2Ss + Tt + 2Pp + 2Qq + Zz = U,$$

* Originally given in his memoir *Mémoires de l'Acad. royale des sciences*, 1777, the memoir itself being dated 1773: see also *Œuvres complètes de Laplace*, t. IX, pp. 5—68.

† *Théorie générale des surfaces*, t. II, pp. 23 *et seq.*

‡ In chapter V of his treatise already (p. 7) quoted.

where R, S, T, P, Q, Z, U, are functions of x and y only. We may take U as zero: for, if U is not zero in any given instance, and if ζ is any particular value of z (no matter how special) which satisfies the equation, then writing

$$z = \zeta + z',$$

the equation for z' is of the same form save that U is zero. We shall therefore assume that U is zero.

Let the independent variables be changed from x and y to u and v: the equation is unaltered in form, so that it is

$$R'r' + 2S's' + T't' + 2P'p' + 2Q'q' + Zz = 0,$$

where p', q', r', s', t' are the derivatives of z with regard to the new variables, and

$$R' = R\left(\frac{\partial u}{\partial x}\right)^2 + 2S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2,$$

$$S' = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y},$$

$$T' = R\left(\frac{\partial v}{\partial x}\right)^2 + 2S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2.$$

The quantities u and v are at our disposal: the simplest form of changed equation depends upon the roots of the quadratic

$$R\mu^2 + 2S\mu + T = 0.$$

Firstly, let the roots of this quadratic be unequal, and denote them by $-m$, $-n$; and then let u and v be determined by the equations

$$\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} + n\frac{\partial v}{\partial y} = 0.$$

Then we have

$$R' = 0, \quad T' = 0;$$

also

$$S' = 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\left(T - \frac{S^2}{R}\right),$$

which is not zero. The equation takes the form

$$\frac{\partial^2 z}{\partial u \partial v} + a\frac{\partial z}{\partial u} + b\frac{\partial z}{\partial v} + cz = 0,$$

where a, b, c are functions of the independent variables alone.

Secondly, let $R=0$, $T=0$: the changed equation will still be of the last form, provided

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=0,\quad \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0,$$

and

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\neq 0.$$

We thus may make

$$u = \text{any function of } x, \text{ and } v = \text{any function of } y:$$

or

$$u = \text{any function of } y, \text{ and } v = \text{any function of } x.$$

For either of these transformations, the deduced form of equation is invariantive.

Thirdly, let the roots of the quadratic be equal, having $-m$ for their common value. We then take

$$u=x,\quad \frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}=0,$$

and we have

$$R'=R,$$

$$S'=(Rm+S)\frac{\partial v}{\partial y}=0,$$

$$T'=0.$$

The equation takes the form

$$\frac{\partial^2 z}{\partial x^2}+a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial v}+cz=0,$$

where a, b, c are functions of the independent variables.

Fourthly, let $S=0$, $T=0$. The equation originally is of the form last obtained: and it remains unchanged by the transformation if

$$\frac{\partial u}{\partial x}\neq 0,\quad \frac{\partial v}{\partial x}=0,$$

that is, if v is any function of y, and if u is any function of x and y which certainly involves x.

Fifthly, let $R=0$, $S=0$: taking

$$u=y,\quad v=x,$$

we again obtain the form in the last case.

Similarly, we can obtain one or other of the forms if only one of the three quantities R, S, T should vanish. Hence in every case, *the linear equation can, by change of the independent variables, be transformed so as to become either*

$$s + ap + bq + cz = 0,$$

or

$$r + ap + bq + cz = 0,$$

where a, b, c are functions of the independent variables alone.

This reduction to one or other of two alternative forms may be compared with the determination of the arguments in the arbitrary functions in the general integral, there supposed to be of finite form and devoid of partial quadratures. From that determination, we infer that, if the equation

$$s + ap + bq + cz = 0$$

possesses a general integral of the specified type, the general integral will contain an arbitrary function of x and an arbitrary function of y in its expression. Also we infer that if, in the equation

$$r + ap + bq + cz = 0,$$

the coefficient b is not zero, a general integral devoid of partial quadratures cannot be of finite form, while, if the coefficient b is zero and if the equation possesses a general integral of finite form and free from partial quadratures, that general integral will involve two arbitrary functions of y in its expression.

Ex. 1. If R, S, T, P, Q, Z are constants, then, by the transformations $u = \alpha x + \beta y$, $v = \alpha' x + \beta' y$, $z = \zeta e^{\alpha'' x + \beta'' y}$, with appropriate determinations of α, β, α', β', α'', β'', the linear equation can be changed to one of the forms

$$r + q = 0, \quad s + z = 0, \quad s + p + q = 0. \qquad \text{(A. Schwartz.)}$$

Ex. 2. Obtain the condition that the system

$$\xi p + \eta q + \zeta z = z_1, \quad \alpha p_1 + \beta q_1 + \gamma z_1 = 0,$$

where ξ, η, ζ, α, β, γ are functions of x and y, may be equivalent to a linear equation of the second order: and, assuming the condition satisfied, integrate the equation. (A. Schwartz.)

190. In the preceding reduction to one or other of two forms, the discrimination is made by the equality or the inequality of the roots of the quadratic

$$R\mu^2 + 2S\mu + T = 0.$$

In many investigations, particularly those concerned with the general theory of surfaces and with characteristics, the inde-

pendent variables are real: and consequently it may be of importance to note the form of the equation according as the roots of the quadratic are real or are complex, the quantities R, S, T being real.

In the first place, let the roots be conjugate complex quantities, so that $RT - S^2 > 0$: write

$$RT - S^2 = \theta^2.$$

Then

$$R'T' - S'^2 = (RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2,$$

so that $R'T' - S'^2$ must be positive, a condition that will be satisfied by taking

$$R' = T', \quad S' = 0.$$

Then

$$R'R = \left(R\frac{\partial u}{\partial x} + S\frac{\partial u}{\partial y}\right)^2 + \theta^2\left(\frac{\partial u}{\partial y}\right)^2,$$

$$T'T = \left(R\frac{\partial v}{\partial x} + S\frac{\partial v}{\partial y}\right)^2 + \theta^2\left(\frac{\partial v}{\partial y}\right)^2,$$

$$0 = \left(R\frac{\partial u}{\partial x} + S\frac{\partial u}{\partial y}\right)\left(R\frac{\partial v}{\partial x} + S\frac{\partial v}{\partial y}\right) + \theta^2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y};$$

and therefore

$$R\frac{\partial u}{\partial x} + S\frac{\partial u}{\partial y} = \mu\frac{\partial v}{\partial y},$$

$$R\frac{\partial v}{\partial x} + S\frac{\partial v}{\partial y} = -\frac{\theta^2}{\mu}\frac{\partial u}{\partial y},$$

where

$$\frac{\mu^2}{\theta^2} = \frac{R}{T},$$

using the relation $R' = T'$. The equations

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{S}{R}\frac{\partial u}{\partial y} + \frac{\mu}{R}\frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= -\frac{T\mu}{R}\frac{\partial u}{\partial y} - \frac{S}{R}\frac{\partial v}{\partial y}\end{aligned}\right\}$$

are of the form considered in §§ 8—13: they give values of u and v. The transformed equation is

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + l\frac{\partial z}{\partial u} + m\frac{\partial z}{\partial v} + nz = 0,$$

that is,

$$r + t + lp + mq + nz = 0,$$

where l, m, n are functions of the independent variables. This is sometimes called the *elliptic* case.

In the second place, let the roots of the quadratic

$$R\mu^2 + 2S\mu + T = 0$$

be real and different: the earlier analysis shews that the transformed equation is

$$s + ap + bq + cz = 0,$$

where now the variables are real. This is sometimes called the *hyperbolic* case.

In the third place, let the roots of the quadratic be equal: they are real. The transformed equation is

$$r + ap + bq + cz = 0,$$

where now the variables are real. This is sometimes called the *parabolic* case.

So far as concerns most of the processes of integration, the distinction between real and complex variables is insignificant: it becomes important in certain applications to physics, to the geometry of ordinary space, and particularly in regard to characteristics. So far as concerns the processes of integration discussed for the linear equation, there is no distinction between real and complex variables: by taking

$$x + iy = x', \quad x - iy = y',$$

we change the elliptic case into the hyperbolic case. Accordingly, as here we are concerned with processes of integration, it will be sufficient to discuss the two forms

$$s + ap + bq + cz = 0,$$
$$r + ap + bq + cz = 0.$$

THE EQUATION $s + ap + bq + cz = 0$: ITS INVARIANTS.

191. We proceed to consider the equation

$$s + ap + bq + cz = 0,$$

one of the two forms to which every linear equation can be reduced. The form of the equation is unaltered if we introduce a new dependent variable such that

$$z = \lambda z',$$

where λ is any function of x and y: it is unaltered if the independent variables are changed to x' and y', where

$$x = \phi(x'), \quad y = \psi(y'),$$

the functions ϕ and ψ being arbitrary: likewise for the transformations

$$x = \psi(y'), \quad y = \phi(x').$$

Consider the effect of these in turn.

Substituting $z = \lambda z'$, we have

$$s' + a'p' + b'q' + c'z' = 0,$$

where

$$a' = a + \frac{1}{\lambda}\frac{\partial \lambda}{\partial y},$$

$$b' = b + \frac{1}{\lambda}\frac{\partial \lambda}{\partial x},$$

$$c' = c + \frac{a}{\lambda}\frac{\partial \lambda}{\partial x} + \frac{b}{\lambda}\frac{\partial \lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2 \lambda}{\partial x \partial y};$$

consequently,

$$c' - a'b' = \frac{\partial^2 (\log \lambda)}{\partial x \partial y} + c - ab,$$

and therefore

$$\frac{\partial a'}{\partial x} + a'b' - c' = \frac{\partial a}{\partial x} + ab - c = h,$$

$$\frac{\partial b'}{\partial y} + a'b' - c' = \frac{\partial b}{\partial y} + ab - c = k.$$

The quantities h and k are thus unaltered for the substitution

$$z = \lambda z'.$$

Making the transformation

$$x = \phi(x'), \quad y = \psi(y'),$$

we find

$$\frac{h'}{h} = \phi'(x')\,\psi'(y') = \frac{k'}{k};$$

and making the transformation

$$x = \psi(y'), \quad y = \phi(x'),$$

we find

$$\frac{h'}{k} = \phi'(x')\,\psi'(y') = \frac{k'}{h}.$$

Consequently, for all the transformations which leave the form of the differential equation unaltered, the combinations of the

coefficients denoted by h and k reproduce themselves, either exactly or save as to a factor which does not depend upon the equation. Accordingly, these quantities h and k are called the *invariants of the equation.*

Ex. Obtain the expressions of these invariants in terms of the coefficients of the original equation

$$Rr + 2Ss + Tt + 2Pp + 2Qq + Zz = 0.$$

(Imschenetsky.)

One important property can be associated with the invariants: it is that, if either of the invariants should vanish, the equation can be immediately integrated. As will be seen, this property is made the basis of Laplace's method of integration.

If h vanishes, then

$$c = ab + \frac{\partial a}{\partial x},$$

and so the equation becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y} + ax\right) + b\left(\frac{\partial z}{\partial y} + az\right) = 0.$$

Hence

$$\frac{\partial z}{\partial y} + az = Ye^{-\int b\,dx},$$

and therefore

$$ze^{\int a\,dy} = X + \int Ye^{\int (a\,dy - b\,dx)}\,dy,$$

where X is an arbitrary function of x and Y of y.

If k vanishes, then

$$c = ab + \frac{\partial b}{\partial y}:$$

proceeding similarly, we find

$$ze^{\int b\,dx} = Y_1 + \int X_1 e^{\int (b\,dx - a\,dy)}\,dx,$$

where X_1 is an arbitrary function of x and Y_1 of y.

In either case, it is clear that the only inverse operations required are quadratures.

192. As it thus appears that different forms of the equation, all of the same type, are obtainable by the transformations, it is convenient to have some *canonical form* to which the equation can

be uniquely reduced. The reduced equation or the canonical form, chosen by Darboux*, is that for which

$$c' = a'b'.$$

When this relation is used, we have

$$\frac{\partial a'}{\partial x} = h, \quad \frac{\partial b'}{\partial y} = k, \quad \frac{\partial^2 \log \lambda}{\partial x \partial y} = ab - c.$$

To obtain a' and b' explicitly, we determine the functions of the variables introduced in the quadratures by the conditions, (i) that a' shall vanish when x has an assigned value x_0 whatever y may be, and (ii) that b' shall vanish when y has an assigned value y_0 whatever x may be. Thus

$$a' = \int_{x_0}^{x} h\,dx, \quad b' = \int_{y_0}^{y} k\,dy;$$

and

$$c' = a'b',$$

so that, as h and k are known from any form of the equation, the coefficients in the reduced form are known. As regards the multiplier λ, we have

$$\frac{\partial \log \lambda}{\partial x} = b' - b = \int_{y_0}^{y} k\,dy - b,$$

$$\frac{\partial \log \lambda}{\partial y} = a' - a = \int_{x_0}^{x} h\,dx - a,$$

and λ is therefore known save as to a constant factor, which is trivial because the equation is linear and homogeneous in z and its derivatives.

Two other forms might be chosen. Thus we might assign

$$a' = 0$$

as characteristic of a reduced form: then

$$-c' = h, \quad \frac{\partial b'}{\partial y} - c' = k,$$

so that

$$c' = -h, \quad b' = \int_{y_0}^{y} (k - h)\,dy,$$

and the reduced form is

$$s' + q' \int_{y_0}^{y} (k - h)\,dy - hz' = 0.$$

* *Théorie générale des surfaces*, t. II, p. 26.

Similarly, if we were to assign

$$b' = 0$$

as characteristic of a reduced form, it would be

$$s' + p' \int_{x_0}^{x} (h - k)\, dx - kz' = 0.$$

Obviously the condition, necessary and sufficient to secure that an equation shall be reducible to a form

$$s = \mu z,$$

is that the invariants shall be equal: their common value is μ, and the multiplier λ needed to lead to this form is e^u, where

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}.$$

There is one other form to which the equation can be conditionally reduced. Suppose that some integral, no matter how particular, is known: let it be denoted by u. Then, when we substitute

$$z = u\zeta,$$

where ζ is a new dependent variable, the equation for ζ is

$$\frac{\partial^2 \zeta}{\partial x \partial y} + \left(a + \frac{1}{u}\frac{\partial u}{\partial y}\right)\frac{\partial \zeta}{\partial x} + \left(b + \frac{1}{u}\frac{\partial u}{\partial x}\right)\frac{\partial \zeta}{\partial y} = 0,$$

say

$$\frac{\partial^2 \zeta}{\partial x \partial y} + a'\frac{\partial \zeta}{\partial x} + b'\frac{\partial \zeta}{\partial y} = 0,$$

where a' and b' are functions of x and y. The term involving ζ alone is absent.

The invariants of this equation are the same as before, as is to be expected: for

$$h' = \frac{\partial a'}{\partial x} + a'b'$$

$$= \frac{\partial a}{\partial x} + ab - c,$$

on reduction: and similarly

$$k' = \frac{\partial b'}{\partial y} + a'b'$$

$$= \frac{\partial b}{\partial x} + ab - c.$$

This form of the equation occurs frequently in the general theory of surfaces. In particular, it is the equation satisfied by each of the Cartesian coordinates of a point on a surface, expressed in terms of the parameters of conjugate directions on the surface.

Relation of the two Laplace-Transformations.

193. Having discussed the simple cases when one or other of the invariants vanishes, we may now suppose that neither vanishes.

In the first place, we take

$$\frac{\partial z}{\partial y} + az = z_1;$$

then

$$\begin{aligned}\frac{\partial z_1}{\partial x} + bz_1 &= \frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + \left(\frac{\partial a}{\partial x} + ab\right) z \\ &= hz,\end{aligned}$$

and therefore

$$\frac{\partial}{\partial y}\left\{\frac{1}{h}\left(\frac{\partial z_1}{\partial x} + bz_1\right)\right\} + \frac{a}{h}\left(\frac{\partial z_1}{\partial x} + bz_1\right) = z_1,$$

so that the equation for z_1 is

$$s_1 + a_1 p_1 + b_1 q_1 + c_1 z_1 = 0,$$

where

$$\begin{aligned} a_1 &= a - \frac{1}{h}\frac{\partial h}{\partial y} = a - \frac{\partial \log h}{\partial y}, \\ b_1 &= b, \\ c_1 &= c - \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - \frac{b}{h}\frac{\partial h}{\partial y}. \end{aligned}$$

The equation for z_1 is of exactly the same type as the equation for z. Let h_1 and k_1 denote its invariants: then

$$\left.\begin{aligned} h_1 &= 2h - k - \frac{\partial^2 \log h}{\partial x \partial y} \\ k_1 &= h \end{aligned}\right\}.$$

In the next place, we take

$$\frac{\partial z}{\partial x} + bz = Z_1,$$

so that

$$\frac{\partial Z_1}{\partial y} + aZ_1 = kz;$$

the equation for Z_1 is similarly found to be

$$S_1 + A_1 P_1 + B_1 Q_1 + C_1 Z_1 = 0,$$

where

$$A_1 = a,$$

$$B_1 = b - \frac{1}{k}\frac{\partial k}{\partial x} = b - \frac{\partial \log k}{\partial x},$$

$$C_1 = c + \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} - \frac{a}{k}\frac{\partial k}{\partial x}.$$

The equation for Z_1 is of the same type as the equation for z. Denoting its invariants by H_1 and K_1, we have

$$\left.\begin{aligned} H_1 &= k \\ K_1 &= 2k - h - \frac{\partial^2 \log k}{\partial x \partial y} \end{aligned}\right\}.$$

It would appear as if a couple of distinct transformations could thus be obtained, independent of one another; as a matter of fact, they are in a sense the inverses of each other. Taking

$$z_1 = \frac{\partial z}{\partial y} + az = \sigma(z),$$

$$Z_1 = \frac{\partial z}{\partial x} + bz = \Sigma(z),$$

we had

$$hz = \frac{\partial z_1}{\partial x} + bz_1 = \Sigma(z_1) = \Sigma\sigma(z),$$

$$kz = \frac{\partial Z_1}{\partial y} + aZ_1 = \sigma(Z_1) = \sigma\Sigma(z).$$

Thus the two transformations, effected in succession upon z, give merely a multiple of z and so (§ 191) lead to an equation with the same invariants as the original equation. Effectively, we may write

$$\Sigma\sigma = h, \quad \sigma\Sigma = k,$$

as operators: or, as multiplication of z by a factor does not affect the invariants, we can regard the operations Σ and σ as inverses of one another.

Hence, when we take any number of these substitutions in turn and in any order, it is unnecessary to frame combinations of σ and Σ in immediate succession, so far as our quest is the form of the invariants of the successively transformed equations. All the independent sets of invariants will be obtained by taking

$$\sigma,\ \sigma^2,\ \sigma^3,\ \ldots,$$
$$\Sigma,\ \Sigma^2,\ \Sigma^3,\ \ldots,$$

where

$$\sigma^2(z) = \sigma\{\sigma(z)\},$$

and so on. Moreover, as the invariants of the equation satisfied by $\Sigma\sigma z$ are the invariants of the equation satisfied by z, we can write

$$\Sigma = \sigma^{-1},\quad \Sigma^2 = \sigma^{-1}(\sigma^{-1}) = \sigma^{-2},$$

and so on: and, similarly,

$$\sigma = \Sigma^{-1},\quad \sigma^2 = \Sigma^{-2},$$

and so on. Thus all the independent sets of invariants will be obtained by effecting upon z the set of operations

$$\ldots,\ \sigma^{-3},\ \sigma^{-2},\ \sigma^{-1},\ 1,\ \sigma,\ \sigma^2,\ \sigma^3,\ \ldots,$$

or the set of operations

$$\ldots,\ \Sigma^{-3},\ \Sigma^{-2},\ \Sigma^{-1},\ 1,\ \Sigma,\ \Sigma^2,\ \Sigma^3,\ \ldots,$$

the two sets being the same as one another in reversed order.

It should be noted, that the coefficient a in the equation is unaltered for the operation Σ or σ^{-1}, and that the coefficient b is unaltered for the operation σ or Σ^{-1}.

This inverse character of the two transformations relatively to each other can be illustrated by the two sets of equations giving the two sets of invariants. We have

$$H_1 = k,$$
$$K_1 = 2k - h - \frac{\partial^2 \log k}{\partial x\, \partial y}.$$

Expressing Σ in the form σ^{-1}, we should naturally express H_1 and K_1 in the forms h_{-1} and k_{-1}: and so

$$h_{-1} = k,$$
$$k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial x\, \partial y},$$

giving a relation between the invariants of an equation and those of the equation next after it in the ascending series of transformations σ. Repeating the relation, we have

$$h = k_1,$$
$$k = 2k_1 - h_1 - \frac{\partial^2 \log k_1}{\partial x\, \partial y},$$

that is,

$$k_1 = h,$$
$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y},$$

which are the equations expressing the invariants of the equation in σz in terms of the invariants of the equation in z.

Invariants of Successively Transformed Equations.

194. Expressions for the invariants of the equation, having $\sigma^n z$ for its dependent variable, can be found in terms of the invariants of the equations that occur earlier in the series. Since

$$h_n = 2h_{n-1} - k_{n-1} - \frac{\partial^2 \log h_{n-1}}{\partial x \partial y},$$
$$k_n = h_{n-1},$$

we have

$$h_n - 2h_{n-1} + h_{n-2} = -\frac{\partial^2 \log h_{n-1}}{\partial x \partial y},$$
$$h_{n-1} - 2h_{n-2} + h_{n-3} = -\frac{\partial^2 \log h_{n-2}}{\partial x \partial y},$$
$$h_{n-2} - 2h_{n-3} + h_{n-4} = -\frac{\partial^2 \log h_{n-3}}{\partial x \partial y},$$
$$\dots\dots\dots\dots\dots\dots = \dots\dots,$$
$$h_1 - 2h + k = -\frac{\partial^2 \log h}{\partial x \partial y};$$

adding, we find

$$h_n - h_{n-1} = h - k - \frac{\partial^2}{\partial x \partial y} \log (h h_1 \dots h_{n-1}).$$

Taking this for $n, n-1, \dots$ and adding, we have

$$h_n - h = n(h-k) - \frac{\partial^2}{\partial x \partial y} \log (h^n h_1{}^{n-1} \dots h^2{}_{n-2} h_{n-1}),$$

that is,

$$h_n = (n+1)h - nk - \frac{\partial^2}{\partial x \partial y} \log (h^n h_1{}^{n-1} \dots h^2{}_{n-2} h_{n-1}).$$

Also

$$k_n = h_{n-1}$$
$$= nh - (n-1)k - \frac{\partial^2}{\partial x \partial y} \log (h^{n-1} h_1^{n-2} \dots h^2_{n-3} h_{n-2}).$$

And then, knowing the invariants, we can write down a canonical form of the equation.

Similar expressions can be obtained for the invariants of the equation having $\Sigma^n z$ for its dependent variable.

It is important also to have, in explicit form, the relation between the dependent variables z and $\sigma^n z$. We have seen that the coefficient b is unaltered by the application of the σ-substitution: also that, if

$$z_{m+1} = \frac{\partial z_m}{\partial y} + a_m z_m = \sigma z_m = \sigma^{m+1} z,$$

then

$$h_m z_m = \frac{\partial z_{m+1}}{\partial x} + b z_{m+1}.$$

Consequently,

$$z_m e^{\int b\,dx} = \frac{1}{h_m} \frac{\partial}{\partial x} (z_{m+1} e^{\int b\,dx}),$$

and therefore

$$z e^{\int b\,dx} = \frac{\partial}{h \partial x} \cdot \frac{\partial}{h_1 \partial x} \cdot \dots \cdot \frac{\partial}{h_{n-1} \partial x} (z_n e^{\int b\,dx}),$$

where z_n, $= \sigma^n z$, is the dependent variable in the nth transformation. Thus z is expressible in terms of z_n by a series of direct operations.

Similarly,

$$z e^{\int a\,dy} = \frac{\partial}{k \partial y} \cdot \frac{\partial}{K_1 \partial y} \cdot \dots \cdot \frac{\partial}{K_{\mu-1} \partial y} (Z_\mu e^{\int a\,dy})$$

is the relation between z and Z_μ, $= \Sigma^\mu z$, the dependent variable in the μth equation in the series of successive transformations Σ.

These expressions are important for the practical integration of the equation because, in this method, it is the variable z_n or Z_μ which is first explicitly obtained.

Ex. 1. The invariants of the equation satisfied by $\sigma^n z$, when the equation satisfied by z is

$$s + \frac{\alpha}{x+y} p + \frac{\beta}{x+y} q + \frac{\gamma}{(x+y)^2} z = 0,$$

where α, β, γ are constants, can be derived from the preceding results.

We have

$$a=\frac{\alpha}{x+y},\quad b=\frac{\beta}{x+y},\quad c=\frac{\gamma}{(x+y)^2}:$$

hence

$$h=\frac{\alpha\beta-\alpha-\gamma}{(x+y)^2},\quad k=\frac{\alpha\beta-\beta-\gamma}{(x+y)^2}.$$

Also

$$\frac{\partial^2\log h}{\partial x\,\partial y}=\frac{2}{(x+y)^2};$$

therefore

$$h_i=\frac{\mu_i}{(x+y)^2},\quad k_i=\frac{\mu_{i-1}}{(x+y)^2},$$

where μ_i is a constant partly dependent upon i. Now

$$\begin{aligned} h_i &= (i+1)\,h-ik-\frac{\partial^2}{\partial x\,\partial y}\log\left(h^i h_1^{\,i-1}\dots h^2_{i-2}h_{i-1}\right)\\ &= (i+1)\,h-ik-\frac{2}{(x+y)^2}\{i+(i-1)+\dots+2+1\}\\ &= (i+1)\,h-ik-\frac{i(i+1)}{(x+y)^2}: \end{aligned}$$

consequently,

$$\begin{aligned} \mu_i &= (i+1)(\alpha\beta-\alpha-\gamma)-i(\alpha\beta-\beta-\gamma)-i(i+1)\\ &= -\gamma+(\alpha+i)(\beta-i-1). \end{aligned}$$

Hence

$$h_n=\frac{(\alpha+n)(\beta-n-1)-\gamma}{(x+y)^2},$$

$$k_n=\frac{(\alpha+n-1)(\beta-n)-\gamma}{(x+y)^2}.$$

In the special case when γ is a prime number, α and β also being integers, neither h_n nor k_n can vanish unless

$$\gamma=\alpha+\beta-2.$$

If both conditions, viz. that γ is a prime number and is equal to $\alpha+\beta-2$, be satisfied, then n is $\beta-2$ when $h_n=0$ or (what is the same thing in effect) n is $\beta-1$ when $k_n=0$.

In the special case, when γ is an integer that is not a prime, then particular forms of α and β may make h_n zero or k_n zero.

In the special case, when the constants α,β,γ (not being integers) are such that, for one (or more than one) value of n, either of the relations

$$\gamma=(\alpha+n)(\beta-n-1),$$

$$\gamma=(\alpha+n-1)(\beta-n),$$

is satisfied, then at the corresponding stage, we have $h_n=0$ or $k_n=0$.

In no other case will either of the invariants vanish at any stage in the successive transformations.

Ex. 2. Shew that, if the invariants of the equation satisfied by σz are the same as those of the equation satisfied by z, both equations are represented by

$$s = z,$$

on making the appropriate changes of variables. (Darboux.)

Ex. 3. Shew that, if the invariants of the equation satisfied by $\sigma^2 z$ are the same as those of the equation satisfied by z, then

$$\frac{\partial^2}{\partial x\, \partial y}\{\log(hk)\} = 0,$$

and that, by appropriate changes of x into a function of itself alone and of y into a function of itself alone, the values of h and k can be deduced from the integral of

$$\frac{\partial^2 \omega}{\partial x\, \partial y} = \sin \omega.$$

Obtain a reduced form of the original differential equation. (Darboux.)

Ex. 4. Shew that, if $h_2 = k$, then $k_2 = h$ subject to a transformation of x into a function of itself and of y also into a function of itself.

Ex. 5. The equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 2l\frac{\partial z}{\partial x} + 2m\frac{\partial z}{\partial y} + nz = 0$$

is transformed by a substitution

$$z_1 = \lambda z,$$

where λ is a function of x and y only: prove that the quantities

$$J = \frac{\partial l}{\partial y} - \frac{\partial m}{\partial x},$$

$$K = \frac{\partial l}{\partial x} + \frac{\partial m}{\partial y} + l^2 + m^2 - n,$$

are invariants for all such transformations. (Burgatti.)

Ex. 6. Shew that, if both the invariants J and K of the preceding equation vanish, the equation is reducible to

$$\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} = 0:$$

if J vanishes but not K, the equation is reducible to

$$\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} + cz_1 = 0:$$

and that, if K vanishes but not J, the equation is reducible to

$$\beta\frac{\partial^2 (\alpha z)}{\partial x^2} + \alpha\frac{\partial^2 (\beta z)}{\partial y^2} = 0,$$

where

$$\alpha = e^{\int l\, dx}, \quad \beta = e^{\int m\, dy}.$$ (Burgatti.)

Ex. 7. Apply the results in Exx. 5, 6 to the equation

$$s + ap + bq + cz = 0.$$

195. From the relation between the dependent variable in the original equation and the dependent variables in the equations that arise by successive applications of one or other of the two transformations, it is clear that the general integral of the original equation can be obtained when once the general integral of one of the transformed equations is known.

The integration is certainly possible for an equation when either of its invariants vanishes. Suppose that the nth equation, in the succession of σ-transformations, is the first of the equations characterised by the possession of a vanishing invariant: then h_n must be the vanishing invariant, because the value of k_n is h_{n-1} which is a non-vanishing invariant of the next earlier equation. Thus

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial z_n}{\partial y}+a_n z_n\right)+b\left(\frac{\partial z_n}{\partial y}+a_n z_n\right)&=\left(\frac{\partial a_n}{\partial x}+a_n b-c_n\right)z_n\\ &=0,\end{aligned}$$

because h_n is zero: the coefficient b is the coefficient b in the original equation which is unaffected by the σ-transformations; and therefore

$$\frac{\partial z_n}{\partial y}+a_n z_n=Ye^{-\int b\,dx},$$

$$z_n=e^{-\int a_n dy}\left\{X+\int Ye^{\int(a_n dy-b\,dx)}dy\right\},$$

where X and Y are arbitrary functions respectively. If

$$-\gamma=\int(a_n dy-b\,dx),\quad e^{-\gamma}=\beta,\quad \alpha=e^{-\int a_n dy},$$

we have

$$z_n=\alpha\left(X+\int Y\beta\,dy\right),$$

where α and β are determinate functions of x and y.

Now the relation between z and z_n is

$$\begin{aligned}ze^{\int b\,dx}&=\frac{\partial}{h\partial x}\cdot\frac{\partial}{h_1\partial x}\cdot\ldots\cdot\frac{\partial}{h_{n-1}\partial x}\left(z_n e^{\int b\,dx}\right)\\ &=\frac{\partial}{h\partial x}\cdot\frac{\partial}{h_1\partial x}\cdot\ldots\cdot\frac{\partial}{h_{n-1}\partial x}\left\{e^{\gamma}\left(X+\int Ye^{-\gamma}dy\right)\right\};\end{aligned}$$

and therefore, effecting the differential operations, we have the value of z expressed in the form

$$\begin{aligned}z=A\left(X+\int Y\beta\,dy\right)+A_1\left(X'+\int Y\frac{\partial\beta}{\partial x}dy\right)+\ldots&\\ \ldots+A_n\left(X^{(n)}+\int Y\frac{\partial^n\beta}{\partial x^n}dy\right),&\end{aligned}$$

where A, A_1, ..., A_n are determinate functions of x and y, and where $X^{(m)}$ is the mth derivative of X with respect to its argument x.

The functions X and Y are arbitrary: consequently, the integral obtained is a general integral and, in the form obtained, it involves indefinite partial quadratures. As Y is arbitrary, we shall have a specialised integral on making Y zero: and this specialised integral is

$$z = AX + A_1 X' + \ldots + A_n X^{(n)},$$

so that an integral exists, involving homogeneously and linearly an arbitrary function and its derivatives, when one of the invariants in the succession of equations, constructed by a repeated σ-transformation, vanishes: and the specialised integral is in finite form, without partial quadratures.

The converse of this result is also true: that is to say, if an equation

$$s + ap + bq + cz = 0$$

possesses an integral

$$z = AX + A_1 X' + \ldots + A_n X^{(n)},$$

where A_1, ..., A_n are determinate functions of x and y, where X is an arbitrary function of x and X', ..., $X^{(n)}$ are its first n derivatives, then the successive application of the σ-transformation will, after n operations at most, produce an equation for which the invariant h is zero. To prove this assertion, let the specified value of z be substituted in the differential equation: the result is

$$B_{n+1} X^{(n+1)} + B_n X^{(n)} + \ldots = 0,$$

where

$$B_{n+1} = \frac{\partial A_n}{\partial y} + aA_n,$$

$$B_n = \frac{\partial A_{n-1}}{\partial y} + aA_{n-1} + \frac{\partial^2 A_n}{\partial x \partial y} + a\frac{\partial A_n}{\partial x} + b\frac{\partial A_n}{\partial y} + cA_n,$$

and, for values of $m \leqslant n$,

$$B_m = \frac{\partial A_{m-1}}{\partial y} + aA_{m-1} + \frac{\partial^2 A_m}{\partial x \partial y} + a\frac{\partial A_m}{\partial x} + b\frac{\partial A_m}{\partial y} + cA_m.$$

As X is an arbitrary function of x, and as the differential equation must be satisfied identically by the postulated value of z, we must have

$$B_{n+1}=0, \quad B_n=0, \; \ldots, \; B_m=0,$$

for all values of $m \leqslant n$. Thus

$$\frac{\partial A_n}{\partial y}+aA_n=0,$$

from $B_{n+1}=0$; and therefore

$$\frac{\partial A_{n-1}}{\partial y}+aA_{n-1}=hA_n.$$

Now let the σ-transformation be applied to z: a new dependent variable z_1 is introduced, and we have

$$\begin{aligned} z_1 &= \frac{\partial z}{\partial y}+az \\ &= \left(\frac{\partial A_n}{\partial y}+aA_n\right)X^{(n)}+\left(\frac{\partial A_{n-1}}{\partial y}+aA_{n-1}\right)X^{(n-1)}+\ldots \\ &= hA_nX^{(n-1)}+\ldots, \end{aligned}$$

so that the order of the highest derivative of X in z_1 is certainly less by unity than it is in z and, if $h=0$, it is certainly less by two units than it is in z.

Similarly, when the corresponding σ-transformation is applied to the equation in z_1, a new variable z_2 is obtained such that the order of the highest derivative of X which it contains is certainly less by unity than the corresponding order in z_1 and, if $h_1=0$, the order is certainly less by two units than the highest order in z_1.

Hence taking these substitutions in succession, we reduce the order of the highest derivative of X in the successive dependent variables by one unit at least in each operation: and therefore, after n operations at most, either we obtain an equation such that the invariant h of the preceding equation vanishes, or we obtain a dependent variable ζ such that

$$z_\mu=\zeta=CX,$$

where C is a determinate function of x and y, and $\mu \leqslant n$. Now an equation of the type under consideration, which is satisfied by z_μ, is

$$s_\mu+a_\mu p_\mu+b_\mu q_\mu+c_\mu z_\mu=0,$$

where

$$a_\mu = -\frac{1}{C}\frac{\partial C}{\partial y}, \quad b_\mu = \frac{1}{C}\frac{\partial C}{\partial x}, \quad c_\mu = -\frac{1}{C}\frac{\partial^2 C}{\partial x \partial y};$$

and therefore

$$h_\mu = \frac{\partial a_\mu}{\partial x} + a_\mu b_\mu - c_\mu$$

$$= 0,$$

on substitution.

Ex. The equation

$$s + \frac{\alpha}{x+y}p + \frac{\beta}{x+y}q + \frac{\gamma}{(x+y)^2}z = 0$$

possesses an integral of the preceding type in finite form involving an arbitrary function X and its first n derivatives, if

$$\gamma = (\alpha + n)(\beta - n - 1),$$

where n is a positive integer.

Obtain the general integral for the conditional value of γ when $n = 1$.

Integrals of Finite Rank.

196. When the series of σ-transformations is finite in the sense that, after a finite number n of operations, an equation is obtained having its invariant h equal to zero, we have seen that an integral of the original equation exists in the form

$$z = AX + A_1 X' + \ldots + A_n X^{(n)}.$$

Conversely, if an integral of the original equation of this form exists, then μ of the σ-transformations (where $\mu \leqslant n$) lead to an equation having its invariant h equal to zero.

If $\mu = n$, the two properties are the exact reciprocals of each other.

If $\mu < n$, then an integral of the original equation exists in the form

$$z = CX_1 + C_1 X_1' + \ldots + C_\mu X_1^{(\mu)},$$

where X_1 is another arbitrary function of x. Hence it is necessary to consider whether expressions of this type can, by change of the arbitrary function, be changed so that the new form involves, in a diminished order, the derivatives of the new arbitrary function.

It is clear that the highest order in such an expression can always be increased by taking

$$X_1 = \alpha X + \alpha_1 X' + \ldots + \alpha_{n-\mu} X^{(n-\mu)},$$

where $\alpha, \alpha_1, \ldots, \alpha_{n-\mu}$ are specific functions of x alone: it is not clear (and it is not, in fact, the case) that it is always possible to decrease the order of such an expression by taking

$$\xi = \beta X + \beta_1 X' + \ldots + \beta_\rho X^{(\rho)},$$

where $\beta, \beta_1, \ldots, \beta_\rho$ are functions of x, specifically at our disposal.

An expression

$$AX + A_1 X' + \ldots + A_n X^{(n)}$$

is declared* to be of *rank* $n+1$, (or to be *irreducible*), when it is not possible, by any transformation

$$\xi = \alpha X + \alpha_1 X' + \ldots + \alpha_\mu X^{(\mu)},$$

to make the order of the highest derivative of ξ in the transformed expression less than n. We may therefore say that, when n is the number of σ-transformations applied in succession and needed to produce the first equation having its invariant h equal to zero, the original equation possesses an integral of rank $n+1$.

All these properties are associated with the σ-transformations of which the first is

$$\frac{\partial z}{\partial y} + az = z_1.$$

Similar considerations occur in association with the Σ- (or σ^{-1}) transformations of which the first is

$$\frac{\partial z}{\partial x} + bz = Z_1.$$

The general result, which can be established in a precisely similar manner, is that, if the equation resulting after m applications of the Σ-transformation is the first in the succession for which an invariant (now k_m) vanishes, then the original equation possesses an integral of rank $m+1$, of the form

$$z = BY + B_1 Y' + \ldots + B_m Y^{(m)},$$

where Y is an arbitrary function of y and $Y', Y'', \ldots$ are its derivatives, and where $B, B_1, \ldots, B_m$ are definite functions of x

* Darboux uses the term *rang*.

and y. Conversely, if an integral of this form is possessed by the original equation, the rank of the integral may be equal to $m+1$ but, if not, it is less than $m+1$.

197. As regards the possible reducibility of a given expression, the rank of which is less than $n+1$ though it involves derivatives of the arbitrary function of order n, it is sufficient to reduce the highest order of derivative by one unit at a time. For if an expression is reducible by a transformation

$$X_1 = \beta X + \alpha X' + X'',$$

where α and β are specific functions of x alone, let a quantity λ be determined by the equation

$$\lambda' + \alpha\lambda - \lambda^2 = \beta,$$

and then take

$$\mu = \alpha - \lambda :$$

if

$$\xi = X' + \lambda X,$$

then

$$X_1 = \xi' + \mu\xi,$$

that is, the original expression is reducible by the transformations

$$\xi = X' + \lambda X,$$
$$X_1 = \xi' + \mu\xi,$$

in succession. Similarly, if the expression in question can have the order of the highest derivative reduced by more than two units, we can secure the reduction by successive reductions of a single unit at a time.

Ex. 1. The condition for reducibility (and, if the condition is satisfied, the reduced form) of an expression Θ, where

$$\Theta = aX + a_1 X' + a_2 X'',$$

where a, a_1, a_2 are functions of x and y, are easily obtained. Let a relation

$$\xi = X' + \lambda X,$$

where λ is a function of x only, represent the arbitrary function in a reduced form of Θ: then

$$\xi' = X'' + \lambda X' + \lambda' X,$$

so that

$$a_2\xi' + p\xi = a_2 X'' + (a_2\lambda + p) X' + (a_2\lambda' + p\lambda) X.$$

In order that this expression may be the same as Θ, we must have

$$a_2\lambda + p = a_1,$$
$$a_2\lambda' + p\lambda = a;$$

and therefore λ satisfies the equation

$$a_2\lambda' = a - a_1\lambda + a_2\lambda^2,$$

while, when λ is known, the value of p is given by

$$p = a_1 - a_2\lambda.$$

Now the equation satisfied by λ is

$$\lambda' - \lambda^2 = \frac{a}{a_2} - \lambda\frac{a_1}{a_2},$$

and λ is a function of x only: hence, taking

$$\lambda = -\frac{1}{u}\frac{du}{dx},$$

where u is a function of x only, we have

$$a_2\frac{d^2u}{dx^2} + a_1\frac{du}{dx} + au = 0.$$

Hence also

$$\frac{\partial a_2}{\partial y}\frac{d^2u}{dx^2} + \frac{\partial a_1}{\partial y}\frac{du}{dx} + \frac{\partial a}{\partial y}u = 0;$$

and therefore

$$\frac{\dfrac{d^2u}{dx^2}}{a_1\dfrac{\partial a}{\partial y} - a\dfrac{\partial a_1}{\partial y}} = \frac{\dfrac{du}{dx}}{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}} = \frac{u}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}.$$

Consequently, necessary and sufficient conditions are

$$\frac{\partial}{\partial y}\left(\frac{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}\right) = 0,$$

$$\frac{a_1\dfrac{\partial a}{\partial y} - a\dfrac{\partial a_1}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}} = \frac{\partial}{\partial x}\left(\frac{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}\right) + \left(\frac{a\dfrac{\partial a_2}{\partial y} - a_2\dfrac{\partial a}{\partial y}}{a_2\dfrac{\partial a_1}{\partial y} - a_1\dfrac{\partial a_2}{\partial y}}\right)^2.$$

The conditions can be simplified by taking

$$a = \beta a_2, \quad a_1 = \beta_1 a_2:$$

they then become

$$\frac{\partial}{\partial y}\left(\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right) = 0,$$

and

$$\frac{\beta_1\dfrac{\partial\beta}{\partial y} - \beta\dfrac{\partial\beta_1}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}} + \frac{\partial}{\partial x}\left(\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right) = \left(\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right)^2,$$

the latter of which can also be written

$$\beta\left(\frac{\partial\beta_1}{\partial y}\right)^2-\beta_1\frac{\partial\beta}{\partial y}\frac{\partial\beta_1}{\partial y}+\left(\frac{\partial\beta}{\partial y}\right)^2=\frac{\partial\beta_1}{\partial y}\frac{\partial^2\beta}{\partial x\,\partial y}-\frac{\partial\beta}{\partial y}\frac{\partial^2\beta_1}{\partial x\,\partial y}.$$

The value of λ is

$$\lambda=\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}},$$

and that of p is

$$p=a_2\left(\beta_1-\frac{\dfrac{\partial\beta}{\partial y}}{\dfrac{\partial\beta_1}{\partial y}}\right)$$

Ex. 2. Shew that the expression

$$X'''+a_1X''+a_2X'+a_3X=0$$

can be transformed into an expression involving only X_1, X_1', X_1'', where X_1 is an arbitrary function, provided the following conditions are satisfied, viz.:—Let quantities A, B, C be defined by the relations

$$A\begin{vmatrix}\dfrac{\partial a_1}{\partial y}, & \dfrac{\partial a_2}{\partial y}\\[2mm] \dfrac{\partial^2 a_1}{\partial y^2}, & \dfrac{\partial^2 a_2}{\partial y^2}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial a_1}{\partial y}, & \dfrac{\partial a_3}{\partial y}\\[2mm] \dfrac{\partial^2 a_1}{\partial y^2}, & \dfrac{\partial^2 a_3}{\partial y^2}\end{vmatrix},$$

$$(A^2-B)\begin{vmatrix}\dfrac{\partial a_1}{\partial y}, & \dfrac{\partial a_2}{\partial y}\\[2mm] \dfrac{\partial^2 a_1}{\partial y^2}, & \dfrac{\partial^2 a_2}{\partial y^2}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial a_2}{\partial y}, & \dfrac{\partial a_3}{\partial y}\\[2mm] \dfrac{\partial^2 a_2}{\partial y^2}, & \dfrac{\partial^2 a_3}{\partial y^2}\end{vmatrix},$$

$$C-3AB+A^3=a_3+a_1(A^2-B)-a_2A;$$

then the conditions are that the equations

$$\frac{\partial A}{\partial y}=0,\quad \frac{\partial A}{\partial x}=B,\quad \frac{\partial B}{\partial x}=C,$$

shall be satisfied.

Shew also that the reduced form is

$$X_1''+(a_1-A)X_1'+(a_2-a_1A+A^2-2B)X_1,$$

where

$$X_1=X'+AX.$$

Ex. 3. Obtain conditions necessary and sufficient to secure that the expression

$$X'''+a_1X''+a_2X'+a_3X=0$$

is of rank not greater than two.

Equations having an Integral of Finite Rank.

198. We can now construct the aggregate of equations which admit an integral of rank $n+1$ obtainable by means of the σ-transformation.

When an integral is of rank $n+1$, the invariant h_n of the equation, which results from n successive applications of the σ-transformation, is equal to zero. Let a_n and b_n be chosen arbitrarily: then, as $h_n = 0$, we have

$$c_n = a_n b_n + \frac{\partial a_n}{\partial x},$$

which determines c_n; and then

$$\begin{aligned} k_n &= \frac{\partial b_n}{\partial y} + a_n b_n - c_n \\ &= \frac{\partial b_n}{\partial y} - \frac{\partial a_n}{\partial x} \\ &= \theta, \end{aligned}$$

say. Now the relations

$$h_{m+1} = 2h_m - k_m - \frac{\partial^2 \log h_m}{\partial x \partial y},$$
$$k_{m+1} = h_m,$$

give

$$h_m = k_{m+1},$$
$$k_m = 2k_{m+1} - h_{m+1} - \frac{\partial^2 \log k_{m+1}}{\partial x \partial y}.$$

Consequently,

$$h_{n-1} = k_n = \theta,$$
$$k_{n-1} = 2k_n - \frac{\partial^2 \log k_n}{\partial x \partial y} = 2\theta - \frac{\partial^2 \log \theta}{\partial x \partial y};$$

all the invariants for all the equations in the series can thus be calculated in backward succession. Also we have

$$b_{j+1} = b_j,$$

so that the coefficient b in what is to be the original equation is given by

$$b = b_n.$$

Again,

$$a_{j+1} = a_j - \frac{\partial \log h_j}{\partial y},$$

and therefore the coefficient a in what is to be the original equation is given by

$$a = a_n + \frac{\partial}{\partial y} \log (hh_1 \dots h_{n-1}).$$

Lastly,

$$c = \frac{\partial a}{\partial x} + ab - h = \frac{\partial b}{\partial y} + ab - k.$$

The coefficients in the equation are thus determined, so that the equation can be regarded as known: its actual expression involves the two arbitrary elements a_n and b_n.

The integral of the equation is of rank $n+1$. Let

$$\gamma = \int (b_n dx - a_n dy),$$

so that γ may be considered known and, in particular,

$$\frac{\partial^2 \gamma}{\partial x \partial y} = \theta;$$

then the actual expression of the full general integral of the equation is

$$ze^{\int b_n dx} = \frac{\partial}{h\partial x} \cdot \frac{\partial}{h_1 \partial x} \cdot \dots \cdot \frac{\partial}{h_{n-1}\partial x} \left\{ e^{\gamma} \left(X + \int Y e^{-\gamma} \, dy \right) \right\},$$

where X and Y denote arbitrary functions of x and of y respectively.

A similar process leads to the aggregate of equations which admit an integral of finite rank $m+1$ obtainable by means of the Σ-transformation. We take arbitrary quantities a_m, b_m: as K_m vanishes, we have

$$c_m = a_m b_m + \frac{\partial b_m}{\partial y}.$$

The successive invariants K_i and H_i are given by the relations

$$H_{i+1} = K_i,$$
$$K_{i+1} = 2K_i - H_i - \frac{\partial^2 \log K_i}{\partial x \partial y};$$

the coefficients in what is to be the original equation are given by

$$a = a_m,$$
$$b = b_m + \frac{\partial}{\partial x} \log (kK_1 \dots K_{m-1}),$$
$$c = \frac{\partial a}{\partial x} + ab - h = \frac{\partial b}{\partial y} + ab - k;$$

and if

$$\delta = \int (a_m dy - b_m dx),$$

the full general integral of the equation is given by

$$ze^{\int a_m dy} = \frac{\partial}{k\partial y} \cdot \frac{\partial}{K_1 \partial y} \cdot \ldots \cdot \frac{\partial}{K_{m-1}\partial y} \left\{ e^{\delta} \left(Y + \int X e^{-\delta} dx \right) \right\},$$

where X and Y denote arbitrary functions of x and of y respectively.

Ex. 1. The equations admitting an integral of rank unity are of the form

$$s + ap + bq + cz = 0,$$

where either

$$c = ab + \frac{\partial a}{\partial x},$$

when h vanishes, and the integral is

$$ze^{\int a\, dy} = X + \int Y e^{\int (a\,dy - b\,dx)}\, dy :$$

or

$$c = ab + \frac{\partial b}{\partial y},$$

when k vanishes, and the integral is

$$ze^{\int b\, dx} = Y + \int X e^{\int (b\,dx - a\,dy)}\, dx.$$

The coefficients a and b are chosen arbitrarily.

Ex. 2. For equations admitting an integral of rank two, we have a couple of forms according as the vanishing invariant arises through the σ-transformation or through the Σ-transformation.

Taking the σ-transformation, the invariant h_1 is to vanish: we choose two arbitrary quantities α and β, these being a_1 and b_1. Then

$$\begin{aligned} \frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} &= k_1 \\ &= h ; \end{aligned}$$

and then

$$\begin{aligned} b &= \beta, \\ a &= \alpha + \frac{\partial (\log h)}{\partial y}, \\ c &= -h + \alpha\beta + \beta \frac{\partial \log h}{\partial y} + \frac{\partial \alpha}{\partial x} + \frac{\partial^2 \log h}{\partial x\, \partial y} . \end{aligned}$$

These give the coefficients of the equation: it is easy to deduce from them the tests as to whether an equation with coefficients a', b', c' belongs to the type. We have

$$h = \frac{\partial a'}{\partial x} + a'b' - c',$$

and then

$$\beta = b',$$
$$a = a' - \frac{\partial \log h}{\partial y},$$

so that, as

$$h = \frac{\partial \beta}{\partial y} - \frac{\partial a}{\partial x},$$

we have

$$h = \frac{\partial b'}{\partial y} - \frac{\partial a'}{\partial x} + \frac{\partial^2 \log h}{\partial x \, \partial y},$$

as the necessary condition*.

For the general integral of the equation

$$s + ap + bq + cz = 0,$$

when it admits an integral of rank two associated with the σ-transformation, let

$$\begin{aligned}\gamma &= \int (\beta dx - a dy) \\ &= \int (b dx - a dy) + \log h \\ &= u + \log h,\end{aligned}$$

say: then, by the general result, the full integral is

$$\begin{aligned} z e^{\int b dx} &= \frac{1}{h} \frac{\partial}{\partial x} \left\{ h e^u \left(X + \int \frac{Y}{h} e^{-u} dy \right) \right\} \\ &= \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) e^u \left(X + \int \frac{Y}{h} e^{-u} dy \right) \\ &\qquad + e^u \left\{ X' + \int Y \frac{\partial}{\partial x} \left(\frac{e^{-u}}{h} \right) dy \right\} \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} z e^{\int a dy} &= X' + \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) X \\ &\quad + \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) \int Y \frac{e^{-u}}{h} dy + \int Y \frac{\partial}{\partial x} \left(\frac{e^{-u}}{h} \right) dy. \end{aligned}$$

The integral of rank two is, of course,

$$z = e^{-\int a dy} X' + e^{-\int a dy} \left(\frac{1}{h} \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} \right) X.$$

Corresponding results hold when the integral of rank two is associated with the Σ-transformation. We take two arbitrary quantities a' and β': as K_1 is zero in this case, we have

$$\frac{\partial a'}{\partial x} - \frac{\partial \beta'}{\partial y} = H_1 = k,$$

* It is, in effect, $h_1 = 0$.

and then

$$a = a',$$

$$b = \beta' + \frac{\partial \log k}{\partial x},$$

$$c = -k + a'\beta' + a'\frac{\partial \log k}{\partial x} + \frac{\partial \beta'}{\partial y} + \frac{\partial^2 \log k}{\partial x \partial y}.$$

The condition that a given equation

$$s + a''p + b''q + c''z = 0$$

should belong to this type is that

$$k = \frac{\partial a''}{\partial x} - \frac{\partial b''}{\partial y} + \frac{\partial^2 \log k}{\partial x \partial y},$$

where

$$k = -c'' + a''b'' + \frac{\partial b''}{\partial y}.$$

Also, the full integral of the equation

$$s + ap + bq + cz = 0,$$

when it admits an integral of rank two associated with the Σ-transformation, is

$$ze^{\int a\,dy} = \frac{1}{k}\frac{\partial}{\partial y}\left\{ke^{v}\left(Y + \int \frac{X}{k} e^{-v} dx\right)\right\},$$

where

$$v = \int (a\,dy - b\,dx):$$

and the integral of rank two is

$$z = e^{-\int b\,dx} Y' + e^{-\int b\,dx}\left(\frac{1}{k}\frac{\partial k}{\partial y} + \frac{\partial v}{\partial y}\right) Y.$$

Ex. 3. Prove that, if an equation

$$s + ap + bq + cz = 0$$

admits an integral of rank $n+1$, the term involving the highest derivative of the arbitrary function is

$$e^{-\int a\,dy} X^{(n)} \quad \text{or} \quad e^{-\int b\,dx} Y^{(n)}$$

in the respective cases.

Ex. 4. Integrate the equations

(i) $s + xp + yq + (1 + xy) z = 0$;

(ii) $s + mxp + nyq + (2m - n + mn\,xy) z = 0$;

(iii) $s + myp + e^{cy}q + (2c + my) e^{cy} z = 0$;

where m, n, c are constants in the last two equations.

Ex. 5. Prove that the equation

$$s + xyq + nxz = 0,$$

where n is a finite integer, possesses an integral of finite rank; and obtain the integral in the two cases $n = 2$, $n = -1$. (Imschenetsky.)

Ex. 6. Let the equation

$$s+ap+bq=0$$

be of rank $m+1$ in one of the variables; prove that the equation

$$s+p(a+u)+q(b+v)=0$$

is of rank $m+2$ in one of the variables, where

$$u=\frac{\partial}{\partial y}\log\left(\frac{z_3 J}{\frac{\partial\sigma}{\partial y}}\right),\quad v=\frac{\partial}{\partial x}\log\left(\frac{z_3 J}{\frac{\partial\sigma}{\partial x}}\right),$$

z_1, z_2, z_3 are three integrals of the original equation, σ is the ratio of either z_1 or z_2 to z_3, and

$$J=\frac{\partial\left(\frac{z_1}{z_3},\ \frac{z_2}{z_3}\right)}{\partial(x,\ y)}.$$

Apply this result to the equations

(i) $s=0$;

(ii) $xys+xp+yq=0$. (R. Liouville.)

Equations having Integrals of Doubly-finite Rank.

199. Hitherto, the equations considered have been such that they have admitted an integral which involves an arbitrary function of one of the variables so as to be of finite rank in that variable: but the general integral, in the actual expressions obtained, involved partial quadratures so far as concerns the occurrence of the other variable. It is manifest that one specially select class of equations will be constituted by those possessing general integrals which are of finite rank in both variables and for which, therefore, both sets of transformations lead to a vanishing invariant after only a finite number of operations in each series effected upon the given equation.

We have seen that, when the equation

$$s+ap+bq+cz=0$$

possesses an integral of rank $n+1$ in the form

$$z=AX+A_1X'+\ldots+A_nX^{(n)},$$

and when the σ-transformation

$$\frac{\partial z}{\partial y}+az=z_1$$

is effected, then the rank of z_1 is n. On the other hand, if the Σ-transformation

$$\frac{\partial z}{\partial x} + bz = Z_1$$

is effected, then the rank of Z_1 in the variable x is obviously $n+2$. Similarly, if the equation possesses an integral of rank $m+1$ in the form

$$z = BY + B_1Y' + \ldots + B_m Y^{(m)},$$

and if the σ-transformation

$$\frac{\partial z}{\partial y} + az = z_1$$

is effected, the rank of z_1 is $m+2$, while if the Σ-transformation

$$\frac{\partial z}{\partial x} + bz = Z_1$$

is effected, the rank of Z_1 is m.

Now suppose that both sets of transformations are finite in the sense that, in each set, only a finite number of operations is needed to produce a vanishing invariant; then obviously the general integral of the equation is

$$z = AX + A_1X' + \ldots + A_nX^{(n)} + BY + B_1Y' + \ldots + B_mY^{(m)}.$$

The effect of the σ-transformation on this quantity z is to increase the rank of the new variable z_1 in y by one unit and to decrease the rank in x by one unit: that is, the integer $m+n$ is the same for z as for z_1, and therefore it is invariantive for the σ-transformation. Similarly, this integer $m+n$ is invariantive for the Σ-transformation. Accordingly, Darboux* calls this invariantive integer $m+n$ the *characteristic number* of the equation.

In preceding investigations, when only a single series of transformations leading to a vanishing invariant was considered, the general integral contained terms of one of the two forms

$$\int Y \frac{\partial^\mu \alpha}{\partial x^\mu} dy, \quad \int X \frac{\partial^\mu \beta}{\partial y^\mu} dx,$$

respectively: it is not impossible that, on integration by parts or through some other process, such terms could be replaced by

* *L.c.*, t. II, p. 38.

quantities that are of finite rank in the respective variables. Hence it is desirable to consider equations which are of finite rank in each of the variables. Accordingly, we assume that the equation

$$s + ap + bq + cz = 0$$

possesses a general integral which is of finite rank $n+1$ in the variable x and of finite rank $m+1$ in the variable y.

Let the σ-transformation be applied to this equation so as to construct n equations in succession: the dependent variable z_n of the last of these equations is of rank 1 in the variable x and of rank $m+n+1$ in the variable y so that, writing

$$\mu = m + n,$$

we have

$$z_n = KX + CY + C_1 Y' + \ldots + C_\mu Y^{(\mu)}.$$

Moreover, the invariant h_n of the last equation in the series vanishes, and so the equation is

$$\frac{\partial}{\partial x}\left(\frac{\partial z_n}{\partial y} + a_n z_n\right) + b_n \left(\frac{\partial z_n}{\partial y} + a_n z_n\right) = 0:$$

also the values of K and C_μ, the coefficients of X and of $Y^{(\mu)}$ in z_n, are* given by

$$K = e^{-\int a_n dy}, \quad C_\mu = e^{-\int b_n dx}.$$

Taking

$$u = e^{\int (a_n dy - b_n dx)},$$

$$\phi = z_n e^{\int a_n dy},$$

the equation is

$$\frac{\partial}{\partial x}\left(\frac{1}{u}\frac{\partial \phi}{\partial y}\right) = 0,$$

and therefore

$$\phi = X + \int u Y_1 dy,$$

where X and Y_1 are arbitrary functions of x and of y respectively. Having regard to the expression for z_n and to the value of K in that expression, we see that the X in z_n and the X in ϕ are the same; and then

$$\int u Y_1 dy = GY + G_1 Y' + \ldots + G_\mu Y^{(\mu)},$$

* Ex. 3, § 198.

where

$$G_r = \frac{C_r}{K}, \qquad (r = 0, 1, \ldots, \mu),$$

and, in particular,

$$G_\mu = \frac{C_\mu}{K} = e^{\int a_n dy - b_n dx} = u.$$

Hence Y_1, which is a function of y only, must be of the form

$$\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)},$$

where $\lambda, \lambda_1, \ldots, \lambda_\mu$ are determinate functions of y, the function Y being arbitrary. Thus

$$\begin{aligned}\frac{\partial}{\partial y}&\{GY + G_1 Y' + \ldots + G_{\mu-1} Y^{(\mu-1)} + u Y^{(\mu)}\}\\ &= uY_1\\ &= u\{\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)}\};\end{aligned}$$

and therefore, as Y is an arbitrary function of y, we must have

$$\begin{aligned} u\lambda_\mu &= G_{\mu-1} + \frac{\partial u}{\partial y},\\ u\lambda_{\mu-1} &= G_{\mu-2} + \frac{\partial G_{\mu-1}}{\partial y},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ u\lambda_r &= G_{r-1} + \frac{\partial G_r}{\partial y},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ u\lambda_1 &= G + \frac{\partial G_1}{\partial y},\\ u\lambda &= \frac{\partial G}{\partial y}.\end{aligned}$$

The elimination of the quantities $G, G_1, \ldots, G_{\mu-1}$ among these equations leads to the relation

$$u\lambda - \frac{\partial}{\partial y}(u\lambda_1) + \frac{\partial^2}{\partial y^2}(u\lambda_2) - \ldots + (-1)^\mu \frac{\partial^\mu}{\partial y^\mu}(u\lambda_\mu) + (-1)^{\mu+1}\frac{\partial^{\mu+1} u}{\partial y^{\mu+1}} = 0,$$

which may be regarded as an equation for the determination of the quantity u, where

$$u = e^{\int (a_n dy - b_n dx)}.$$

Moreover, when u is known (on the assumption that $\lambda, \lambda_1, \ldots, \lambda_\mu$ are known), then $G_{\mu-1}, \ldots, G_1, G$ are immediately derivable from the foregoing equations. The equation, having z_n for its dependent

variable, can be constructed: and thence, by the inverse substitutions repeated n times in succession, we construct the original equation.

A similar result follows if we proceed from the original equation, supposed to possess a general integral of what may be called doubly-finite rank, by using the Σ (or σ^{-1}) transformations. For the construction of the equation, it is necessary to solve an ordinary differential equation, still of order $\mu + 1$ and having x for its independent variable: the problem is of the same order of difficulty as under the preceding process.

Accordingly, continuing the solution of the problem under the former analysis, we choose the quantities $\lambda, \lambda_1, \ldots, \lambda_\mu$ at will as functions of y: the variable x remains parametric in the determination of u as an integral of the ordinary equation of order $\mu + 1$: and then, u being known, the quantities $G, G_1, \ldots, G_{\mu-1}$ are obtained (in reverse order) merely by differential operations. The knowledge of these quantities gives the value of ϕ: we return, by the inverted σ-transformations, to the original equation and to the value of z.

We have seen that an expression, involving an arbitrary function and derivatives of that function up to any order p, may have its rank less than p: if such be the case, one or other of the transformations considered in connection with the equations under discussion will lead to a vanishing invariant after a number of applications which is less than p. In the present case, the same question arises as to the rank of the quantity ϕ: the expression

$$GY + G_1 Y' + \ldots + G_{\mu-1} Y^{(\mu-1)} + uY^{(\mu)}$$

must not be reducible because otherwise the rank of ϕ in the variable y would be less than $\mu + 1$. The expression can only be reducible if

$$\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)}$$

is reducible, that is, if the equation

$$\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)} = 0$$

is reducible. The quantities $\lambda, \lambda_1, \ldots, \lambda_\mu$ are at our disposal: they therefore must be chosen so that the preceding equation is irreducible: and this choice can always be made*.

* A method for constructing irreducible equations has been given by Frobenius, *Crelle*, t. LXXX (1875), p. 332: see vol. IV of this work, § 80.

200. Now that the existence of the result is definitely established, its form can be materially simplified by the use of other properties. Let $y_1, \ldots, y_{\mu+1}$ be $\mu+1$ linearly independent integrals of the equation

$$\lambda v + \lambda_1 v' + \ldots + \lambda_\mu v^{(\mu)} + v^{(\mu+1)} = 0,$$

so that the determinant

$$\Delta = \begin{vmatrix} y_1 & , & y_2 & , & \ldots, & y_{\mu+1} \\ y_1' & , & y_2' & , & \ldots, & y'_{\mu+1} \\ \ldots & & \ldots & & \ldots & \ldots \\ y_1^{(\mu)}, & & y_2^{(\mu)}, & & \ldots, & y_{\mu+1}^{(\mu)} \end{vmatrix}$$

does not vanish, its actual value being $Ae^{-\int \lambda_\mu dy}$, where A is a non-vanishing constant. Then, for each of these integrals, the expression

$$\frac{\partial}{\partial y}(Gy_i + G_1 y_i' + \ldots + G_{\mu-1} y_i^{(\mu-1)} + u y_i^{(\mu)})$$

vanishes, so that

$$Gy_i + G_1 y_i' + \ldots + G_{\mu-1} y_i^{(\mu-1)} + u y_i^{(\mu)} + \xi_i = 0,$$

where ξ_i is a function of x alone, and i has the values $1, \ldots, \mu+1$. Solving these $\mu+1$ linear equations for $G, G_1, \ldots, G_{\mu-1}$, and for u (which is G_μ), and substituting their values in

$$\begin{aligned} \phi &= X + \int u Y_1 dy \\ &= X + GY + G_1 Y' + \ldots + G_\mu Y^{(\mu)}, \end{aligned}$$

we have

$$\phi\Delta = \begin{vmatrix} X & , & Y & , & Y' & , & \ldots, & Y^{(\mu)} \\ \xi_1 & , & y_1 & , & y_1' & , & \ldots, & y_1^{(\mu)} \\ \xi_2 & , & y_2 & , & y_2' & , & \ldots, & y_2^{(\mu)} \\ \ldots & & \ldots & & \ldots & & \ldots & \ldots \\ \xi_{\mu+1}, & & y_{\mu+1}, & & y'_{\mu+1}, & & \ldots, & y_{\mu+1}^{(\mu)} \end{vmatrix} = \Theta,$$

say: and then

$$\begin{aligned} z_n &= e^{-\int a_n dy}\phi \\ &= \frac{\Theta}{\Delta} e^{-\int a_n dy}. \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= u Y_1 \\ &= u(\lambda Y + \lambda_1 Y' + \ldots + \lambda_\mu Y^{(\mu)} + Y^{(\mu+1)}); \end{aligned}$$

substituting for ϕ and equating coefficients of $Y^{(\mu+1)}$, we have

$$u\Delta = (-1)^{\mu+1} \begin{vmatrix} \xi_1, & y_1, & y_1', & \dots, & y_1^{(\mu-1)} \\ \xi_2, & y_2, & y_2', & \dots, & y_2^{(\mu-1)} \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{\mu+1}, & y_{\mu+1}, & y'_{\mu+1}, & \dots, & y_{\mu+1}^{(\mu-1)} \end{vmatrix},$$

which gives the value of u. Also, since the expression

$$Y^{(\mu+1)} + \lambda Y + \lambda_1 Y' + \dots + \lambda_\mu Y^{(\mu)}$$

vanishes when $Y = y_1, y_2, \dots, y_{\mu+1}$, we have

$$\begin{aligned} \Delta\,(Y^{(\mu+1)} + \lambda Y + \lambda_1 Y' + \dots + \lambda_\mu Y^{(\mu)}) \\ = (-1)^{\mu+1} \begin{vmatrix} Y, & Y', & \dots, & Y^{(\mu)}, & Y^{(\mu+1)} \\ y_1, & y_1', & \dots, & y_1^{(\mu)}, & y_1^{(\mu+1)} \\ \dots & \dots & \dots & \dots & \dots \\ y_{\mu+1}, & y'_{\mu+1}, & \dots, & y_{\mu+1}^{(\mu)}, & y_{\mu+1}^{(\mu+1)} \end{vmatrix} \\ = (-1)^{\mu+1}\Phi, \end{aligned}$$

say, so that

$$\frac{\partial\phi}{\partial y} = (-1)^{\mu+1}\frac{u}{\Delta}\Phi,$$

where u and Δ do not involve the arbitrary function Y.

It is to be noted that no linear relation can exist among the quantities $\xi_1, \xi_2, \dots, \xi_{\mu+1}$: if such a relation existed, there would be a linear relation among the quantities G of the form

$$\begin{aligned} G &= \alpha_1 G_1 + \dots + \alpha_{\mu-1} G_{\mu-1} + \alpha_\mu G_\mu \\ &= \alpha_1 G_1 + \dots + \alpha_{\mu-1} G_{\mu-1} + \alpha_\mu u, \end{aligned}$$

which by the use of the relations between u and these quantities G, would lead to a linear equation of order μ satisfied by u, contrary to the property that the linear equation of order $\mu+1$ satisfied by u is irreducible.

We can now proceed to the construction of the original equation as well as to the derivation of its general integral. We have

$$ze^{\int b\,dx} = \frac{\partial}{h\partial x}\cdot\frac{\partial}{h_1\partial x}\cdot\dots\cdot\frac{\partial}{h_{n-1}\partial x}(z_n e^{\int b\,dx}),$$

and therefore

$$z = Az_n + A_1\frac{\partial z_n}{\partial x} + \dots + A_n\frac{\partial^n z_n}{\partial x^n}.$$

When the value

$$z_n = \frac{\Theta}{\Delta} e^{-\int a_n dy}$$

is substituted, we see at once that terms involving $X, X', \dots, X^{(n)}$ will occur linearly: moreover, there will be terms involving Y and its derivatives linearly, and it is known that the highest derivative of Y that should occur is $Y^{(m)}$. Hence we have

$$z = \alpha X + \alpha_1 X' + \dots + \alpha_n X^{(n)} + \beta Y + \beta_1 Y' + \dots + \beta_m Y^{(m)},$$

where the coefficients $\alpha, \dots, \alpha_n, \beta, \dots, \beta_m$ have yet to be determined.

Owing to the presence of the determinant Θ in the expression for z_n, it is clear that z_n vanishes when

$$X = \xi_i, \quad Y = y_i,$$

simultaneously; and this is true for all values of i. Also forming the derivatives of z_n with regard to x, it is clear that every one of these derivatives vanishes similarly when

$$X = \xi_i, \quad Y = y_i,$$

simultaneously: consequently, as z is a linear combination of z_n and these derivatives, z itself also vanishes in these circumstances. Hence

$$\alpha\xi_i + \alpha_1\xi_i' + \dots + \alpha_n\xi_i^{(n)} + \beta y_i + \beta_1 y_i' + \dots + \beta_m y_i^{(m)} = 0,$$

for $i = 1, 2, \dots, \mu + 1$, where $\mu = m + n$. These $m + n + 1$ relations, linear and homogeneous in the $m + n + 2$ coefficients α and β, determine the ratios of these quantities and can be regarded as determining all these $m + n + 2$ coefficients, save as to an unknown common factor. Consequently z is known, save as to this factor.

But the differential equation, so far as concerns its two invariants, is unaffected by the association of any factor with z: and therefore, within this range, we can neglect the factor or, what is the same thing, we can make it unity. Hence, writing

$$\begin{vmatrix} X, & X', & \dots, & X^{(n)}, & Y, & Y', & \dots, & Y^{(m)} \\ \xi_1, & \xi_1', & \dots, & \xi_1^{(n)}, & y_1, & y_1', & \dots, & y_1^{(m)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \xi_{\mu+1}, & \xi'_{\mu+1}, & \dots, & \xi^{(n)}_{\mu+1}, & y_{\mu+1}, & y'_{\mu+1}, & \dots, & y^{(m)}_{\mu+1} \end{vmatrix} = Z,$$

we may take

$$z = Z;$$

and in this expression, $y_1, \ldots, y_{\mu+1}$ are $\mu+1$ linearly independent functions of y, while $\xi_1, \ldots, \xi_{\mu+1}$ are $\mu+1$ linearly independent functions of x.

Next, to obtain the differential equation of the second order satisfied by z, we suppose that Z is expanded and, when expanded, has the form

$$Z = \alpha X + \alpha_1 X' + \ldots + \alpha_n X^{(n)} + \beta Y + \beta_1 Y' + \ldots + \beta_m Y^{(m)},$$

where the coefficients $\alpha, \ldots, \alpha_n, \beta, \ldots, \beta_m$ are now known functions of x and y, and, in particular,

$$\alpha = \begin{vmatrix} \xi_1', & \ldots, & \xi_1^{(n)}, & y_1', & \ldots, & y_1^{(m)} \\ \multicolumn{6}{c}{\dotfill} \\ \xi'_{\mu+1}, & \ldots, & \xi^{(n)}_{\mu+1}, & y'_{\mu+1}, & \ldots, & y^{(m)}_{\mu+1} \end{vmatrix},$$

with similar expressions for α_n, β, β_m. With the foregoing value Z of z, we have

$$\frac{\partial z}{\partial x} = \frac{\partial \alpha}{\partial x} X + \ldots + \alpha_n X^{(n+1)} + \frac{\partial \beta}{\partial x} Y + \ldots + \frac{\partial \beta_m}{\partial x} Y^{(m)},$$

$$\frac{\partial z}{\partial y} = \frac{\partial \alpha}{\partial y} X + \ldots + \frac{\partial \alpha_n}{\partial y} X^{(n)} + \frac{\partial \beta}{\partial y} Y + \ldots + \beta_m Y^{(m+1)},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 \alpha}{\partial x \partial y} X + \ldots + \frac{\partial \alpha_n}{\partial y} X^{(n+1)} + \frac{\partial^2 \beta}{\partial x \partial y} Y + \ldots + \frac{\partial \beta_m}{\partial x} Y^{(m+1)};$$

and therefore, taking the equation for z in the form

$$s + ap + bq + cz = 0,$$

and substituting these values of s, p, q, we have a relation

$$\frac{\partial \alpha_n}{\partial y} + a\alpha_n = 0,$$

from the coefficient of $X^{(n+1)}$, a relation

$$\frac{\partial \beta_m}{\partial x} + b\beta_m = 0,$$

from the coefficient of $Y^{(m+1)}$, and a relation

$$\frac{\partial^2 \alpha}{\partial x \partial y} + a \frac{\partial \alpha}{\partial x} + b \frac{\partial \alpha}{\partial y} + c\alpha = 0,$$

from the coefficient* of X. These three relations give the values of a, b, c: and so the differential equation is fully known.

It thus appears that equations having general integrals which are doubly finite in rank can be constructed. The integrals are formed by means of a number of functions $\xi_1, \ldots, \xi_{\mu+1}$ of x, having no linear relations with one another, and a number of functions $y_1, \ldots, y_{\mu+1}$ of y, likewise having no linear relations with one another. The number of different kinds of integrals is the same as the partition of the integer μ into two positive integers n and m: each integral, thus provided by a partition of the integer μ, determines a differential equation

$$s + ap + bq + cz = 0$$

uniquely. The integer μ is the characteristic number of the equations: and thus there are $\mu + 1$ different types of equations having one and the same characteristic number.

Ex. 1. From the values of α and β, expressed in the forms of determinants of $\xi_1, \ldots, \xi_{\mu+1}, y_1, \ldots, y_{\mu+1}$ and of their derivatives, verify that the relations

$$\frac{\partial^2 \alpha}{\partial x \partial y} + a\frac{\partial \alpha}{\partial x} + b\frac{\partial \alpha}{\partial y} + c\alpha = 0,$$

$$\frac{\partial^2 \beta}{\partial x \partial y} + a\frac{\partial \beta}{\partial x} + b\frac{\partial \beta}{\partial y} + c\beta = 0,$$

are equivalent to one another.

Ex. 2. Prove that the expression for Z given in the text can vanish identically only when X is a linear combination of $\xi_1, \ldots, \xi_{\mu+1}$ with constant coefficients and, at the same time, Y is the same linear combination of $y_1, \ldots, y_{\mu+1}$. (Darboux.)

Ex. 3. There is only one equation with the characteristic number zero, for there is only one partition of 0.

To construct the equation and its integral, we require a single function of x and a single function of y, say ξ and η respectively. Then

$$Z = \begin{vmatrix} X, & Y \\ \xi, & \eta \end{vmatrix} = \xi\eta\left(\frac{X}{\xi} - \frac{Y}{\eta}\right):$$

* There is also a relation

$$\frac{\partial^2 \beta}{\partial x \partial y} + a\frac{\partial \beta}{\partial x} + b\frac{\partial \beta}{\partial y} + c\beta = 0$$

from the coefficient of Y. Both of these relations involving c are expressions of the condition that α and β are solutions of the equation obtained by taking $X=1$, $Y=0$, and $X=0$, $Y=1$, respectively.

we know that any factor associated with z can be neglected without affecting the invariants of the equation, and so we neglect the factor $\xi\eta$: and then, taking

$$X_1 = \frac{X}{\xi}, \quad Y_1 = \frac{Y}{\eta},$$

as new arbitrary functions of x and of y respectively, we have

$$z = X_1 - Y_1.$$

The differential equation is

$$s = 0;$$

and we have

$$h = 0, \quad k = 0.$$

The equation $s = 0$ is, in fact, the simplest reduced form of equation for which $h = 0$, $k = 0$: the less simple form is

$$s + p\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial x} + z\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x\,\partial y}\right) = 0,$$

where u is any function of the variables.

Ex 4. There are two types of equation with the characteristic number unity, corresponding to the partitions $1+0$ and $0+1$ of 1. We take two functions ξ_1 and ξ_2 of x, and two functions η_1 and η_2 of y.

For one type, corresponding to partition $1+0$, the value of Z is

$$Z = \begin{vmatrix} X, & X', & Y \\ \xi_1, & \xi_1', & \eta_1 \\ \xi_2, & \xi_2', & \eta_2 \end{vmatrix} = aX + a_1 X' + \beta Y.$$

No loss of generality as regards the invariants of the equation is caused by taking

$$\xi_2 = x, \quad \eta_2 = y;$$

and then

$$a = y\xi_1' - \eta_1, \quad a_1 = x\eta_1 - y\xi_1, \quad \beta = \xi_1 - x\xi_1',$$

so that

$$a = -\frac{1}{a_1}\frac{\partial a_1}{\partial y} = -\frac{x\eta_1' - \xi_1}{x\eta_1 - y\xi_1},$$

$$b = -\frac{1}{\beta}\frac{\partial \beta}{\partial x} = \frac{x\xi_1''}{\xi_1 - x\xi_1'},$$

$$c = ab,$$

the last being obtained either from the equation

$$\frac{\partial^2 \beta}{\partial x\,\partial y} + a\frac{\partial \beta}{\partial x} + b\frac{\partial \beta}{\partial y} + c\beta = 0,$$

on noticing that b does not involve y, or from the condition

$$k = 0,$$

so that

$$c = ab + \frac{\partial b}{\partial y} = ab.$$

The value of h is given by

$$h=\frac{\partial a}{\partial x}+ab-c$$
$$=\frac{\partial a}{\partial x}$$
$$=-\frac{(x\xi_1'-\xi_1)(y\eta_1'-\eta_1)}{(x\eta_1-y\xi_1)^2};$$

and then

$$h_1=2h-k-\frac{\partial^2(\log h)}{\partial x\,\partial y}$$
$$=2\frac{\partial a}{\partial x}+2\frac{\partial^2}{\partial x\,\partial y}(\log a_1)$$
$$=2\frac{\partial a}{\partial x}-2\frac{\partial a}{\partial x}=0.$$

For the other type of equation, corresponding to the partition $0+1$, the value of Z is

$$Z=\begin{vmatrix} X, & Y, & Y' \\ \xi_1, & \eta_1, & \eta_1' \\ \xi_2, & \eta_2, & \eta_2' \end{vmatrix}=\alpha X+\beta Y+\beta_1 Y'.$$

As in the previous case, we may take

$$\xi_2=x,\quad \eta_2=y;$$

and then

$$\alpha=\eta_1-y\eta_1',\quad \beta=x\eta_1'-\xi_1,\quad \beta_1=y\xi_1-x\eta_1,$$

so that

$$a=-\frac{1}{\alpha}\frac{\partial \alpha}{\partial y}=\frac{y\eta_1''}{\eta_1-y\eta_1'},$$
$$b=-\frac{1}{\beta}\frac{\partial \beta}{\partial x}=-\frac{y\xi_1'-\eta_1}{y\xi_1-x\eta_1},$$
$$c=ab.$$

And, for this type of equation, we have

$$h=0,\quad K_1=0.$$

Ex. 5. Integrate the equations

(i) $$s-\frac{1}{y}p+\frac{k}{x}q-\frac{k}{xy}z=0,$$

k being a constant;

(ii) $$s+\left(\frac{1}{y}-\frac{1}{x-y}\right)p-\frac{2}{x}q-\frac{2}{x}\left(\frac{1}{y}-\frac{1}{x-y}\right)z=0;$$

(iii) $$s+\frac{2}{x-y}p-\frac{2}{x-y}q-\frac{4}{(x-y)^2}z=0.$$

Ex. 6. Obtain the three types of equation

$$s+ap+bq+cz=0,$$

which have 2 for their characteristic number.

Is it possible to determine c so that 2 shall be the characteristic number, if

$$a=\frac{(x-y)\,\eta''}{\xi-\eta-(x-y)\,\eta'},\quad b=\frac{(x-y)\,\xi''}{\xi-\eta-(x-y)\,\eta'},$$

where ξ and η are functions of x and of y respectively?

Ex. 7. Prove that the equation

$$s+mxyp+nyz=0$$

is of finite rank in one of the variables if $m \div n$ is an integer, m and n being constants.

Obtain the successive invariants when this condition is satisfied; and integrate the equation so as to obtain the general integral.

Ex. 8. Transform the equation

$$r+2\lambda s+(\lambda^2-\mu^2)\,t+\alpha p+\beta q=0,$$

where λ, μ, α, β are functions of the independent variables, so that it becomes

$$s+ap+bq=0;$$

and express the invariants of the latter in terms of λ, μ, α, β.

Apply this method to the equation

$$r+\tfrac{3}{2}s+\tfrac{1}{2}t-\frac{2}{x}(p+q)=0,$$

shewing that the transformed equation is of doubly-finite rank.

(Winckler.)

Ex. 9. Solve the equation

$$r-t=\frac{1}{x}(p-q),$$

by making a transformation similar to that in the last example.

Ex. 10. Prove that the equation

$$s+\frac{p+q}{x+y}-\tfrac{1}{4}\,\frac{n(n+1)}{(x+y)^2}\,z=0$$

is of doubly-finite rank when n is a positive integer (including zero).

Ex. 11. Shew that the integral of the equation

$$r-t=2n\frac{p}{x},$$

where n is a positive integer, is

$$z=\sum_{m=0}^{m=n}(-1)^m\,\frac{2^m}{m!}\,\frac{\dbinom{n}{m}}{\dbinom{2n}{m}}\,x^m\,[\phi^{(m)}(x-y)+\psi^{(m)}(x+y)],$$

where ϕ and ψ are arbitrary functions; and that the integral of the equatio

$$r-t=-2n\frac{p}{x},$$

where n is a positive integer, is

$$z=\sum_{m=0}^{m=n-1}(-1)^m\frac{2^m}{m!}\frac{\binom{n-1}{m}}{\binom{2n-2}{m}}x^{-2n+m+1}[\phi^{(m)}(x-y)+\psi^{(m)}(x+y)].$$

Deduce these results also by transforming both equations into Laplace' linear form. (Sersawy.

DARBOUX'S MODIFIED FORMS.

201. Darboux has given* another form for the succession o invariants and for the equation itself, when it is of finite rank i either of the variables.

Suppose that a linear equation is of finite rank in the variabl x, and assume that the invariant h of the nth equation in th succession, obtained by the use of the σ-transformation, is zer Thus, as $h_n=0$, we have

$$\begin{aligned}k_n &= \frac{\partial b_n}{\partial y}+a_nb_n-c_n\\ &= \frac{\partial b_n}{\partial y}-\frac{\partial a_n}{\partial x}\\ &= -\frac{\partial^2\log\alpha}{\partial x\partial y},\end{aligned}$$

where

$$\alpha=e^{\int(a_ndy-b_ndx)};$$

and the value of z_n is given by

$$z_ne^{\int a_ndy}=X+\int\alpha Ydy.$$

Now

$$\begin{aligned}h_{n-1} &= k_n\\ &= -\frac{\partial^2\log\alpha}{\partial x\partial y};\end{aligned}$$

and

$$h_{m+1}-2h_m+h_{m-1}=-\frac{\partial^2\log h_m}{\partial x\partial y},$$

* *Théorie générale des surfaces*, t. II, pp. 123, et seq.

for all values of m, so that, taking $m = n - 1$, we have

$$\begin{aligned} h_{n-2} &= \quad 2h_{n-1} - \frac{\partial^2 \log h_{n-1}}{\partial x \partial y} \\ &= -2\frac{\partial^2 \log \alpha}{\partial x \partial y} - \frac{\partial^2}{\partial x \partial y}\left\{\log\left(\frac{1}{\alpha}\frac{\partial^2 \alpha}{\partial x \partial y} - \frac{1}{\alpha^2}\frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y}\right)\right\} \\ &= -\frac{\partial^2}{\partial x \partial y}\left\{\log\left(\alpha\frac{\partial^2 \alpha}{\partial x \partial y} - \frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y}\right)\right\}. \end{aligned}$$

This result suggests a new form: Darboux introduces quantities

$$H_0 = \quad \alpha \,,$$

$$H_1 = \begin{vmatrix} \alpha \,, & \dfrac{\partial \alpha}{\partial x} \\ \dfrac{\partial \alpha}{\partial y} \,, & \dfrac{\partial^2 \alpha}{\partial x \partial y} \end{vmatrix},$$

$$H_2 = \begin{vmatrix} \alpha \,, & \dfrac{\partial \alpha}{\partial x} \,, & \dfrac{\partial^2 \alpha}{\partial x^2} \\ \dfrac{\partial \alpha}{\partial y} \,, & \dfrac{\partial^2 \alpha}{\partial x \partial y} \,, & \dfrac{\partial^3 \alpha}{\partial x^2 \partial y} \\ \dfrac{\partial^2 \alpha}{\partial y^2} \,, & \dfrac{\partial^3 \alpha}{\partial x \partial y^2} \,, & \dfrac{\partial^4 \alpha}{\partial x^2 \partial y^2} \end{vmatrix},$$

and so on. The expressions, by means of these quantities, of the two invariants already obtained are

$$\begin{aligned} h_n &= \quad 0, \\ h_{n-1} &= -\frac{\partial^2}{\partial x \partial y}(\log H_0), \\ h_{n-2} &= -\frac{\partial^2}{\partial x \partial y}(\log H_1); \end{aligned}$$

and it is natural to inquire whether the expression of h_{n-s} is

$$h_{n-s} = -\frac{\partial^2}{\partial x \partial y}(\log H_{s-1}).$$

This suggested expression for h_{n-s} is actually valid: to establish the validity, we proceed as follows.

Writing

$$\frac{\partial^{r+s} \alpha}{\partial x^r \partial y^s} = \alpha_{r,s},$$

consider the determinant

$$H_{p+1} = \begin{vmatrix} \alpha_{0,0} \ , & \alpha_{1,0} \ , & \dots, & \alpha_{p+1,0} \\ \alpha_{0,1} \ , & \alpha_{1,1} \ , & \dots, & \alpha_{p+1,1} \\ \dots & \dots & \dots & \dots \\ \alpha_{0,p+1}, & \alpha_{1,p+1}, & \dots, & \alpha_{p+1,p+1} \end{vmatrix}$$

Let $A_{i,j}$ denote the minor of $\alpha_{i,j}$ in H_{p+1}: then, by a well-known property, we have

$$\begin{vmatrix} A_{p,p} \ , & A_{p+1,p} \\ A_{p,p+1}, & A_{p+1,p+1} \end{vmatrix} = H_{p+1} \begin{vmatrix} \alpha_{0,0} \ , & \dots, & \alpha_{p-1,0} \\ \dots & \dots & \dots \\ \alpha_{0,p-1}, & \dots, & \alpha_{p-1,p-1} \end{vmatrix}$$

$$= H_{p+1} H_{p-1}.$$

Now

$$A_{p+1,p+1} = H_p,$$

$$A_{p+1,p} = \frac{\partial H_{p+1}}{\partial \alpha_{p+1,p}} = -\frac{\partial H_p}{\partial y},$$

$$A_{p,p+1} = \frac{\partial H_{p+1}}{\partial \alpha_{p,p+1}} = -\frac{\partial H_p}{\partial x},$$

$$A_{p,p} = \frac{\partial H_{p+1}}{\partial \alpha_{p,p}} = \frac{\partial^2 H_p}{\partial x \partial y};$$

and therefore

$$H_{p+1} H_{p-1} = H_p \frac{\partial^2 H_p}{\partial x \partial y} - \frac{\partial H_p}{\partial x} \frac{\partial H_p}{\partial y}$$

$$= H_p^2 \frac{\partial^2}{\partial x \partial y} (\log H_p).$$

Now suppose that

$$h_{n-s-1} = -\frac{\partial^2}{\partial x \partial y} (\log H_s),$$

for $s = 0, 1, \dots, p$; then

$$H_{p+1} H_{p-1} = - H_p^2 h_{n-p-1},$$

so that

$$\frac{\partial^2}{\partial x \partial y} (\log H_{p+1})$$

$$= -\frac{\partial^2}{\partial x \partial y} (\log H_{p-1}) + 2 \frac{\partial^2}{\partial x \partial y} (\log H_p) + \frac{\partial^2}{\partial x \partial y} (\log h_{n-p-1})$$

$$= -2h_{n-p-1} + h_{n-p} + \frac{\partial^2}{\partial x \partial y} (\log h_{n-p-1})$$

$$= -h_{n-p-2},$$

so that the form holds for $s=p+1$, if it holds for $s=0, 1, \dots, p$. It is known to hold for $s=0, 1$, so that it holds generally; and therefore

$$h_{n-i} = -\frac{\partial^2}{\partial x \partial y}(\log H_{i-1}),$$

for all the values of i.

202. Similar analysis can be employed to construct the whole series of equations that occur through the successive applications of the Laplace σ-transformation. The integral of the nth equation is

$$\begin{aligned} z_n e^{\int a_n dy} &= X + \int \alpha Y dy \\ &= z_0, \end{aligned}$$

say. Let z_m denote the determinant

$$z_m = \begin{vmatrix} z_0, & \dfrac{\partial z_0}{\partial x}, & \dots, & \dfrac{\partial^m z_0}{\partial x^m} \\ \alpha, & \dfrac{\partial \alpha}{\partial x}, & \dots, & \dfrac{\partial^m \alpha}{\partial x^m} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial^{m-1}\alpha}{\partial y^{m-1}}, & \dfrac{\partial^m \alpha}{\partial x \partial y^{m-1}}, & \dots, & \dfrac{\partial^{2m-1}\alpha}{\partial x^m \partial y^{m-1}} \end{vmatrix};$$

let Z_μ denote the minor of $\dfrac{\partial^\mu z_0}{\partial x^\mu}$ in z_m, and let $C_{i,j}$ denote the minor of $\alpha_{i,j}$ in z_m. Then, using the same theorem as before, we have

$$\begin{vmatrix} Z_{m-1}, & Z_m \\ C_{m-1,m-1}, & C_{m,m-1} \end{vmatrix} = (-1)^{m-1} z_m H_{m-2};$$

also

$$\begin{aligned} Z_m &= (-1)^m H_{m-1}, \\ Z_{m-1} &= (-1)^{m-1} \frac{\partial H_{m-1}}{\partial x}, \\ C_{m-1,m-1} &= -\frac{\partial z_{m-1}}{\partial x}, \\ C_{m,m-1} &= z_{m-1}; \end{aligned}$$

and therefore

$$z_m H_{m-2} = z_{m-1}\frac{\partial H_{m-1}}{\partial x} - H_{m-1}\frac{\partial z_{m-1}}{\partial x}.$$

Again,

$$\begin{vmatrix} Z_{m-1}, & Z_m \\ C_{m-1,m-2}, & C_{m,m-2} \end{vmatrix} = (-1)^m z_m \begin{vmatrix} \alpha, & \dfrac{\partial\alpha}{\partial x}, & \ldots, & \dfrac{\partial^{m-2}\alpha}{\partial x^{m-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{m-3}\alpha}{\partial y^{m-3}}, & & \ldots, & \dfrac{\partial^{2m-5}\alpha}{\partial x^{m-2}\partial y^{m-3}} \\ \dfrac{\partial^{m-1}\alpha}{\partial y^{m-1}}, & & \ldots, & \dfrac{\partial^{2m-3}\alpha}{\partial x^{m-2}\partial y^{m-1}} \end{vmatrix}$$

$$= (-1)^m z_m \frac{\partial H_{m-2}}{\partial y};$$

and therefore, as

$$C_{m,m-2} = -\frac{\partial z_{m-1}}{\partial y}, \quad C_{m-1,m-2} = \frac{\partial^2 z_{m-1}}{\partial x \partial y},$$

we have

$$z_m \frac{\partial H_{m-2}}{\partial y} = \frac{\partial z_{m-1}}{\partial y}\frac{\partial H_{m-1}}{\partial x} - H_{m-1}\frac{\partial^2 z_{m-1}}{\partial x \partial y}.$$

Hence

$$\frac{\partial^2 z_{m-1}}{\partial x \partial y} - \frac{\partial(\log H_{m-1})}{\partial x}\frac{\partial z_{m-1}}{\partial y} = -\frac{1}{H_{m-1}}\frac{\partial H_{m-2}}{\partial y} z_m$$

$$= -\frac{1}{H_{m-1}}\frac{\partial(\log H_{m-2})}{\partial y}\left\{z_{m-1}\frac{\partial H_{m-1}}{\partial x} - H_{m-1}\frac{\partial z_{m-1}}{\partial x}\right\},$$

by the former relation: consequently,

$$\frac{\partial^2 z_{m-1}}{\partial x \partial y} - \frac{\partial z_{m-1}}{\partial x}\frac{\partial(\log H_{m-2})}{\partial y} - \frac{\partial z_{m-1}}{\partial y}\frac{\partial(\log H_{m-1})}{\partial x}$$

$$+ z_{m-1}\frac{\partial(\log H_{m-1})}{\partial x}\frac{\partial(\log H_{m-2})}{\partial y} = 0.$$

This is a linear equation of the second order satisfied by z_{m-1}; and it has the canonical form constructed in § 192, for which $c = ab$. Moreover, its invariants are

$$-\frac{\partial^2}{\partial x \partial y}(\log H_{m-2}), \quad = h_{n-m+1},$$

and

$$-\frac{\partial^2}{\partial x \partial y}(\log H_{m-1}), \quad = h_{n-m}$$

$$= k_{n-m+1},$$

which are the invariants of the equation that is deduced from the original equation by $n - m + 1$ successive applications of the

Laplace σ-transformation. Now an equation in canonical form is uniquely determined by its invariants; hence the foregoing equation, which is satisfied by z_{m-1}, is the $(n-m+1)$th equation in the series derived from the original equation by the successive application of the Laplace σ-transformations.

Moreover, an expression for z_{m-1} (which is the general integral of the equation) has been given which involves the two arbitrary functions X and Y of x and of y respectively, through the quantity z_0 and its derivatives. It also involves the quantity α, which is not known initially but belongs to the last equation in the series. If, however, α can be obtained (and α must be determined, if this process of integration is to be effective in practice), then the whole series of equations is known and the integral of every equation in the series is known.

When α is assumed at will, the preceding equation satisfied by z_{m-1} is the typical expression of equations which are of finite rank $m-1$ in the variable x. If, however, the equation is given, then the invariant h is known; and, writing

$$\log U = -\iint h\,dx\,dy,$$

we have

$$H_{m-2} = U,$$

which is an equation satisfied by α: but, in the absence of other information, the value of m is not known, and the determination of α is not practicable by this method.

Ex. Prove that

$$z_{m-1}\frac{\partial^2 z_{m-1}}{\partial x\,\partial y} - \frac{\partial z_{m-1}}{\partial x}\frac{\partial z_{m-1}}{\partial y} = z_m z_{m-2},$$

$$z_{m-1}\frac{\partial H_{m-2}}{\partial y} - H_{m-2}\frac{\partial z_{m-1}}{\partial y} = H_{m-1} z_{m-2}.$$

(Darboux.)

203. If however the equation is of finite rank in both variables, say of rank n in the variable x and of rank m in the variable y, then the complete series of equations can be represented by equating the successive expressions

$$\sigma^{-m}F,\ \sigma^{-(m-1)}F,\ \ldots,\ \sigma^{-1}F,\ F,\ \sigma F,\ \ldots,\ \sigma^n F$$

to zero. In that case, we can begin with the equation

$$\sigma^n F = 0,$$

as before, assuming a quantity α; and we can construct gradually the invariants of the equations in backward succession. Writing

$$m + n = \mu - 1,$$

we know that the invariants of $\sigma^{-m}F$ are

$$-\frac{\partial^2 \log H_{\mu-1}}{\partial x \partial y}, \qquad -\frac{\partial^2 \log H_{\mu-2}}{\partial x \partial y};$$

as the series terminates with the mth repetition of the transformation σ^{-1}, the former vanishes. Now

$$H_{p+1} H_{p-1} = H_p{}^2 \frac{\partial^2}{\partial x \partial y} (\log H_p),$$

for all values of p; and therefore

$$\begin{aligned} H_\mu H_{\mu-2} &= H^2{}_{\mu-1} \frac{\partial^2}{\partial x \partial y} (\log H_{\mu-1}) \\ &= 0, \end{aligned}$$

or, since $H_{\mu-2}$ does not vanish, we must have

$$H_\mu = 0,$$

that is,

$$\begin{vmatrix} \alpha\ , & \dfrac{\partial \alpha}{\partial x}\ , & \ldots, & \dfrac{\partial^\mu \alpha}{\partial x^\mu} \\ \dfrac{\partial \alpha}{\partial y}\ , & \dfrac{\partial^2 \alpha}{\partial x \partial y}\ , & \ldots, & \dfrac{\partial^{\mu+1} \alpha}{\partial x^\mu \partial y} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^\mu \alpha}{\partial y^\mu}\ , & \dfrac{\partial^{\mu+1} \alpha}{\partial x \partial y^\mu}\ , & \ldots, & \dfrac{\partial^{2\mu} \alpha}{\partial x^\mu \partial y^\mu} \end{vmatrix} = 0,$$

which is an equation for the determination of α.

The equation shews that a relation

$$x_0 \alpha + x_1 \frac{\partial \alpha}{\partial x} + \ldots + x_\mu \frac{\partial^\mu \alpha}{\partial x^\mu} = 0$$

exists, such that $x_0, \ldots, x_\mu$ are functions of x alone. Suppose that $\xi_1, \ldots, \xi_\mu$ are μ linearly independent integrals of such an equation; then the most general value of α is

$$\alpha = \eta_1 \xi_1 + \eta_2 \xi_2 + \ldots + \eta_\mu \xi_\mu,$$

where $\eta_1, \ldots, \eta_\mu$ are independent of x, that is, are functions of y. The equation also shews that a relation

$$y_0 \alpha + y_1 \frac{\partial \alpha}{\partial y} + \ldots + y_\mu \frac{\partial^\mu \alpha}{\partial y^\mu} = 0$$

exists, such that $y_0, \ldots, y_\mu$ are functions of y alone; the same argument as before shews that the most general value of α is

$$\alpha = \eta_1 \xi_1 + \eta_2 \xi_2 + \ldots + \eta_\mu \xi_\mu,$$

where $\xi_1, \ldots, \xi_\mu$ are μ linearly independent functions of x and $\eta_1, \ldots, \eta_\mu$ are μ linearly independent functions of y.

The value of z_m, the integral of the last equation of the series, depends upon z_0, which is

$$X + \int Y\alpha\, dy,$$

and which thus appears to involve partial quadratures. In the case just considered when the equations are of finite rank in each of the variables, z_m should be expressible without the use of such quadratures: the actual expression can be obtained as follows.

The quantity α, where

$$\alpha = \xi_1 \eta_1 + \ldots + \xi_\mu \eta_\mu,$$

is the most general integral of the equation

$$P(\alpha) = y_0 \alpha + y_1 \frac{\partial \alpha}{\partial y} + \ldots + y_\mu \frac{\partial^\mu \alpha}{\partial y^\mu} = 0,$$

where $\eta_1, \ldots, \eta_\mu$ are μ linearly independent particular integrals. Let

$$Q(\beta) = y_0' \beta + y_1' \frac{\partial \beta}{\partial y} + \ldots + y_\mu' \frac{\partial^\mu \beta}{\partial y^\mu} = 0$$

be the equation which is the adjoint of $P(\alpha) = 0$: then*

$$\beta P(\alpha) - \alpha Q(\beta) = \frac{dU}{dy},$$

where U is a quantity free from partial quadratures and of order $\mu - 1$ in the derivatives of β. Now replace Y in the integral $\int Y\alpha\, dy$ by $-Q(Y)$, which is permissible because Y is quite arbitrary: and let α be the foregoing integral of $P(\alpha) = 0$: then

$$\begin{aligned} \frac{dU}{dy} &= \beta P(\alpha) - \alpha Q(\beta) \\ &= -\alpha Q(Y), \end{aligned}$$

* See vol. IV of this work, § 82.

on taking $\beta = Y$; and now

$$z_0 = X - \int \alpha Q(Y)\,dy$$
$$= X + U,$$

where U now is of order $\mu - 1$ in the derivatives of y. This being the value of z_0, the value of z_m is explicitly free from partial quadratures: the construction of other forms of z_m can be made exactly as before, in §§ 201, 202.

Goursat's Theorem on the Rank of an Equation.

204. We have seen that the general integral of the equation

$$s + ap + bq + cz = 0$$

involves its arbitrary elements in linear fashion. It can be particularised in many ways: each particular form is an integral: and any linear combination of such particular integrals with constant coefficients is also an integral of the equation. If, in any aggregate of such integrals, no one of them can be expressed as a linear combination of the others with constant coefficients, they are said to be linearly distinct.

Now suppose that n successive applications of the σ-transformation lead to a vanishing invariant h_n, being the first vanishing invariant of the forward series of equations: the preceding theory shews that the original equation possesses an integral

$$z = AX + A_1X' + \ldots + A_nX^{(n)},$$

where X is an arbitrary function of x, and $A, A_1, \ldots, A_n$ are determinate functions of x and y. Let $n+2$ linearly independent and arbitrary functions $X_1, \ldots, X_{n+2}$ be chosen in such a way that the $n+2$ integrals, which they determine, are linearly distinct. Denoting these integrals by $z_1, \ldots, z_{n+2}$, and eliminating $A, A_1, \ldots, A_n$ by means of their expressions, we have

$$\begin{vmatrix} z_1, & X_1, & \ldots, & X_1^{(n)} \\ z_2, & X_2, & \ldots, & X_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ z_{n+2}, & X_{n+2}, & \ldots, & X_{n+2}^{(n)} \end{vmatrix} = 0,$$

a relation in which the coefficients of $z_1, \ldots, z_{n+2}$ are functions of x only. Hence, upon the supposition that n applications of the σ-transformation lead to a vanishing invariant h_n, we have found a linear relation, the coefficients in which are functions of the variables, among $n+2$ linearly distinct integrals.

The converse* is also true, viz.: *If* $n+1$ *linearly distinct integrals of the equation are connected by a homogeneous linear relation the coefficients of which are functions of one of the variables only, then* $n-1$ *applications at most of one of the Laplace transformations will lead to a vanishing invariant.* (The σ- or the Σ-transformation should be applied according as the coefficients in the homogeneous linear relation are functions of x or are functions of y).

First, let $n=1$, so that, if z_1 and z_2 are two linearly distinct integrals, we have

$$z_2 = z_1 u,$$

where u is a function of x only. Substituting z_2 in the equation

$$s + ap + bq + cz = 0,$$

and remembering that z_1 also is an integral, we have

$$\frac{du}{dx}\left(\frac{\partial z_1}{\partial y} + az_1\right) = 0,$$

that is, since u is a function of x,

$$\frac{\partial z_1}{\partial y} + az_1 = 0.$$

Consequently,

$$\frac{\partial^2 z_1}{\partial x \partial y} + a\frac{\partial z_1}{\partial x} + \frac{\partial a}{\partial x} z_1 = 0,$$

and therefore

$$-b\frac{\partial z_1}{\partial y} + \left(\frac{\partial a}{\partial x} - c\right) z_1 = 0,$$

or, inserting the earlier value of $\frac{\partial z_1}{\partial y}$ and removing the non-vanishing factor z_1, we have

$$h = \frac{\partial a}{\partial x} + ab - c = 0,$$

which verifies the theorem for $n=1$.

* The theorem is due to Goursat, *Amer. Jour. Math.*, t. XVIII (1896), p. 348.

Take now the general case of a homogeneous linear relation between $n+1$ integrals, and assume that the coefficients are functions of x only: dividing by the coefficient of z_{n+1}, we have the relation in the form

$$z_{n+1} = \xi_1 z_1 + \xi_2 z_2 + \ldots + \xi_n z_n.$$

It may be assumed that the coefficients $\xi_1, \ldots, \xi_n$ are not connected by a linear relation with constant coefficients: if any relation exists, such as

$$\xi_n = a_1 \xi_1 + \ldots + a_{n-1} \xi_{n-1} + a_n,$$

then

$$z_{n+1} - a_n z_n = \xi_1 (z_1 + a_1 z_n) + \xi_2 (z_2 + a_2 z_n) + \ldots + \xi_{n-1} (z_{n-1} + a_{n-1} z_n),$$

a homogeneous linear relation between n linearly distinct integrals $z_1 + a_1 z_n, \ldots, z_{n-1} + a_{n-1} z_n, z_{n+1} - a_n z_n$. Thus the assumption that the coefficients $\xi_1, \ldots, \xi_n$ are linearly independent of one another is really an assumption that $n+1$ integrals is the smallest number between which a homogeneous linear relation exists. Such an assumption is no limitation but only makes the problem more precise: it will therefore be made.

Let the Laplace σ-transformation be applied to the equation, and write

$$\zeta_r = \frac{\partial z_r}{\partial y} + a z_r,$$

for $r = 1, \ldots, n$: thus $\zeta_1, \ldots, \zeta_n$ are integrals of the transformed equation. Now

$$z_{n+1} = \sum_{r=1}^{n} \xi_r z_r,$$

so that

$$\frac{\partial z_{n+1}}{\partial y} = \sum_{r=1}^{n} \xi_r \frac{\partial z_r}{\partial y},$$

$$\frac{\partial z_{n+1}}{\partial x} = \sum_{r=1}^{n} \left(\xi_r \frac{\partial z_r}{\partial x} + z_r \frac{d\xi_r}{dx} \right),$$

$$\frac{\partial^2 z_{n+1}}{\partial x \partial y} = \sum_{r=1}^{n} \left(\xi_r \frac{\partial^2 z_r}{\partial x \partial y} + \frac{\partial z_r}{\partial y} \frac{d\xi_r}{dx} \right):$$

substituting z_{n+1} for z in

$$s + ap + bq + cz = 0,$$

and remembering that z_r is an integral for $r = 1, \ldots, n$, we have

$$\sum_{r=1}^{n} \left(\frac{\partial z_r}{\partial y} + a z_r \right) \frac{d\xi_r}{dx} = 0,$$

that is,

$$\zeta_1 \frac{d\xi_1}{dx} + \zeta_2 \frac{d\xi_2}{dx} + \ldots + \zeta_n \frac{d\xi_n}{dx} = 0.$$

Now no one of the quantities $\xi_1, \ldots, \xi_n$ is a constant: otherwise the original relation would effectively be a relation between only n integrals of the original equation: hence we can divide by $\frac{d\xi_n}{dx}$, and we have

$$\zeta_n = \xi_1'\zeta_1 + \xi_2'\zeta_2 + \ldots + \xi'_{n-1}\zeta_{n-1}.$$

We have assumed that there is no relation of the form

$$\xi_n = a_1\xi_1 + \ldots + a_{n-1}\xi_{n-1} + a_n,$$

and therefore there is no relation of the form

$$\frac{d\xi_n}{dx} = a_1 \frac{d\xi_1}{dx} + \ldots + a_{n-1} \frac{d\xi_{n-1}}{dx}:$$

consequently there is no relation of the form

$$a_1\xi_1' + \ldots + a_{n-1}\xi'_{n-1} + 1 = 0.$$

It therefore follows that a relation of the indicated type, among $n+1$ linearly distinct integrals of the original equation, leads to a relation of the same type among n integrals of the equation, which is the result of applying to the original equation the Laplace σ-transformation.

Applying the Laplace σ-transformation to the equation which has $\zeta_1, \zeta_2, \ldots, \zeta_n$ for its integrals, we obtain a new equation and are led to a relation, of the same type as before, existing among $n-1$ integrals of the new equation; and applying it to successive equations $n-1$ times in all, we are led at the end to a homogeneous linear relation between two integrals of the last equation.

From what has already been proved, we know that the h-invariant of this last equation is zero: that is,

$$h_n = 0.$$

A similar argument holds when the coefficients in the homogeneous linear relation between $n+1$ integrals are functions of y alone: the successive application of the Laplace Σ-transformation, $n-1$ times in all, leads to a vanishing invariant for the last equation: that is,

$$K_n = 0,$$

in this case.

Thus Goursat's theorem is established.

Ex. Shew that three integrals of the equation

$$s + \frac{2z}{(x-y)^2} = 0$$

are connected by an equation

$$\begin{vmatrix} z_1, & a_1x^2+2b_1x+c_1, & a_1x+b_1 \\ z_2, & a_2x^2+2b_2x+c_2, & a_2x+b_2 \\ z_3, & a_3x^2+2b_3x+c_3, & a_3x+b_3 \end{vmatrix} = 0;$$

and obtain them. Shew also that three integrals ζ_1, ζ_2, ζ_3 are connected by a relation

$$\begin{vmatrix} \zeta_1, & a_1y^2+2b_1y+c_1, & a_1y+b_1 \\ \zeta_2, & a_2y^2+2b_2y+c_2, & a_2y+b_2 \\ \zeta_3, & a_3y^2+2b_3y+c_3, & a_3y+b_3 \end{vmatrix} = 0;$$

and obtain them also. (Goursat.)

Lévy's Transformation.

205. In the two Laplace transformations

$$z_1 = \frac{\partial z}{\partial y} + az, \quad Z_1 = \frac{\partial z}{\partial x} + bz,$$

the only quantities that occur are those which appear in the differential equation: and it might be deemed possible to secure a more general transformation by taking new variables of the form

$$z_1' = \frac{\partial z}{\partial y} + uz, \quad Z_1' = \frac{\partial z}{\partial x} + vz,$$

where u and v are quantities initially at our disposal. As a matter of fact, such a transformation is not so effective as the Laplace transformation: its main importance lies, partly in the analytical forms obtained, partly in some geometrical applications. As it does not lead to any new process for the integration of the equation, only a brief outline will be given.

The suggested transformation is adopted by Lévy* in the form

$$z' = \frac{\partial z}{\partial y} + (a + \alpha)\,z,$$

* *Journ. de l'Éc. Polytechnique*, t. XXXVII, Cah. LVI (1886), p. 67.

where α is the disposable quantity, the original equation still being

$$s + ap + bq + cz = 0.$$

Proceeding as before, we easily find that the equation for z' is

$$s' + a'p' + b'q' + c'z' + Az = 0,$$

where

$$\begin{aligned} a' &= a - \frac{1}{\alpha}\frac{\partial \alpha}{\partial y}, \\ b' &= b, \\ c' &= c + k - h - \frac{\partial \alpha}{\partial x} - \frac{b}{\alpha}\frac{\partial \alpha}{\partial y}, \\ \frac{A}{\alpha} &= \frac{\partial^2 \log \alpha}{\partial x \partial y} - \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right) + k - h. \end{aligned}$$

In order that the transformed equation may have the same character as the original equation, we take

$$A = 0,$$

and then we have

$$s' + a'p' + b'q' + c'z' = 0.$$

Denoting the invariants of this equation by h' and k', we have

$$\begin{aligned} h' &= \frac{\partial a'}{\partial x} + a'b' - c' \\ &= 2h - k + \frac{\partial \alpha}{\partial x} - \frac{\partial^2 \log \alpha}{\partial x \partial y} \\ &= h + \frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right), \end{aligned}$$

on account of $A = 0$: and

$$\begin{aligned} k' &= \frac{\partial b'}{\partial y} + a'b' - c' \\ &= h + \frac{\partial \alpha}{\partial x}. \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{k'}{\alpha}\right) &= \frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right) + \frac{\partial^2 \log \alpha}{\partial x \partial y} \\ &= \frac{\partial \alpha}{\partial x} + h - k \\ &= k' - k, \end{aligned}$$

so that

$$k = k' - \frac{\partial}{\partial y}\left(\frac{k'}{\alpha}\right).$$

It is clear that k' cannot vanish unless $k = 0$; hence, in so far as Lévy's transformation aims at securing an integrable form of equation, it is unnecessary to take further account of the invariant k'.

Let h_1 be the h-invariant belonging to the equation obtained by applying the Laplace σ-transformation to the original equation, so that

$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y}:$$

then

$$\begin{aligned} h' - h_1 &= \frac{\partial \alpha}{\partial x} - \frac{\partial^2 \log \alpha}{\partial x \partial y} + \frac{\partial^2 \log h}{\partial x \partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\alpha}{h} h\right) + \frac{\partial}{\partial x}\left\{\frac{\alpha}{h}\frac{\partial}{\partial y}\left(\frac{h}{\alpha}\right)\right\} \\ &= \frac{\partial}{\partial x}\left(\frac{\alpha h'}{h}\right). \end{aligned}$$

It is clear that h' can vanish only if $h_1 = 0$: and therefore the Lévy transformation does not provide a vanishing invariant unless a vanishing invariant is provided by the Laplace σ-transformation. Hence from the point of view of integrating the equation, no particular advantage accrues from the Lévy transformation.

Ex. 1. Shew that, if the Laplace σ-transformation be applied any number of times in succession to the equation

$$s' + a'p' + b'q' + c'z' = 0,$$

as transformed by Lévy, and if h_1', h_2', h_3', ... are the h-invariants of the successive equations, then

$$h_n = h'_{n-1} - \frac{\partial}{\partial x}\left(\frac{\alpha h' h_1' \ldots h'_{n-1}}{h h_1 \ldots h_{n-1}}\right),$$

$$h_n' = h_n \quad + \frac{\partial}{\partial y}\left(\frac{h h_1 \ldots h_n}{\alpha h' h_1' \ldots h'_{n-1}}\right),$$

where $h_1, \ldots, h_n$ denote, as usual, the h-invariants of the equations obtained by the successive application of the σ-transformation to the original equation. (Lévy.)

Ex. 2. Shew that the Laplace σ-transformation and a Lévy transformation are permutable with each other in the sense that the invariants of the doubly transformed equation are independent of the order of application of the two transformations. (Lévy.)

Ex. 3. Shew that, if ζ is any particular solution of the original equation, then the Lévy transformation can be expressed in the form

$$\frac{z'}{\zeta} = \frac{\partial}{\partial y}\left(\frac{z}{\zeta}\right). \qquad \text{(Lévy.)}$$

THE EQUATION $r + 2\alpha p + 2\beta q + \gamma z = 0$.

206. We now come to consider, more briefly, the equation

$$r + a'p + b'q + c'z = 0;$$

this was found (§ 189) to be the alternative of

$$s + ap + bq + cz = 0$$

as a form to which every linear equation can be changed. It will be convenient to take it in a form

$$r + 2\alpha p + 2\beta q + \gamma z = 0,$$

where α, β, γ are functions of the independent variables x and y.

When we take

$$z = \lambda\zeta,$$

the new equation having ζ for its dependent variable is

$$\frac{\partial^2\zeta}{\partial x^2} + 2\alpha'\frac{\partial\zeta}{\partial x} + 2\beta'\frac{\partial\zeta}{\partial y} + \gamma'\zeta = 0,$$

where

$$\beta' = \beta,$$

$$\alpha' = \alpha + \frac{1}{\lambda}\frac{\partial\lambda}{\partial x},$$

$$\gamma' = \gamma + \frac{2\alpha}{\lambda}\frac{\partial\lambda}{\partial x} + \frac{2\beta}{\lambda}\frac{\partial\lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2\lambda}{\partial x^2}.$$

Obviously β is an invariant for the transformation in question. Again,

$$\gamma' - \frac{\partial\alpha'}{\partial x} - \alpha'^2 = \gamma - \frac{\partial\alpha}{\partial x} - \alpha^2 + \frac{2\beta}{\lambda}\frac{\partial\lambda}{\partial y},$$

and therefore

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta'}\left(\gamma' - \frac{\partial\alpha'}{\partial x} - \alpha'^2\right)\right\} - 2\frac{\partial\alpha'}{\partial y}$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial\alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial\alpha}{\partial y},$$

on the assumption that β does not vanish. Hence, if

$$I = \beta,$$

$$J = \frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial \alpha}{\partial y},$$

then I *and* J *are invariants for transformations of the type*

$$z' = \lambda z,$$

which leave the form of the equation unaltered.

Further, if β should vanish or if λ should involve x only, then

$$\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2$$

is an invariant for all transformations of the type $z' = \lambda z$.

Again, the equation conserves its form when the independent variables are changed according to a law

$$x' = \text{a function of } x, \quad y' = \text{a function of } y,$$

and the form is conserved only for such a law. If we take

$$x' = \xi, \quad y' = \eta,$$

ξ being a function of x only, and η a function of y only, and if the new equation be ·

$$r' + 2\alpha' p' + 2\beta' q' + \gamma' z = 0,$$

we have

$$\beta' = \frac{\beta \eta'}{\xi'^2},$$

$$\alpha' = \frac{\alpha}{\xi'} + \frac{1}{2}\frac{\xi''}{\xi'^2},$$

$$\gamma' = \frac{\gamma}{\xi'};$$

and therefore

$$I' = \beta' = \frac{\eta'}{\xi'^2} I,$$

$$J' = \frac{\partial}{\partial \xi}\left\{\frac{1}{\beta'}\left(\gamma' - \frac{\partial \alpha'}{\partial \xi} - \alpha'^2\right)\right\} - 2\frac{\partial \alpha'}{\partial \eta}$$

$$= \frac{1}{\xi'\eta'} J - \frac{1}{2\xi'\eta'}\frac{\partial}{\partial x}\left[\frac{\{\xi, x\}}{\beta}\right],$$

where $\{\xi, x\}$ is the Schwarzian derivative of ξ.

Thus I is invariantive for all changes of the independent variables that conserve the form of the equation. In order that J may be invariantive, we must have

$$\frac{\partial}{\partial x}\left[\frac{\{\xi, x\}}{\beta}\right] = 0,$$

of which obviously there are four cases, viz.

(i) if β is a function of x alone, the transformations of x are limited by the relation

$$\{\xi, x\} = k\beta,$$

where k is any constant:

(ii) if β is a function of y alone, the transformations of x are limited by the relation

$$\{\xi, x\} = -\tfrac{1}{2}k^2,$$

where k is an arbitrary constant, that is,

$$\xi = \frac{ae^{kx} + b}{ce^{kx} + d},$$

where a, b, c, d, k are arbitrary constants:

(iii) if β involves x and y and is expressible in a form

$$\beta = XY,$$

where X is a function of x alone and Y is a function of y alone, then the transformations of x are limited by the relation

$$\{\xi, x\} = kX,$$

where k is any constant:

(iv) if β involves x and y and is not expressible in a form

$$\beta = XY,$$

or if β is zero, the transformations of x are limited by the relation

$$\{\xi, x\} = 0,$$

that is,

$$\xi = \frac{ax + b}{cx + d},$$

where a, b, c, d are arbitrary constants.

Thus the quantity J, which is invariantive for all transformations of the form

$$z' = \lambda z,$$

where λ is any function of x and y, is invariantive for only certain transformations of the independent variables. The more important transformations, however, are those which change the dependent variable only.

207. If the invariant I vanishes, the equation is of the form

$$\frac{\partial^2 z}{\partial x^2} + 2\alpha \frac{\partial z}{\partial x} + \gamma z = 0,$$

effectively an ordinary linear equation for which y is parametric. The one invariant is

$$K = \gamma - \frac{\partial \alpha}{\partial x} - \alpha^2,$$

being the usual invariant of the ordinary equation of the second order for transformations of the kind considered: writing

$$z' = ze^{\int \alpha\, dx},$$

we have the equation in the form

$$\frac{\partial^2 z'}{\partial x^2} + Kz' = 0.$$

The case needs no special consideration in the present connection.

If the invariant J vanishes, that is, if

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right)\right\} - 2\frac{\partial \alpha}{\partial y} = 0,$$

then a function θ of x and y exists such that

$$\alpha = \frac{\partial \theta}{\partial x},$$

$$\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2 = 2\beta \frac{\partial \theta}{\partial y}.$$

When these values of α and γ are introduced into the equation, it is easily transformed to

$$\frac{\partial^2}{\partial x^2}(ze^\theta) + 2\beta \frac{\partial}{\partial y}(ze^\theta) = 0.$$

Various cases arise according to the form of β.

If neither I nor J should vanish, it is always possible to transform the equation so that it shall not contain the first derivative of z with regard to x. To make this transformation, take two quantities θ and ϕ such that

$$\theta = \int_{x_0}^{x} \alpha\, dx,$$

$$\phi = \int_{x_0}^{x} J\, dx;$$

then

$$\alpha = \frac{\partial \theta}{\partial x},$$

$$J = \frac{\partial \phi}{\partial x},$$

and consequently

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right)\right\} - 2\,\frac{\partial^2 \theta}{\partial x \partial y} = J = \frac{\partial \phi}{\partial x},$$

so that we may take

$$\frac{1}{\beta}\left(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\right) - 2\,\frac{\partial \theta}{\partial y} = \phi,$$

the arbitrary additive function of y being deemed included in ϕ. Then, writing

$$z' = ze^{\theta},$$

we easily find

$$\frac{\partial^2 z'}{\partial x^2} + 2\beta\,\frac{\partial z'}{\partial y} + 2\beta\phi z' = 0:$$

the invariant J' of this equation is

$$J' = \frac{\partial \phi}{\partial x},$$

which, in fact, is J.

Ex. 1. Suppose that the invariant J of the equation vanishes and that β does not involve x: then, if

$$\zeta = ze^{\theta}, \quad dt = -\frac{dy}{2\beta},$$

the equation takes the well-known form

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial \zeta}{\partial t}.$$

If an integral of this equation is required which is such as to give

$$\zeta = f(t), \quad \frac{\partial \zeta}{\partial x} = g(t),$$

when $x=a$, it is easily obtainable in the form

$$\zeta=f(t)+\frac{(x-a)^2}{2\,!}f'(t)+\frac{(x-a)^4}{4\,!}f''(t)+\dots$$
$$+(x-a)g(t)+\frac{(x-a)^3}{3\,!}g'(t)+\frac{(x-a)^5}{5\,!}g''(t)+\dots.$$

The construction of an integral of the original equation, which is such as to give

$$z=F(y),\quad \frac{\partial z}{\partial x}=G(y),$$

when $x=a$, is now only a matter of transformation of the variables.

Ex. 2. Obtain the integral of the equation

$$\frac{\partial^2 z}{\partial x^2}=x\frac{\partial z}{\partial y},$$

such that

$$z=f(y),\quad \frac{\partial z}{\partial x}=g(y),$$

when $x=0$.

Ex. 3. The equation

$$\frac{\partial^2 z}{\partial x^2}+2a\frac{\partial z}{\partial x}+2\beta\frac{\partial z}{\partial y}+\gamma z=0$$

is transformed by a substitution

$$z_1=\frac{\partial z}{\partial x}+(a+u)z$$

into the equation

$$\frac{\partial^2 z_1}{\partial x^2}+2a_1\frac{\partial z_1}{\partial x}+2\beta_1\frac{\partial z_1}{\partial y}+\gamma_1 z_1=0:$$

prove that u satisfies the equation

$$\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\left(\frac{\partial u}{\partial x}-u^2\right)\right\}+2\frac{\partial u}{\partial y}=J,$$

and that a_1, β_1, γ_1 are given by

$$a_1=a-\frac{1}{2\beta}\frac{\partial\beta}{\partial x},$$
$$\beta_1=\beta,$$
$$\gamma_1=\gamma-\frac{a}{\beta}\frac{\partial\beta}{\partial x}+\frac{u}{\beta}\frac{\partial\beta}{\partial x}-2\frac{\partial u}{\partial x}.$$

Prove also that J_1, the J-invariant of the new equation, is given by

$$J_1-J=\frac{\partial}{\partial y}\left(\frac{1}{\beta}\frac{\partial\beta}{\partial x}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left\{\frac{1}{\beta}\frac{\partial}{\partial x}\left(\frac{1}{\beta}\frac{\partial\beta}{\partial x}\right)\right\}-\frac{\partial}{\partial x}\left\{\frac{1}{u}\frac{\partial}{\partial x}\left(\frac{u^2}{\beta}\right)\right\}.$$

Verify independently this last result for the special case

$$u=0,\quad J=0.$$

208. The somewhat cumbrous forms, occurring in the last example given, are the forms which are necessary for the maintenance of the type of the equation: and they suggest that no series of successive transformations, similar to the Laplace σ- and Σ-transformations, can usefully be constructed for the equation or can lead to types of equations the integrals of which are expressible in finite form.

Moreover, taking an equation for which neither of the invariants vanishes, we have seen that it can be transformed so as to become

$$\frac{\partial^2 z'}{\partial x^2} + 2\beta \frac{\partial z'}{\partial y} + 2\beta\phi z' = 0,$$

where neither β nor ϕ vanishes. The conclusions of § 186 and the application of Cauchy's theorem alike shew that there is a general integral involving two arbitrary functions in its expression, both of them having y for their argument: moreover, owing to the particular linear form of the equation, each of the functions and its derivatives enters linearly into the expression of the integral, when the integral is given explicitly. It is easy to see that, *when these functions are quite arbitrary, the integral cannot be expressed in finite form which is completely explicit and free from partial quadratures.* If possible, let such an integral be

$$z' = \alpha Y + \alpha_1 Y' + \ldots + \alpha_n Y^{(n)},$$

where Y is an arbitrary function of y: the substitution of this value of z' introduces, through $2\beta \dfrac{\partial z'}{\partial y}$, a term

$$2\beta\alpha_n Y^{(n+1)}$$

which cannot be cancelled, for $Y^{(n+1)}$ is not introduced elsewhere in the substitution.

It is thus useless to seek for finite explicit forms for the most general integral provided by Cauchy's theorem for an unconditioned equation of the present type; but for particular equations finite forms may be obtainable when the arbitrary functions are specialised. In the latter instances, the earlier argument does not hold: for in the case of a specialised function Y, the term

$$2\beta\alpha_n Y^{(n+1)}$$

may be balanced by other terms from

$$\frac{\partial^2 \alpha_n}{\partial x^2} Y^{(n)} + 2\beta \frac{\partial \alpha_n}{\partial y} Y^{(n)} + 2\beta\phi\alpha_n Y^{(n)}.$$

The result is limited to integrals that occur in explicit form: it does not necessarily hold for integrals the expressions of which involve partial quadratures.

It therefore appears that for the unconditioned equation

$$\frac{\partial^2 z}{\partial x^2} + 2\alpha \frac{\partial z}{\partial x} + 2\beta \frac{\partial z}{\partial y} + \gamma z = 0,$$

the only integrals, which are of a general type and which are expressed explicitly, are not finite in form. Such integrals are given by Cauchy's theorem; and they are such that, when $x = a$, the quantity z assumes an assigned value $f(y)$ and the quantity $\frac{\partial z}{\partial x}$ assumes another assigned value $g(y)$, subject to conditions as to regularity—connected with the forms of α, β, γ, $f(y)$, $g(y)$. By taking all the functions f and g that are regular within a selected domain, and by taking all the domains within which the coefficients α, β, γ are regular, we obtain all the regular integrals of the equation. But the aggregate of integrals thus obtained is restricted to those which are regular; and if the integrals be constructed by the analysis used to obtain Cauchy's theorem, no one of them is in finite form.

Borel's Expression for Regular Integrals not finite in form.

209. In connection with these integrals, and as illustrating the generality of some integrals with partial quadratures, it is worthy of notice that Borel has shewn* that the aggregate of these regular integrals can be represented by one form which requires partial quadratures. His investigation deals with equations in n variables and of order p, which are algebraically resoluble with regard to $\frac{\partial^p z}{\partial x_1^p}$: it will be sufficiently explained for the present purpose by taking $n = 2$, $p = 2$.

* *Bull. des sciences math.*, Sér. II, t. XIX (1895), p. 122: the idea of this mode of representation of an aggregate of integrals was suggested to him by some remarks of Delassus, *ib.*, pp. 51 et seq.

Take any positions which are ordinary positions for the equation in the planes of x and of y respectively, and make these the origins. The integral given by Cauchy's theorem is such that $z=f(y)$ and $\frac{\partial z}{\partial x}=g(y)$, when $x=0$, the functions f and g being regular functions in the domain of $y=0$ and otherwise arbitrary. The initial conditions therefore give the values of

$$\frac{\partial^n z}{\partial y^n}, \quad \frac{\partial^{n+1} z}{\partial x \partial y^n},$$

when $x=0$ and $y=0$, for all values of n: let these quantities be arranged in any sequence such that the total order of derivation does not decrease in the sequence and denote them, so arranged, by $\mu_1, \mu_2, \mu_3, \ldots$.

Now take a function u such that

$$u=\frac{1}{\left(1-\frac{x}{r}\right)\left(1-\frac{y}{r}\right)}:$$

and, forming the values of

$$\frac{\partial^n u}{\partial y^n}, \quad \frac{\partial^{n+1} u}{\partial x \partial y^n},$$

when $x=0$ and $y=0$, for all values of n, arrange them in the same sequence as the derivatives of z above; denote them, so arranged, by $\sigma_1, \sigma_2, \sigma_3, \ldots$. Owing to the limitations of regularity imposed on $f(y)$ and $g(y)$, it is always possible to choose a value r' of r such that, for values of m equal to or greater than some selected integer k,

$$\sigma_m > |\mu_m|:$$

we shall assume that r is greater than r'.

Construct the Cauchy integral of the given linear differential equation of the second order resoluble with regard to $\frac{\partial^2 z}{\partial x^2}$, assigning as the initial conditions that

$$z=u, \quad \frac{\partial z}{\partial x}=\frac{\partial u}{\partial x},$$

when $x=0$. On account of the linear form, this integral is of the form

$$z=\sigma_1\psi_1+\sigma_2\psi_2+\sigma_3\psi_3+\ldots,$$

where the quantities $\psi_1, \psi_2, \psi_3, \ldots$ are determinate regular functions of x and y in the domains considered. This integral is absolutely and uniformly convergent.

By means of the quantity z, so obtained, construct a quantity θ defined by the equation

$$\theta(x, y, \alpha) = \sigma_1\psi_1 \cos\alpha + \sigma_2\psi_2 \cos 2\alpha + \sigma_3\psi_3 \cos 3\alpha + \ldots,$$

where α is a real parameter: thus θ is convergent, and it is determinate, containing no arbitrary function. Also take

$$F(\alpha) = \frac{\mu_1}{\sigma_1}\cos\alpha + \frac{\mu_2}{\sigma_2}\cos 2\alpha + \frac{\mu_3}{\sigma_3}\cos 3\alpha + \ldots,$$

which is absolutely and uniformly convergent: as $F(\alpha)$ contains the quantities $\mu_1, \mu_2, \mu_3, \ldots$ derived through the arbitrary functions $f(y)$ and $g(y)$ initially given, it clearly is an arbitrary function. It is possible to integrate the product $\theta F(\alpha)$ with regard to α, on account of the character of the convergence of θ and of $F(\alpha)$. Obviously

$$\frac{1}{\pi}\int_0^{2\pi} \theta(x, y, \alpha) F(\alpha)\, d\alpha = \mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 + \ldots,$$

and, in the integral with partial quadratures, $\theta(x, y, \alpha)$ is determinate while $F(\alpha)$ is arbitrary. Now just as the integral, determined by the assignment of initial values $\sigma_1, \sigma_2, \sigma_3, \ldots$ to certain derivatives of the dependent variable, is given by

$$\sigma_1\psi_1 + \sigma_2\psi_2 + \sigma_3\psi_3 + \ldots,$$

so the integral, determined by the assignment of $\mu_1, \mu_2, \mu_3, \ldots$ to the same derivatives of the dependent variable, is given by

$$\mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 + \ldots.$$

But the latter aggregate of assignments is the equivalent of the initial conditions in Cauchy's theorem which require that z and $\dfrac{\partial z}{\partial x}$ shall acquire the values $f(y)$ and $g(y)$ respectively, when $x = 0$, the functions f and g being regular within the domain of $y = 0$ and being otherwise arbitrary. Denoting this integral by Z, we have

$$\begin{aligned} Z &= \mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 + \ldots \\ &= \frac{1}{\pi}\int_0^{2\pi} \theta(x, y, \alpha) F(\alpha)\, d\alpha, \end{aligned}$$

where $\theta(x, y, \alpha)$ is a determinate function, and $F(\alpha)$ is an arbitrary function because of the arbitrary quality in its coefficients.

Thus the Cauchy integral, associated with the two arbitrary functions in the initial conditions, can be represented by an expression, requiring partial quadratures and involving only a single arbitrary function *.

Ex. 1. A simple example will shew how Borel's construction of the integral with partial quadratures works out in practice. Consider the equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}:$$

when the initial conditions are that $z = f(y)$ and $\frac{\partial z}{\partial x} = g(y)$ when $x = 0$, on the assumption $f(y)$ and $g(y)$ are regular functions in a domain (say) of $y = 0$: the explicit integral (Ex. 1, § 207) is

$$Z = f(y) + \frac{x^2}{2!} f'(y) + \frac{x^4}{4!} f''(y) + \dots$$
$$+ x g(y) + \frac{x^3}{3!} g'(y) + \frac{x^5}{5!} g''(y) + \dots$$

For the actual expressions of $f(y)$ and $g(y)$, let

$$f(y) = \sum_0 \frac{\nu_m}{m!} y^m,$$

$$g(y) = \sum_0 \frac{\lambda_m}{m!} y^m;$$

and in order to secure uniform convergence, let the domain of y for each of these functions be a circle of radius r, where $r > r' > 1$, r' itself being a quantity that is greater than unity. Then if M be the greatest value of $|f(y)|$ within the domain, and if M' be the greatest value of $|g(y)|$ within the domain, it is known † that

$$|\nu_m| \leqslant M r^{-m}, \quad |\lambda_m| \leqslant M' r^{-m},$$

or if N denote a quantity greater than M and M', then

$$|\nu_m| < N r^{-m}, \quad |\lambda_m| < N r^{-m}.$$

When values of $f(y)$ and $g(y)$ are substituted so as to give a doubly-infinite series which obviously converges, we have

$$Z = \sum_{n=0} \sum_{m=0} \frac{y^n}{n!} \left\{ \frac{x^{2m}}{2m!} \nu_{n+m} + \frac{x^{2m+1}}{(2m+1)!} \lambda_{n+m} \right\},$$

which is merely another form of the integral satisfying the initial conditions.

* The explanation of the paradox in the present case, if it be regarded as a paradox, lies in the fact that the two sets of arbitrary coefficients arising from the two functions $f(y)$ and $g(y)$ respectively are included in the single function $F(\alpha)$.

† *T. F.*, § 22.

Now construct the integral of the equation which is such that

$$z = Ne^{\nu}, \quad \frac{\partial z}{\partial x} = Ne^{\nu},$$

when $x=0$. Keeping for the moment coefficients ν_n', λ_n' so as to correspond with ν_n, λ_n respectively, where, in fact,

$$\nu_n' = N, \quad \lambda_n' = N,$$

we have the integral (say ζ) in the form

$$\zeta = \sum_{n=0} \sum_{m=0} \frac{y^n}{n\,!} \left\{ \frac{x^{2m}}{2m\,!} \nu'_{n+m} + \frac{x^{2m+1}}{(2m+1)\,!} \lambda'_{n+m} \right\}.$$

In accordance with the notation of the text, we take

$$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \ldots = \nu_0', \lambda_0', \nu_1', \lambda_1', \nu_2', \lambda_2', \ldots,$$

and

$$\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \ldots = \nu_0, \lambda_0, \nu_1, \lambda_1, \nu_2, \lambda_2, \ldots;$$

and we construct a function $\theta(x, y, a)$ from ζ by changing σ_μ in ζ into $\sigma_\mu \cos \mu a$, so that

$$\begin{aligned} \theta(x, y, a) = \sum_{n=0} \sum_{m=0} \frac{y^n}{n\,!} \Bigg[& \frac{x^{2m}}{2m\,!} \nu'_{n+m} \cos\{(2n+2m+1)a\} \\ & + \frac{x^{2m+1}}{(2m+1)\,!} \lambda'_{n+m} \cos\{(2n+2m+2)a\} \Bigg]. \end{aligned}$$

Inserting the values of ν'_{n+m} and λ'_{n+m}, we have

$$\begin{aligned} \theta(x, y, a) &= N \sum_{n=0} \sum_{m=0} \frac{y^n}{n\,!} \left[\frac{x^{2m}}{2m\,!} \cos\{(2n+2m+1)a\} + \frac{x^{2m+1}}{(2m+1)\,!} \cos\{(2n+2m+2)a\} \right] \\ &= N \sum_{n=0} \sum_{p=0} \frac{y^n}{n\,!} \frac{x^p}{p\,!} \cos\{(2n+p+1)a\} \\ &= Ne^{x\cos a + y\cos 2a} \cos(a + x\sin a + y\sin 2a). \end{aligned}$$

It is evident, on simple substitution, that the relation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial y}$$

is satisfied.

The function $F(a)$ is now required: in accordance with the construction in the text, we take

$$\begin{aligned} F(a) &= \frac{\mu_1}{\sigma_1} \cos a + \frac{\mu_2}{\sigma_2} \cos 2a + \ldots \\ &= \frac{\nu_0}{\nu_0'} \cos a + \frac{\lambda_0}{\lambda_0'} \cos 2a + \frac{\nu_1}{\nu_1'} \cos 3a + \frac{\lambda_1}{\lambda_1'} \cos 4a + \ldots \\ &= \frac{1}{N} \sum_{m=0} [\nu_m \cos\{(2m+1)a\} + \lambda_m \cos\{(2m+2)a\}]. \end{aligned}$$

Because

$$\frac{|\nu_m|}{N} < \frac{1}{r^m}, \quad \frac{|\lambda_m|}{N} < \frac{1}{r^m},$$

and r is greater than a quantity which itself is greater than unity, this series for $F(a)$ converges absolutely and uniformly. Hence, writing

$$G(a) = \sum_{m=0} [\nu_m \cos\{(2m+1)a\} + \lambda_m \cos\{(2m+2)a\}],$$

we have

$$Z = \frac{1}{\pi}\int_0^{2\pi} e^{x\cos a + y\cos 2a} \cos(a + x\sin a + y\sin 2a)\, G(a)\, da,$$

which is an expression, requiring partial quadratures and involving the one arbitrary function $G(a)$, for the integral of the equation determined by the initial conditions that

$$\left.\begin{aligned} Z &= f(y) = \Sigma \frac{\nu_m}{m!} y^m \\ \frac{\partial Z}{\partial x} &= g(y) = \Sigma \frac{\lambda_m}{m!} y^m \end{aligned}\right\},$$

when $x=0$, and when the suppositions as to the domains of $f(y)$ and $g(y)$ are satisfied.

If the domains, within which assigned functions $f(y)$ and $g(y)$ are regular, be circles of radius r, where r is less than unity, the case can be changed to the case already discussed by taking new variables such that

$$\frac{x''}{r''} = \frac{x}{r}, \quad \frac{y''}{r''} = \frac{y}{r},$$

where r'' is greater than a quantity which itself is greater than unity.

Evidently

$$\nu_m = \frac{1}{\pi}\int_0^{2\pi} \cos\{(2m+1)a\}\, G(a)\, da,$$

$$\lambda_m = \frac{1}{\pi}\int_0^{2\pi} \cos\{(2m+2)a\}\, G(a)\, da.$$

Ex. 2. Prove that an integral of the equation in the preceding example, viz. of

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y},$$

is given by

$$z = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x + 2ay^{\frac{1}{2}})\, e^{-a^2}\, da,$$

where ϕ denotes any function of its argument. (Fourier.)

Ex. 3. The functions

$$f(y) = \sum_{m=0} \frac{\nu_m}{m!} y^m, \quad g(y) = \sum_{m=0} \frac{\lambda_m}{m!} y^m,$$

are regular functions of y in a domain, round the origin, of radius greater than a quantity which itself is greater than unity ; and $G(a)$ is given by

$$G(a) = \sum_{m=0} [\nu_m \cos\{(2m+1)a\} + \lambda_m \cos\{(2m+2)a\}],$$

which is a uniformly converging series. Prove that an integral of the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

such that z and $\frac{\partial z}{\partial x}$ are equal to $f(y)$ and $g(y)$ respectively, when $x=0$, is given by

$$z = \frac{1}{2\pi} \int_0^{2\pi} \theta(x, y, a)\, G(a)\, da,$$

where

$$\theta(x, y, a) = e^{y\cos 2a - x\sin 2a}\{\cos(a + y\sin 2a + x\cos 2a) + \sin(y\sin 2a + x\cos 2a)\}$$
$$+ e^{y\cos 2a + x\cos 2a}\{\cos(a + y\sin 2a - x\cos 2a) - \sin(y\sin 2a - x\cos 2a)\}.$$

CHAPTER XIV.

Adjoint Equations: Linear Equations having Equal Invariants.

The present chapter is devoted, partly to the consideration of the adjoint equation and of Riemann's investigation whereby the adjoint equation is employed to further the integration of the original equation, partly to the consideration of equations which are self-adjoint and are of finite rank in both of the variables. In the account here given, much use has been made of the exposition given by Darboux*: and the method devised by Moutard, in the fragment of the memoir quoted in § 218, is explained and some illustrations are given. Reference may also be made to a memoir† by R. Liouville.

210. Not a few of the characteristic properties of a linear ordinary equation are expressible by means of the properties of the associated equation which is usually called Lagrange's adjoint equation. It proves similarly convenient to associate an adjoint equation with a linear partial equation which, for the present purpose, will be limited to the form

$$s + ap + bq + cz = 0.$$

Generalising the usual definition‡ of the expression adjoint to an ordinary linear expression, which is the linear condition to be satisfied by v in order that

$$vP(w), \quad = v\left(P_0 \frac{d^n w}{dz^n} + \ldots + P_n w\right),$$

may be an exact differential, we say that, if $F(z)$ is a quantity which is linear in z and its derivatives with regard to x and y, then

* *Théorie générale des surfaces*, t. II, pp. 71—163.

† *Journ. de l'Éc. Polytechnique*, t. XXXVII (1886), pp. 7—62.

‡ See vol. IV of this work, § 82.

the linear expression adjoint to $F(z)$ is the quantity which must vanish in order that the double integral

$$\iint uF(z)\,dx\,dy$$

can be expressed as a simple integral. The condition therefore is that u should be such as to secure that a relation

$$uF(z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

should hold, where M and N are free from all quadrature operations. For our purpose, we take

$$F(z) = s + ap + bq + cz;$$

and then, if

$$uF(z) = u(s + ap + bq + cz) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y},$$

M and N must, quà functions of z, be of the forms

$$M = A\frac{\partial z}{\partial y} + Bz, \quad N = C\frac{\partial z}{\partial x} + Dz,$$

respectively, where A, B, C, D are functions of x, y, u, and do not involve z. Substituting these values of M and N, and equating coefficients of the derivatives of z, we have

$$u = A + C,$$

$$au = B + \frac{\partial C}{\partial y},$$

$$bu = \frac{\partial A}{\partial x} + D,$$

$$cu = \frac{\partial B}{\partial x} + \frac{\partial D}{\partial y}.$$

Evidently,

$$\frac{\partial}{\partial x}(au) + \frac{\partial}{\partial y}(bu) - cu = \frac{\partial^2 C}{\partial x \partial y} + \frac{\partial^2 A}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial x \partial y},$$

so that, writing

$$G(u) = \frac{\partial^2 u}{\partial x \partial y} - a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + \left(c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)u,$$

the necessary condition is

$$G(u) = 0.$$

When this condition is satisfied, the above four equations become equivalent to three only that are independent of one another: consequently, they are inadequate for the precise determination of A, B, C, D, and therefore M and N cannot be precisely determined. The latter result is, however, only a fair expectation from the form adopted; for $\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}$ will remain unaltered in value if M and N be increased by $\frac{\partial \phi}{\partial y}$ and $-\frac{\partial \phi}{\partial x}$ respectively, where ϕ is any function of x, y, z, u, which is linear and homogeneous in z and u.

We therefore may satisfy all the equations and still may make one assumption that is not inconsistent with them: the simplest appears to be the assumption that $A = C$. Then

$$A = C = \tfrac{1}{2}u;$$

and so

$$B = au - \tfrac{1}{2}\frac{\partial u}{\partial y},$$

$$D = bu - \tfrac{1}{2}\frac{\partial u}{\partial x},$$

so that

$$M = \tfrac{1}{2}\left(u\frac{\partial z}{\partial y} - z\frac{\partial u}{\partial y}\right) + auz,$$

$$N = \tfrac{1}{2}\left(u\frac{\partial z}{\partial x} - z\frac{\partial u}{\partial x}\right) + buz.$$

Then, with these values of M and N, we have

$$uF(z) - \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = zG(u),$$

that is,

$$\left.\begin{aligned} uF(z) - zG(u) &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \\ zG(u) - uF(z) &= -\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \end{aligned}\right\}.$$

The former shews that $G(u) = 0$ is the condition that the double integral $\iint uF(z)\,dx\,dy$ should be expressible as a simple integral: the latter shews that $F(z) = 0$ is the condition that the double

integral $\iint zG(u)\,dxdy$ should be expressible as a simple integral. Hence *the two equations*

$$F(z)=0, \quad G(u)=0,$$

are adjoint to each other, a property which is the same as for ordinary linear equations.

Also, $G(u)=0$ is a linear equation, similar in type to $F(z)=0$ and of the same order. Denoting the invariants of $F(z)=0$ by h_f and k_f, and the invariants of $G(u)=0$ by h_g and k_g, we have

$$h_g=-\frac{\partial a}{\partial x}+(-a)(-b)-\left(c-\frac{\partial a}{\partial x}-\frac{\partial b}{\partial y}\right)=k_f,$$

$$k_g=-\frac{\partial b}{\partial y}+(-a)(-b)-\left(c-\frac{\partial a}{\partial x}-\frac{\partial b}{\partial y}\right)=h_f.$$

Thus the invariants of either of two equations, which are reciprocally adjoint, are the invariants of the other merely interchanged: and it will be noticed that the coefficients a and b in the two equations differ only in sign.

Further, if $F(z)=0$ and $G(u)=0$ are effectively the same, so that each is self-adjoint, then $a=0$, $b=0$: the equation is

$$s+cz=0,$$

that is, the equation has equal invariants. The converse of this property is obviously true.

Relations between Adjoint Equations under Laplace Transformations.

211. Consider now the two series of equations derived from $F=0$ by the successive applications of the two Laplace transformations. The two sets of quantities, which (by vanishing) give the equations of the series, are expressible in the form

$$\ldots,\ \sigma^{-3}F,\ \sigma^{-2}F,\ \sigma^{-1}F,\ F,\ \sigma F,\ \sigma^2 F,\ \sigma^3 F,\ \ldots.$$

The invariants h_i and k_i of $\sigma^i F$, where i is positive, are given according to the law

$$h_{\mu+1}=2h_\mu-h_{\mu-1}-\frac{\partial^2\log h_\mu}{\partial x\partial y},$$

$$k_{\mu+1}=h_\mu;$$

and the invariants H_i and K_i of $\sigma^{-i}F$, where i is positive, are given according to the law

$$H_{\mu+1} = K_\mu,$$
$$K_{\mu+1} = 2K_\mu - K_{\mu-1} - \frac{\partial^2 \log K_\mu}{\partial x \partial y}.$$

Let the h-invariant of F be denoted by h_0 and the k-invariant be denoted by h_{-1}. Then

$$H_1 = k = h_{-1},$$
$$K_1 = 2k - h - \frac{\partial^2 \log k}{\partial x \partial y}$$
$$= 2h_{-1} - h_0 - \frac{\partial^2 \log h_0}{\partial x \partial y},$$

so that we may write

$$K_1 = h_{-2},$$

in conformity with the general law: and then

$$H_2 = K_1 = h_{-2},$$

and so on. Thus the invariants of the equations may be arranged in the form

$$\ldots, \begin{matrix} h_{-2} \\ h_{-3} \end{matrix}\Bigg\}, \begin{matrix} h_{-1} \\ h_{-2} \end{matrix}\Bigg\}, \begin{matrix} h \\ h_{-1} \end{matrix}\Bigg\}, \begin{matrix} h_1 \\ h \end{matrix}\Bigg\}, \begin{matrix} h_2 \\ h_1 \end{matrix}\Bigg\}, \ldots,$$

that is, we have a succession of functions

$$\ldots, h_{-3}, \quad h_{-2}, \quad h_{-1}, \quad h, \quad h_1, \quad h_2, \quad h_3, \ldots.$$

Now let the Laplace transformations be applied to the equation $G(u) = 0$ any number of times in succession. The two sets of quantities, which (by vanishing) give the equations of the series, are expressible in the form

$$\ldots, \sigma^{-3}G, \quad \sigma^{-2}G, \quad \sigma^{-1}G, \quad G, \quad \sigma G, \quad \sigma^2 G, \quad \sigma^3 G, \ldots.$$

The invariants of G are known to be k and h, being those of F interchanged. For σG, the coefficients are

$$a_1' = -a - \frac{\partial \log k}{\partial y},$$
$$b_1' = -b,$$
$$c_1' = c - 2\frac{\partial b}{\partial y} + b\frac{\partial \log k}{\partial y};$$

hence, for σG,

$$\begin{aligned} h_1' &= \frac{\partial a_1'}{\partial x} + a_1'b_1' - c_1' \\ &= 2k - h - \frac{\partial^2 \log k}{\partial x \partial y} \\ &= K_1 \\ &= h_{-2}, \end{aligned}$$

and

$$\begin{aligned} k_1' &= \frac{\partial b_1'}{\partial y} + a_1'b_1' - c_1' \\ &= k \\ &= h_{-1}; \end{aligned}$$

and so on. Thus the invariants of the equations may be arranged in the form

$$\ldots, \left.\begin{matrix} h_2 \\ h_3 \end{matrix}\right\}, \left.\begin{matrix} h_1 \\ h_2 \end{matrix}\right\}, \left.\begin{matrix} h \\ h_1 \end{matrix}\right\}, \left.\begin{matrix} h_{-1} \\ h \end{matrix}\right\}, \left.\begin{matrix} h_{-2} \\ h_{-1} \end{matrix}\right\}, \ldots;$$

that is, we have a succession of functions

$$\ldots, h_3, \quad h_2, \quad h_1, \quad h, \quad h_{-1}, \quad h_{-2}, \ldots,$$

being the same succession of functions as before, but in reversed order.

Thus the invariants of $\sigma^n G$ are those of $\sigma^{-n} F$ merely interchanged; and likewise for $\sigma^{-n} G$ and $\sigma^n F$.

Again, if one of the Laplace transformations is of finite rank for F because it leads to a vanishing invariant, the other of the Laplace transformations is of the same finite rank for G.

Again, if both of the Laplace transformations are of finite rank for F because each of them leads to a vanishing invariant, both of them are of finite rank for G, the orders of the finite ranks being interchanged: the characteristic number is the same for the two equations.

Hence, if an equation is integrable by Laplace's method, in the sense that a finite number of either of the transformations leads to an equation which admits of direct quadrature, the adjoint equation is also integrable by the method through the use of the complementary transformation: the number of operations is the same for the two equations.

212. In §§ 199—203 it was shewn how, when an equation is of finite rank in both variables, it was possible to construct all the equations derived from the original equation by both the Laplace transformations and also to give the integral of each of the equations. The method there given can be applied also to the adjoint equation, because then it also is of finite rank in both variables: and, as the tale of invariants for the two sets of equations is the same save only for reversal of order, practically the whole of the former calculations can be used.

The invariant h_n of $\sigma^{-n} G(u)$ vanishes; and so, as this is the vanishing invariant of $\sigma^n F(z)$, we have

$$h_n = -\frac{\partial^2 \log \alpha}{\partial x \partial y}.$$

In order to use the earlier results, we interchange x and y; and we begin as before. Let

$$\zeta_0 = Y + \int \alpha X \, dx,$$

and

$$\zeta_m = \begin{vmatrix} \zeta_0 \,, & \dfrac{\partial \zeta_0}{\partial y} \,, \ldots, & \dfrac{\partial^m \zeta_0}{\partial y^m} \\ \alpha \,, & \dfrac{\partial \alpha}{\partial y} \,, \ldots, & \dfrac{\partial^m \alpha}{\partial y^m} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial^{m-1} \alpha}{\partial x^{m-1}}, & \dfrac{\partial^m \alpha}{\partial x^{m-1} \partial y}, \ldots, & \dfrac{\partial^{2m-1} \alpha}{\partial x^{m-1} \partial y^m} \end{vmatrix};$$

then ζ_m satisfies the equation

$$\frac{\partial^2 \zeta_m}{\partial x \partial y} - \frac{\partial \zeta_m}{\partial x} \frac{\partial (\log H_m)}{\partial x} - \frac{\partial \zeta_m}{\partial y} \frac{\partial (\log H_{m-1})}{\partial x} + \zeta_m \frac{\partial (\log H_m)}{\partial x} \frac{\partial (\log H_{m-1})}{\partial y} = 0,$$

on making the changes indicated. It is the linear equation of the second order satisfied by ζ_m; and it is, in fact, the equation

$$\sigma^{-(n-m)} G(u) = 0,$$

having the integral $u = \zeta_m$.

Now this equation has the same invariants as the equation for z_m, except that their order is reversed: hence it is equivalent to the adjoint of that equation for z_m, which is

$$\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial y}(\log H_{m-1}) + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x}(\log H_m)$$
$$+ \left[\frac{\partial(\log H_m)}{\partial x}\frac{\partial(\log H_{m-1})}{\partial y} + \frac{\partial^2}{\partial x \partial y}\{\log(H_m H_{m-1})\}\right]\phi = 0.$$

Thus

$$\zeta_m = \lambda\phi,$$

so that

$$-\frac{\partial}{\partial y}(\log H_m) + \frac{1}{\lambda}\frac{\partial \lambda}{\partial y} = \frac{\partial}{\partial y}(\log H_{m-1}),$$
$$-\frac{\partial}{\partial x}(\log H_{m-1}) + \frac{1}{\lambda}\frac{\partial \lambda}{\partial x} = \frac{\partial}{\partial x}(\log H_m),$$

and therefore

$$\lambda = H_m H_{m-1}.$$

Hence the equation, which is satisfied by

$$\frac{\zeta_m}{H_m H_{m-1}},$$

is the adjoint of the equation satisfied by z_m.

Ex. 1. Integrate the equations:—

(i) $s - xp - yq - (1 - xy)z = 0$;

(ii) $s - mxp - nyq + (m - 2n + mnxy)z = 0$,
where m and n are constants;

(iii) $s - myp - e^{cy}q + (c + my)e^{cy}z = 0$,
where m and c are constants;

(iv) $s - xyq + mxz = 0$,
where m is a finite integer;

(v) $s + \frac{1}{y}p - \frac{c}{x}q - \frac{c}{xy}z = 0$,
where c is a constant;

(vi) $s - \frac{2}{x-y}p + \frac{2}{x-y}q = 0$;

(vii) $s + \left(\frac{1}{x-y} - \frac{1}{y}\right)p + \frac{2}{x}q - z\left\{\frac{2}{xy} - \frac{2}{x(x-y)} - \frac{1}{(x-y)^2}\right\} = 0$;

all of these being of finite rank.

Ex. 2. Form the equation which is adjoint to

$$s+\frac{m}{x+y}p+\frac{n}{x+y}q+\frac{l}{(x+y)^2}z=0:$$

construct its invariants and thence verify that, if the original equation is of finite rank in either variable, the adjoint equation is of finite rank in the other variable.

Ex. 3. Prove that, by using the adjoint equation, any linear equation

$$s+ap+bq+cz=0$$

can be expressed in the form

$$(a+\beta)\,s+\frac{\partial a}{\partial y}p+\frac{\partial \beta}{\partial x}q=0.$$

If z' and z'' be any two integrals of this equation, supposed of rank n in one of the variables, shew that the equation

$$\frac{\partial}{\partial y}\left(p\frac{q''}{q'}\right)-\frac{\partial}{\partial x}\left(q\frac{p''}{p'}\right)=0$$

is of rank $n+1$ in that variable; shew also that its integral can be obtained, merely by quadratures, from the integral of the original equation.

(R. Liouville.)

Ex. 4. With the notation and assumptions of the preceding example, shew that, if ζ be an integral of the equation

$$(a+\beta)\,s+\frac{\partial \beta}{\partial y}p+\frac{\partial a}{\partial x}q=0,$$

and if

$$\lambda=\int\left(\beta\frac{\partial\zeta}{\partial x}dx-a\frac{\partial\zeta}{\partial y}dy\right),$$

then the equation

$$(a+\beta)\,s+\frac{\lambda-\beta\zeta}{\lambda+a\zeta}\frac{\partial a}{\partial y}p+\frac{\lambda+a\zeta}{\lambda-\beta\zeta}\frac{\partial\beta}{\partial x}q=0$$

is of rank $n+1$. (R. Liouville.)

Riemann's use of Adjoint Equations.

213. If, of two given equations which are reciprocally adjoint, the integral of either in finite form has been obtained by any process, the integral of the other can be deduced. When the obtained integral is free from quadratures, the earlier investigations have shewn that it can be constructed by Laplace's method through the use of a finite number of one of the transformations; the result established in § 210 shews that the adjoint equation can also be integrated by means of the same number of applications of the

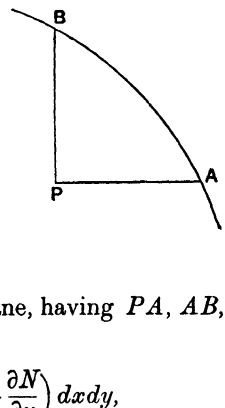
other of the transformations. When the obtained integral is not free from quadratures, the integral of the adjoint equation can be obtained by a method due to Riemann* who, indeed, was the first to use the equation that is adjoint to a given partial equation.

The object of the investigation is to determine an integral of the equation

$$F(z) = s + ap + bq + cz = 0,$$

which shall be the integral determined by the initial conditions in Cauchy's theorem in its most general form, that is, the variable z and one of its derivatives are to assume assigned functions of the variables as their values along a given curve in the plane.

Let AB represent the curve and, taking any point P in the plane, draw PA and PB parallel to the axes and in their positive directions. Let z denote an integral of the equation, and let u denote an integral of the adjoint equation



$$G(u) = 0;$$

then, with the notation of § 210, we have

$$uF(z) - zG(u) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

We can consider $PABP$ as an area in the plane, having PA, AB, BP for its boundary: hence

$$\iint \{uF(z) - zG(u)\}\, dx dy = \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy,$$

the integrals being taken over the area. Hence, when z and u are integrals of their respective equations, we have

$$\iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy = 0,$$

and therefore†, as

$$\iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy = \int (M dy - N dx),$$

* It occurs in a memoir, published in 1860 and apparently written the year before: see *Ges. Werke* (1876), pp. 158 et seq. The exposition of Riemann's method as given by Darboux, *Théorie générale des surfaces*, t. II, pp. 75 et seq., is adopted in the account which follows.

† *T. F.*, § 16.

the single integral being taken positively round the boundary of the area over which the double integral is taken, we have

$$\int_A^B (M\,dy - N\,dx) + \int_B^P M\,dy + \int_P^A (-N)\,dx = 0,$$

that is,

$$\int_A^B (M\,dy - N\,dx) - \int_P^B M\,dy - \int_P^A N\,dx = 0.$$

Now the variable z and one of its derivatives, say $\frac{\partial z}{\partial x}$, possess assigned values along the curve AB: moreover, the relation

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

is always satisfied: and, along the curve, dy is known in terms of dx; hence $\frac{\partial z}{\partial y}$ is known* along the curve. If then, u is completely known, that is, if we know an integral of the adjoint equation subject to any conditions we choose to assign, then

$$\int_A^B (M\,dy - N\,dx)$$

can be regarded as a known quantity in connection with that integral and with the assigned initial conditions for z. Again,

$$\begin{aligned}\int_P^A N\,dx &= \int_P^A \left\{\tfrac{1}{2}\left(u\frac{\partial z}{\partial x} - z\frac{\partial u}{\partial x}\right) + buz\right\} dx \\ &= \int_P^A \left\{\tfrac{1}{2}\frac{\partial}{\partial x}(uz) - z\left(\frac{\partial u}{\partial x} - bu\right)\right\} dx \\ &= \tfrac{1}{2}(uz)_A - \tfrac{1}{2}(uz)_P - \int_P^A z\left(\frac{\partial u}{\partial x} - bu\right) dx;\end{aligned}$$

and, similarly,

$$\int_P^B M\,dy = \tfrac{1}{2}(uz)_B - \tfrac{1}{2}(uz)_P - \int_P^B z\left(\frac{\partial u}{\partial y} - au\right) dy;$$

* If the curve be $\theta(x, y) = 0$, and if the values acquired by z and $\frac{\partial z}{\partial x}$ along the curve be $\zeta(x, y)$ and $\pi(x, y)$ respectively, then the value of $\frac{\partial z}{\partial y}$ along the curve is given by the equation

$$\left(\frac{\partial z}{\partial y} - \frac{\partial \zeta}{\partial y}\right)\frac{\partial \theta}{\partial x} = \left(\pi - \frac{\partial \zeta}{\partial x}\right)\frac{\partial \theta}{\partial y}.$$

whence, substituting these values, we have

$$(uz)_P = \tfrac{1}{2}(uz)_A + \tfrac{1}{2}(uz)_B - \int_A^B (M\,dy - N\,dx)$$
$$- \int_P^A z\left(\frac{\partial u}{\partial x} - bu\right) dx - \int_P^B z\left(\frac{\partial u}{\partial y} - au\right) dy.$$

In the last two quadratures, the quantity z occurs and it is as yet unknown; accordingly, we choose

$$\frac{\partial u}{\partial x} - bu = 0, \text{ along } PA,$$

and

$$\frac{\partial u}{\partial y} - au = 0, \text{ along } PB.$$

Hence, assuming that u is determined as an integral of the equation $G(u) = 0$ satisfying these conditions, we have the value of z at any point P of the plane given by

$$(uz)_P = \tfrac{1}{2}(uz)_A + \tfrac{1}{2}(uz)_B - \int_A^B (M\,dy - N\,dx).$$

Denote the coordinates of P by x, y; and let ξ, η denote current coordinates for u. Then since

$$\frac{\partial u}{\partial x} - bu = 0$$

along PA, that is, along the line $\eta = y$, we have

$$u = u_P\, e^{\int_{x,y}^{\xi,y} b\,dX},$$

for any point along that line. Similarly

$$u = u_P\, e^{\int_{x,y}^{x,\eta} a\,dY},$$

for any point along PB, which is the line $\xi = x$. It is clear that, without affecting the differential conditions imposed upon u, or without affecting the equation $G(u) = 0$, or without affecting the equation which gives z_P, we can remove the factor u_P: this removal can be effected by taking it as equal to unity. Thus the aggregate of conditions imposed upon u is:

(i) it satisfies the equation $G(u) = 0$;

(ii) it acquires the value unity at P;

(iii) when $\eta = y$, then $u = e^{\int_x^{\xi} b\,dX}$;

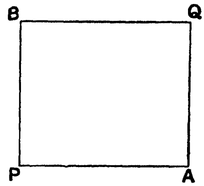
(iv) when $\xi = x$, then $u = e^{\int_y^{\eta} a\,dY}$.

When u is thus known, the value of z, in connection with the assigned initial conditions, is given by

$$z_P = \tfrac{1}{2}(uz)_A + \tfrac{1}{2}(uz)_B - \int_A^B (M\,dy - N\,dx):$$

that is, the integration of $F(z) = 0$ under arbitrarily assigned conditions is made to depend upon the integration of the adjoint equation $G(u) = 0$ under specified particular conditions. This is Riemann's result*.

214. Further significance can be given to the result by developing the reciprocal properties of the equations $F(z) = 0$ and $G(u) = 0$. For this purpose, choose the curve AB so that it shall consist of two straight lines AQ and QB, parallel to the axes; and let x', y' denote the coordinates of Q. Then



$$\begin{aligned}\int_A^B M\,dy &= \int_A^Q M\,dy \\ &= \int_A^Q \left\{\tfrac{1}{2}\left(u\frac{\partial z}{\partial y} - z\frac{\partial u}{\partial y}\right) + auz\right\} dy \\ &= \int_A^Q \left\{-\tfrac{1}{2}\frac{\partial}{\partial y}(uz) + u\left(\frac{\partial z}{\partial y} + az\right)\right\} dy \\ &= \tfrac{1}{2}(uz)_A - \tfrac{1}{2}(uz)_Q + \int_A^Q u\left(\frac{\partial z}{\partial y} + az\right) dy;\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_A^B N\,dx &= \int_Q^B N\,dx \\ &= -\int_B^Q N\,dx \\ &= \tfrac{1}{2}(uz)_Q - \tfrac{1}{2}(uz)_B - \int_B^Q u\left(\frac{\partial z}{\partial x} + bz\right) dx.\end{aligned}$$

Substituting, we have

$$z_P = (uz)_Q - \int_A^Q u\left(\frac{\partial z}{\partial y} + az\right) dy - \int_B^Q u\left(\frac{\partial z}{\partial x} + bz\right) dx.$$

Now a definite curve AQ, QB has been chosen: and though initial (or boundary) conditions are imposed on z, nothing has

* *L.c.*, p. 161.

been said as to their explicit form in the present connection. Accordingly, we shall assume that z is given along AQ and QB by the conditions, (i), that

$$z_Q = 1;$$

(ii), along AQ, we are to have

$$\frac{\partial z}{\partial y} + az = 0,$$

that is,

$$z = e^{-\int_\eta^{y'} a\,dY},$$

along AQ; and, (iii), along BQ, we are to have

$$\frac{\partial z}{\partial x} + bz = 0,$$

that is,

$$z = e^{-\int_\xi^{x'} b\,dX}.$$

(It may be noted that, in passing from an equation $F(z) = 0$ to its adjoint equation $G(u) = 0$, the signs of a and b change though their values are unaltered otherwise: thus these conditions for z are the exact reciprocal of the conditions for u). Substituting the values thus obtained in the preceding relation, we find

$$z_P = u_Q;$$

or, as x and y are variables for z while x' and y' are parametric, and as x' and y' are variables for u while x and y are parametric, we can write the result in the form

$$z(x, y;\ x', y') = u(x', y';\ x, y).$$

It therefore appears that, if we can obtain an integral of either of two adjoint equations subject to selected associate conditions, we can obtain the integral of the other of the two equations subject to reciprocally selected associate conditions. In fact, *in order to effect the integration of both of the adjoint equations, it is sufficient to determine the function which has been denoted by*

$$u(x', y';\ x, y).$$

Ex. 1. Consider the equation

$$F(z) = s - \frac{1}{x+y}p - \frac{1}{x+y}q = 0.$$

The equation $G(u) = 0$, when the independent variables are x' and y', is

$$\frac{\partial^2 u}{\partial x' \partial y'} + \frac{1}{x'+y'}\frac{\partial u}{\partial x'} + \frac{1}{x'+y'}\frac{\partial u}{\partial y'} - \frac{2}{(x'+y')^2}u = 0.$$

There are three conditions imposed upon u by the preceding process :—

(i) when $x'=x$, $y'=y$, we must have
$$u=1\,;$$

(ii) when $y'=y$, we must have
$$u=e^{\int_x^{x'}\frac{-dX}{X+y}}=\frac{x+y}{x'+y}$$

(iii) when $x'=x$, we must have
$$u=e^{\int_y^{y'}\frac{-dY}{x+Y}}=\frac{x+y}{x+y'},$$

the first of which is in accord with the second and the third.

Any form of u, which satisfies the equation $G(u)=0$ and is in accordance with these conditions, will be sufficient for our purpose; and the form, suggested by the conditions, is that of a rational fraction. Now the equation $G(u)=0$ is satisfied by
$$u=(x'+y')^m,$$
provided
$$m(m-1)+2m-2=0,$$
that is,
$$m=1, \quad \text{or} \quad m=-2.$$
Accordingly, we choose a form
$$u=\frac{a+\beta x'+\gamma y'+\delta x'y'}{(x'+y')^2},$$
where a, β, γ, δ do not involve x' or y': they can be functions of x and y and, having regard to the values of u when $x'=x$ and when $y'=y$ respectively we may expect that δ will be a pure constant, β and γ will be of the first degree in x and y, and a will be of the second degree in x and y. But the first requisite is that u should satisfy the equation $G(u)=0$: substituting, we find that the one necessary and sufficient condition is
$$\beta+\gamma=0,$$
so that
$$u=\frac{a+\gamma(y'-x')+\delta x'y'}{(x'+y')^2}.$$

Having regard to the condition that $u=\dfrac{x+y}{x'+y}$ when $y=y'$, we see that
$$(x+y)(x'+y)=\delta x'y+\gamma(y-x')+a:$$
so that, as a, γ, δ are independent of x', we must have
$$\left.\begin{array}{l}\delta y-\gamma=x+y\\ \gamma y+a=(x+y)\,y\end{array}\right\}.$$
Similarly, from the condition that $u=\dfrac{x+y}{x+y'}$ when $x=x'$, we find that
$$\left.\begin{array}{l}\delta x+\gamma=x+y\\ -\gamma x+a=(x+y)\,x\end{array}\right\}.$$

The four equations involving a, γ, δ are satisfied by

$$\delta=2, \quad \gamma=y-x, \quad a=2xy\,;$$

and therefore

$$u=\frac{2(x'y'+xy)+(y-x)(y'-x')}{(y'+x')^2},$$

where x' and y' are current coordinates for u, and x and y are the coordinates of the point where the value of z is desired. We thus have found a value of u which, while completely known, satisfies the assigned conditions along the lines PA and PB which are given by

$$y'=y, \quad x'=x,$$

respectively.

By way of application, let it be required to find an integral z such that

$$z=\frac{(x'+y)^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial x}=3\frac{(x'+y)^2}{(x'+y')^3},$$

when $x=x'$, and

$$z=\frac{(x+y')^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial y}=3\frac{(x+y')^2}{(x'+y')^3},$$

when $y=y'$. Along AQ, let Y be the variable of integration; and along BQ, let X be the variable of integration. Hence, along AQ, we have

$$z=\frac{(x'+Y)^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial y}=3\frac{(x'+Y)^2}{(x'+y')^3},$$

$$u=\frac{2(x'Y+xy)+(Y-x')(y-x)}{(x'+Y)^2};$$

and therefore

$$\int_A^Q u\left(\frac{\partial z}{\partial y}+az\right)dY$$

$$=\int_y^{y'} u\left(\frac{\partial z}{\partial y}-\frac{1}{x'+Y}z\right)dY$$

$$=\frac{2}{(x'+y')^3}\int_y^{y'}\{Y(2x'+y-x)+2xy-x'y+x'x\}\,dY$$

$$=\frac{y'-y}{(x'+y')^3}\{(2x'+y-x)(y'+y)+4xy-2x'y+2x'x\}$$

$$=\frac{1}{(x'+y')^3}\{-y^3-3xy^2+xy(4y'-2x')+y(y'^2-2x'y')+x(2x'y'-y'^2)+2x'y'^2\}.$$

Again, along BQ, we have

$$z=\frac{(X+y')^3}{(x'+y')^3}, \quad \frac{\partial z}{\partial x}=3\frac{(X+y')^2}{(x'+y')^3},$$

$$u=\frac{2(Xy'+xy)+(y'-X)(y-x)}{(X+y')^2};$$

and therefore

$$\int_B^Q u\left(\frac{\partial z}{\partial x}+bz\right)dX$$

$$=\int_x^{x'} u\left(\frac{\partial z}{\partial x}-\frac{1}{X+y'}z\right)dX$$

$$=\frac{2}{(x'+y')^3}\int_x^{x'}\{X(2y'+x-y)+2xy+yy'-xy'\}\,dX$$

$$=\frac{x'-x}{(x'+y')^3}\{(2y'+x-y)(x'+x)+4xy+2yy'-2xy'\}$$

$$=\frac{1}{(x'+y')^3}\{-x^3-3x^2y+xy(4x'-2y')+x(x'^2-2x'y')-y(x'^2-2x'y')+2x'^2y'\}.$$

Now

$$u_Q=\frac{2(x'y'+xy)+(y'-x')(y-x)}{(x'+y')^2},$$

$$z_Q=1\,;$$

consequently

$$z_P=(uz)_Q-\int_A^Q u\left(\frac{\partial z}{\partial y}+az\right)dY-\int_B^Q u\left(\frac{\partial z}{\partial x}+bz\right)dX$$

$$=\frac{(x+y)^3}{(x'+y')^3},$$

on reduction.

Ex. 2. Shew that the equation

$$\frac{\partial^2\theta}{\partial x\partial y}-\frac{m}{x-y}\frac{\partial\theta}{\partial x}-\frac{n}{x-y}\frac{\partial\theta}{\partial y}+\frac{l}{(x-y)^2}\theta=0$$

can, by a change of the dependent variable represented by

$$\theta=(x-y)^a z,$$

where a is an appropriately determined constant, be transformed into the equation

$$\frac{\partial^2 z}{\partial x\partial y}-\frac{\beta'}{x-y}\frac{\partial z}{\partial x}+\frac{\beta}{x-y}\frac{\partial z}{\partial y}=0.$$

Obtain integrals of this last equation in the form

$$z=x^\lambda F\left(-\lambda,\ \beta',\ 1-\beta-\lambda,\ \frac{y}{x}\right),$$

$$z=x^{-\beta}y^{\beta+\lambda}F\left(\beta,\ \beta'+\beta+\lambda,\ 1+\beta+\lambda,\ \frac{y}{x}\right),$$

where F is the ordinary symbol for the hypergeometric series, and λ is any arbitrary constant.

Hence shew that, unless β is a positive integer greater than unity, the transformed equation possesses an infinitude of polynomial integrals.

(Euler, Darboux.)

Ex. 3. Shew that, if $\phi(x, y)$ is an integral of the equation

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\beta'}{x-y}\frac{\partial z}{\partial x} + \frac{\beta}{x-y}\frac{\partial z}{\partial y} = 0,$$

then

$$z = (ax+b)^{-\beta}(ay+b)^{-\beta'}\phi\left(\frac{cx+d}{ax+b}, \frac{cy+d}{ay+b}\right)$$

is also an integral of the equation, where a, b, c, d are arbitrary constants. (Darboux.)

Ex. 4. Denoting by u the variable of the equation which is adjoint to

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\beta'}{x-y}\frac{\partial z}{\partial x} + \frac{\beta}{x-y}\frac{\partial z}{\partial y} = 0,$$

and writing

$$u = (x-y)^{\beta+\beta'} v,$$

prove that v satisfies the equation

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{\beta}{x-y}\frac{\partial v}{\partial x} + \frac{\beta'}{x-y}\frac{\partial v}{\partial y} = 0.$$

Shew that, if

$$\sigma = \frac{(x-x')(y-y')}{(x-y')(y-x')},$$

then an integral u of the adjoint equation is given by

$$u = (y'-x)^{-\beta'}(y-x')^{-\beta}(y-x)^{\beta+\beta'} F(\beta, \beta', 1, \sigma);$$

and verify that this integral satisfies the conditions imposed upon the function u in the Riemann method of constructing an integral of the original equation in z. (Darboux.)

Note. Further properties of these equations are given by Darboux*.

Ex. 5. In the equation

$$s + ap + bq + cz = 0,$$

let a, b, c be functions of x and y which are regular within a domain $|x| \leqslant 1$, $|y| \leqslant 1$; and let $\phi(x)$, $\psi(y)$ be two functions of x and y which also are regular within those domains. Also, let M, N, P, H, K be not less than the greatest values of $|a|$, $|b|$, $|c|$, $|\phi(x)|$, $|\psi(y)|$ respectively, within those domains.

Prove that the original equation possesses an integral, which is a regular function of x and y within the assigned domains, which reduces to $\phi(x)$ when $y=0$, and which reduces to $\psi(y)$ when $x=0$, provided the equation

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{M}{1-x-y}\frac{\partial Z}{\partial x} + \frac{N}{1-x-y}\frac{\partial Z}{\partial y} + \frac{P}{(1-x-y)^2} Z$$

* *Théorie générale des surfaces*, t. II, pp. 54—70, 81—91.

possesses* a regular integral, which reduces to $\dfrac{H}{1-x}$, when $y=0$, and which reduces to $\dfrac{K}{1-y}$, when $x=0$.

Apply the results in the preceding examples to shew that the latter equation does possess the regular integral indicated. (Darboux.)

Ex. 6. Shew that the equation adjoint to

$$f(z)=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+2l\frac{\partial z}{\partial x}+2m\frac{\partial z}{\partial y}+nz=0$$

is

$$g(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2L\frac{\partial u}{\partial x}+2M\frac{\partial u}{\partial y}+Nu=0,$$

where

$$L=-l,\quad M=-m,\quad N=n-2\frac{\partial l}{\partial x}-2\frac{\partial m}{\partial y}.$$

Let J' and K' denote the invariants of $g(u)=0$, which correspond to the invariants J and K of $f(z)=0$ (see § 194, Ex. 5): verify that

$$J'=-J,\quad K'=K.$$

Hence shew that, if J vanishes, $f(z)=0$ and $g(u)=0$ are effectively one and the same equation; and that, if K' vanishes but not J, the equation $g(u)=0$ can be expressed in the form

$$\alpha\frac{\partial^2}{\partial x^2}\left(\frac{u}{\beta}\right)+\beta\frac{\partial^2}{\partial y^2}\left(\frac{u}{\alpha}\right)=0,$$

where

$$\alpha=e^{\int l dx},\quad \beta=e^{\int m dy}.$$

Also prove, in general, that

$$uf(z)-zg(u)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y},$$

where

$$\left.\begin{aligned}P&=u\frac{\partial z}{\partial x}-z\frac{\partial u}{\partial x}+2luz\\ Q&=u\frac{\partial z}{\partial y}-z\frac{\partial u}{\partial y}+2muz\end{aligned}\right\}.$$

(Burgatti.)

Ex. 7. Shew that the equation adjoint to

$$f(z)=\frac{\partial^2 z}{\partial x^2}+2\alpha\frac{\partial z}{\partial x}+2\beta\frac{\partial z}{\partial y}+\gamma z=0$$

is

$$g(u)=\frac{\partial^2 u}{\partial x^2}+2A\frac{\partial u}{\partial x}+2B\frac{\partial u}{\partial y}+\Gamma u=0,$$

* In connection with this existence-theorem, a memoir by Goursat, *Annales de Toulouse*, 2^{me} Sér., t. v (1903), pp. 405—436, may be consulted. It also contains a number of references to other memoirs on the subject.

where

$$A = -\alpha, \quad B = -\beta, \quad \Gamma = \gamma - 2\frac{\partial \alpha}{\partial x} - 2\frac{\partial \beta}{\partial y}.$$

Denoting the invariants of $g(u)=0$ by I' and J', which correspond (§ 206) to the invariants I and J of $f(z)=0$, prove that

$$I' = -I,$$

$$J' = -J + \frac{\partial^2 \log I}{\partial x \partial y};$$

and verify that

$$uf(z) - zg(u) = \frac{\partial T}{\partial x} + \frac{\partial U}{\partial y},$$

where

$$T = u\frac{\partial z}{\partial x} - 2\frac{\partial u}{\partial x} + 2\alpha uz,$$

$$U = 2\beta uz.$$

Ex. 8. Indicate how the results in the two preceding examples can be used to obtain integrals of the respective original differential equations satisfying the initial conditions imposed in Cauchy's existence-theorem.

Ex. 9. Shew that, if ζ is any integral of the equation

$$s = z \sin \phi,$$

where ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial x \partial y} + \cos \phi = 0,$$

then

$$-\frac{\partial \zeta}{\partial x}\frac{\partial \phi}{\partial x}dx + \zeta \cos \phi \, dy$$

and

$$\zeta \cos \phi \, dx - \frac{\partial \zeta}{\partial y}\frac{\partial \phi}{\partial y}dy$$

are perfect differentials. Denoting these perfect differentials by $d\rho$ and $d\sigma$ respectively, prove that

$$\rho\frac{\partial \phi}{\partial x} - \frac{\partial^2 \zeta}{\partial x^2}, \quad \sigma\frac{\partial \phi}{\partial y} - \frac{\partial^2 \zeta}{\partial y^2},$$

are integrals of the original equation to be satisfied by z. (Guichard.)

Ex. 10. Four linearly distinct integrals of the equation

$$F(z) = s + ap + bq + cz = 0$$

are denoted by z_1, z_2, z_3, z_4; and quantities u_1, u_2, u_3, u_4 are determined by the equations

$$z_1 u_1 + z_2 u_2 + z_3 u_3 + z_4 u_4 = 0,$$

$$p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4 = 0,$$

$$q_1 u_1 + q_2 u_2 + q_3 u_3 + q_4 u_4 = 0.$$

Prove that these quantities u satisfy an equation

$$G(u) = \frac{\partial^2 u}{\partial x \partial y} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u = 0.$$

Prove, further, that, if v be the integral of the equation adjoint to $G(u)=0$, then

$$v\left(\frac{\partial u_\mu}{\partial x}+\beta u_\mu\right)dx+u_\mu\left(\frac{\partial v}{\partial y}-av\right)dy$$

is an exact differential dw_μ; and that, if

$$Z=z_1w_1+z_2w_2+z_3w_3+z_4w_4,$$

then Z is an integral of $F(z)=0$. (Darboux.)

Ex. 11. Two integrals, independent of one another and belonging to the equation

$$s+ap+bq=0,$$

are denoted by z_1 and z_2; and z_1z_2 is also an integral of the equation. Prove that

$$\frac{\partial}{\partial x}\left(a+\frac{b}{\rho}\right)=\frac{\partial}{\partial y}(a\rho+b),$$

where

$$\rho=ie^{\int(b\,dx-a\,dy)}.$$

Apply this result to obtain a relation among the coefficients of the equation

$$s+ap+bq+cz=0,$$

when four linearly independent integrals z_1, z_2, z_3, z_4 satisfy a relation

$$z_1^2+z_2^2+z_3^2+z_4^2=0.$$

Equations with Equal Invariants.

215. One of the more important classes of linear equations of the second order, in that their properties are more fully developed than those of the other classes, is composed of those equations which have their *invariants equal* to one another. As has already (§ 192) been proved, the equation can be changed so as to acquire the form

$$s=\lambda z,$$

where λ is the common value of the equal invariants: and this form is canonical for equations with equal invariants.

The equation, which is adjoint to

$$\frac{\partial^2 z}{\partial x\partial y}=\lambda z,$$

is

$$\frac{\partial^2 u}{\partial x\partial y}=\lambda u,$$

as is seen at once by taking the general form, and making a and b zero: thus *the equation with equal invariants is self-adjoint.* Moreover, *it is the only equation which is self-adjoint*: for if $F(z)=0$,

$G(u)=0$, be two reciprocally adjoint equations, their invariants are given by

$$h_g = k_f, \quad k_g = h_f,$$

and therefore, if $F(z)=0$ and $G(u)=0$ are effectively the same equation so that they have the same invariants,

$$h_f = h_g = k_f, \quad k_f = k_g = h_f,$$

that is, the invariants are equal.

Suppose that a self-adjoint equation is of finite rank in one of the variables, say in x; then the set of equations obtained by the Laplace σ-transformations is finite, in the sense that after a finite number of the transformations in succession an equation is obtained having a vanishing invariant. Let

$$F = \frac{\partial^2 z}{\partial x \partial y} - \lambda z = 0$$

be the original equation: let the last equation of the set derived by the σ-transformation be

$$\sigma^n F = 0.$$

We know that, in the case of any equation of finite rank in the variable x, the adjoint equation is of equal finite rank in the variable y. In the present case, F is self-adjoint; and therefore it is of finite rank in the variable y, and the first equation of the set given by means of the Laplace σ^{-1}-transformation, which has a vanishing invariant, is

$$\sigma^{-n} F = 0.$$

Consequently, when a self-adjoint equation is of finite rank in one of the variables, it is of equal finite rank in the other variable.

It therefore follows that, if a self-adjoint equation can be integrated by Laplace's method, the integral can be expressed in such a way that the two arbitrary functions are free from partial quadratures. Moreover, the integral is of the same finite rank in y as it is in x, and thus it must be of the form

$$z = AX + A_1 X' + \ldots + A_m X^{(m)} + BY + B_1 Y' + \ldots + B_m Y^{(m)};$$

we shall return later to the consideration of this form of the integral. The condition that a given self-adjoint equation

$$\frac{\partial^2 z}{\partial x \partial y} - \lambda z = 0$$

should be of finite rank $m+1$ in each of the variables is derivable at once from the earlier investigations: we have

$$h = \lambda = k,$$

$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y} = \lambda - \frac{\partial^2 \log \lambda}{\partial x \partial y},$$

$$h_2 = 2h_1 - h - \frac{\partial^2 \log h_1}{\partial x \partial y} = \lambda - \frac{\partial^2}{\partial x \partial y}\{\log(\lambda^2 h_1)\},$$

....

$$h_m = \lambda - \frac{\partial^2}{\partial x \partial y}\{\log(\lambda^m h_1^{m-1} h_2^{m-2} \dots h^2_{m-2} h_{m-1})\},$$

and therefore, as the necessary and sufficient condition is that $h_m = 0$, this condition is

$$\lambda = \frac{\partial^2}{\partial x \partial y}\{\log(\lambda^m h_1^{m-1} h_2^{m-2} \dots h^2_{m-2} h_{m-1})\},$$

being an equation satisfied by λ. The equation is of order $2m$.

216. The construction of all the linear equations, which have equal invariants and can be integrated in finite terms by Laplace's method, can be effected if this differential equation of order $2m$ for the determination of λ can be integrated in general. This integration has been obtained through a process, devised first by Moutard, of passing from one equation to another of contiguous rank; an exposition of the process will be given almost immediately. Meanwhile, the results (§§ 201—203) obtained by Darboux for a linear equation, which is of finite rank in both variables, can be applied when the linear equation is of the self-adjoint type.

The equations of the double set, derived from a self-adjoint equation $F=0$, can be represented by equating the expressions

$$\sigma^{-n}F,\ \ \sigma^{-n+1}F,\ \dots,\ \sigma^{-1}F,\ \ F,\ \ \sigma F,\ \dots,\ \sigma^{n-1}F,\ \ \sigma^n F$$

to zero; hence the quantity μ of § 203 is given by

$$2n = \mu - 1.$$

The invariant k_n of $\sigma^n F$ is the invariant h_{n-1} of $\sigma^{n-1}F$, that is,

$$k_n = -\frac{\partial^2 \log \alpha}{\partial x \partial y},$$

where α is of the form

$$\alpha = \xi_1 \eta_1 + \dots + \xi_{2n+1}\eta_{2n+1}:$$

and in this expression, $\eta_1, \ldots, \eta_{2n+1}$ are $2n+1$ linearly independent functions of y, while $\xi_1, \ldots, \xi_{2n+1}$ are $2n+1$ linearly independent functions of x. Again, the invariant k_{-n} of $\sigma^{-n}F$ is zero; and its invariant h_{-n} is the invariant k_{-n+1} of $\sigma^{-n+1}F$, that is,

$$h_{-n} = -\frac{\partial^2 \log \beta}{\partial x \partial y},$$

where β is of the form

$$\beta = x_1 y_1 + \ldots + x_{2n+1} y_{2n+1}:$$

and in this expression, $y_1, \ldots, y_{2n+1}$ are $2n+1$ linearly independent functions of y, while $x_1, \ldots, x_{2n+1}$ are $2n+1$ linearly independent functions of x. Now the invariants of $\sigma^m F$ are the same as those of $\sigma^{-m}F$ except as to order; and therefore

$$k_n = h_{-n},$$

that is,

$$\frac{\partial^2 \log \alpha}{\partial x \partial y} = \frac{\partial^2 \log \beta}{\partial x \partial y},$$

so that

$$\alpha = \beta \xi \eta,$$

where ξ and η are functions of x and of y, unrestricted so far as concerns this relation. As $x_1, \ldots, x_{2n+1}$ are $2n+1$ linearly independent functions of x, being $2n+1$ linearly independent integrals of an ordinary equation in x of order $2n+1$, the function ξ can be absorbed into each of them without affecting their linear independence; and similarly, the function η can be absorbed into each of the quantities $y_1, \ldots, y_{2n+1}$ without affecting their linear independence. Let this absorption take place in both cases: then we have

$$\alpha = \beta.$$

Adopting the same notation for the functional determinants in derivatives of α as before, we note that the invariant h_n of $\sigma^n F$ is zero, and that the equation $F=0$ is removed from $\sigma^n F = 0$ by n of the σ-transformations; hence $F=0$ can be expressed in the form

$$s - p\frac{\partial}{\partial y}(\log H_{n-1}) - q\frac{\partial}{\partial x}(\log H_n) + z\frac{\partial}{\partial y}(\log H_{n-1})\frac{\partial}{\partial x}(\log H_n) = 0.$$

Again, the invariant k_{-n} of $\sigma^{-n}F$ is zero, and the equation $F=0$ is removed from $\sigma^{-n}F=0$ by n of the σ^{-1}-transformations; hence, constructing quantities K from the magnitude β in the same way

as the quantities H are constructed from the magnitude α, the equation $F=0$ can be expressed in the form

$$s - q\frac{\partial}{\partial x}(\log K_{n-1}) - p\frac{\partial}{\partial y}(\log K_n) + z\frac{\partial}{\partial x}(\log K_{n-1})\frac{\partial}{\partial y}(\log K_n) = 0.$$

Now we have

$$h_{n-s-1} = -\frac{\partial^2}{\partial x\partial y}(\log H_s),$$

$$k_{-(n-s'-1)} = -\frac{\partial^2}{\partial x\partial y}(\log K_{s'});$$

hence

$$h_{-m} = -\frac{\partial^2}{\partial x\partial y}(\log H_{m+n-1}),$$

$$\begin{aligned} h_{-\mu-1} &= k_{-\mu} \\ &= -\frac{\partial^2}{\partial x\partial y}(\log K_{n-\mu-1}). \end{aligned}$$

We therefore have

$$\frac{\partial^2}{\partial x\partial y}(\log K_n) = h_0 = -\frac{\partial^2}{\partial x\partial y}(\log H_{n-1}),$$

$$\frac{\partial^2}{\partial x\partial y}(\log K_{n-1}) = h_{-1} = -\frac{\partial^2}{\partial x\partial y}(\log H_n);$$

and therefore a common form of the equation, whether derived from $\sigma^n F=0$ or from $\sigma^{-n}F=0$, is

$$\begin{aligned} s - p\frac{\partial}{\partial y}(\log H_{n-1}) - q\frac{\partial}{\partial x}(\log K_{n-1}) & \\ + z\frac{\partial}{\partial y}(\log H_{n-1})\frac{\partial}{\partial x}(\log K_{n-1}) &= 0. \end{aligned}$$

The equation is in one of the canonical forms adopted in § 192. Its two invariants are

$$-\frac{\partial^2}{\partial x\partial y}(\log H_{n-1}), \quad -\frac{\partial^2}{\partial x\partial y}(\log K_{n-1});$$

also K_{n-1} is the same function of β as H_{n-1} is of α, and $\alpha=\beta$: hence the invariants are equal. When the equation is expressed in the binomial form, it is

$$\frac{\partial^2 Z}{\partial x\partial y} = -Z\frac{\partial^2 \log H_{n-1}}{\partial x\partial y},$$

where

$$z = ZH_{n-1}.$$

217. In order to find the integral of this equation, we use the two methods of derivation of the equation and of its integral. When we construct it from $\sigma^n F = 0$ as the initial equation, we have

$$z = \begin{vmatrix} z_0 , & \dfrac{\partial z_0}{\partial x} & , \ldots, & \dfrac{\partial^n z_0}{\partial x^n} \\ \alpha , & \dfrac{\partial \alpha}{\partial x} & , \ldots, & \dfrac{\partial^n \alpha}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{n-1} \alpha}{\partial y^{n-1}} , & \dfrac{\partial^n \alpha}{\partial x \partial y^{n-1}} & , \ldots, & \dfrac{\partial^{2n-1} \alpha}{\partial x^n \partial y^{n-1}} \end{vmatrix},$$

where

$$z_0 = X + U,$$

and U is homogeneous and linear in the derivatives of Y; hence the part of z dependent upon the derivatives of X only is

$$\theta(X) = \begin{vmatrix} X , & X' & , \ldots, & X^{(n)} \\ \alpha , & \dfrac{\partial \alpha}{\partial x} & , \ldots, & \dfrac{\partial^n \alpha}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{n-1} \alpha}{\partial y^{n-1}} , & \dfrac{\partial^n \alpha}{\partial x \partial y^{n-1}} & , \ldots, & \dfrac{\partial^{2n-1} \alpha}{\partial x^n \partial y^{n-1}} \end{vmatrix},$$

and the coefficient of $X^{(n)}$ in this expression is $(-1)^n H_{n-1}$. As

$$z = Z H_{n-1},$$

the coefficient of $X^{(n)}$ in Z is unity save as to sign.

Similarly, when we construct the equation from $\sigma^{-n} F = 0$ as the initial equation, we have

$$z' = \begin{vmatrix} \zeta_0 , & \dfrac{\partial \zeta_0}{\partial y} & , \ldots, & \dfrac{\partial^n \zeta_0}{\partial y^n} \\ \alpha , & \dfrac{\partial \alpha}{\partial y} & , \ldots, & \dfrac{\partial^n \alpha}{\partial y^n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{n-1} \alpha}{\partial x^{n-1}} , & \dfrac{\partial^n \alpha}{\partial x^{n-1} \partial y} & , \ldots, & \dfrac{\partial^{2n-1} \alpha}{\partial x^{n-1} \partial y^n} \end{vmatrix},$$

where

$$\zeta_0 = Y + V,$$

and V is homogeneous and linear in the derivatives of X; hence the part of z' dependent upon the derivatives of Y only is

$$\vartheta(Y) = \begin{vmatrix} Y & , & Y' & ,\ldots, & Y^{(n)} \\ \alpha & , & \dfrac{\partial\alpha}{\partial y} & ,\ldots, & \dfrac{\partial^n\alpha}{\partial y^n} \\ \cdots & & \cdots & & \cdots \\ \dfrac{\partial^{n-1}\alpha}{\partial x^{n-1}} & , & \dfrac{\partial^n\alpha}{\partial x^{n-1}\partial y} & ,\ldots, & \dfrac{\partial^{2n-1}\alpha}{\partial x^{n-1}\partial y^n} \end{vmatrix},$$

and the coefficient of $Y^{(n)}$ in this expression is $(-1)^n H_{n-1}$. As

$$z' = ZH_{n-1},$$

the coefficient of $Y^{(n)}$ in Z is unity save as to sign.

We therefore take

$$\begin{aligned} Z &= \frac{1}{H_{n-1}}\{\theta(X) + \vartheta(Y)\} \\ &= X^{(n)} - \frac{\partial \log H_{n-1}}{\partial x} X^{(n-1)} + \ldots \\ &\quad + Y^{(n)} - \frac{\partial \log H_{n-1}}{\partial y} Y^{(n-1)} + \ldots, \end{aligned}$$

absorbing the doubtful signs into X and Y respectively: and this is the integral of the equation

$$\frac{\partial^2 Z}{\partial x \partial y} = -Z\frac{\partial^2 \log H_{n-1}}{\partial x \partial y}.$$

It should however be borne in mind that, in establishing this result, the relation

$$\alpha = \beta\xi\eta$$

was changed to

$$\alpha = \beta,$$

by absorbing the quantities ξ and η into the sets of quantities used for the composition of α and β. When this change is not made, then we no longer have $K_{n-1} = H_{n-1}$: but as K_{n-1} is the same function of β as H_{n-1} is of α, we have

$$H_{n-1} = K_{n-1}\xi_1\eta_1,$$

where ξ_1 and η_1 are functions of x and of y respectively. The equation

$$\frac{\partial^2 \log H_{n-1}}{\partial x \partial y} = \frac{\partial^2 \log K_{n-1}}{\partial x \partial y}$$

still holds; and the invariants of the equation as obtained are equal.

In that case, the value of Z satisfying the equation

$$\frac{\partial^2 Z}{\partial x \partial y} = -Z \frac{\partial^2 \log H_{n-1}}{\partial x \partial y}$$

is found to have the form

$$Z = \frac{\theta(X)}{H_{n-1}\xi_2} + \frac{\vartheta(Y)}{K_{n-1}\eta_2},$$

where ξ_2 and η_2 are functions of x and of y only*.

The quantity α, which is subsidiary to the construction of the equation, occurs in the form

$$\alpha = \xi_1\eta_1 + \xi_2\eta_2 + \ldots + \xi_{2n+1}\eta_{2n+1},$$

where $\xi_1, \xi_2, \ldots, \xi_{2n+1}$ are $2n+1$ linearly independent functions of x, and $\eta_1, \eta_2, \ldots, \eta_{2n+1}$ are $2n+1$ linearly independent functions of y; and it possesses this form as being an integral of the equation

$$H_{2n+1} = 0.$$

The appropriate values of α are obtained by Darboux† through a consideration of the properties of the ordinary linear equation satisfied by $\xi_1, \ldots, \xi_{2n+1}$, regard being paid to the source of that equation. He proves that this equation is self-adjoint, and shews how the equations can be constructed for successive values of n.

Ex. 1. The simplest case occurs when $n=0$: then H_{n-1}, $=H_{-1}$ in this case, is zero; the equation is

$$\frac{\partial^2 Z}{\partial x\, \partial y} = 0,$$

and the value of Z is

$$Z = X + Y.$$

Ex. 2. The next simplest case occurs when $n=1$. Then

$$\alpha = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$

where ξ_1, ξ_2, ξ_3 are three linearly independent functions of x, and likewise η_1, η_2, η_3 are three linearly independent functions of y.

Thus ξ_1, ξ_2, ξ_3 are a set of independent integrals of an ordinary linear equation of the third order which, as is known‡, can always be expressed in a form

$$\frac{d^3\xi}{dx^3} + R\xi = 0.$$

* See, on this matter, § 219 hereafter.

† *L.c.*, t. II, pp. 140 et seq.

‡ See a paper by the author, *Phil. Trans.*, A, 1888, p. 441.

As the equation is to be self-adjoint, R must be zero: and so we may take

$$\xi_1=\tfrac{1}{2}x^2,\quad \xi_2=x,\quad \xi_3=1.$$

Similarly

$$\eta_1=1,\quad \eta_2=y,\quad \eta_3=\tfrac{1}{2}y^2:$$

and

$$a=\tfrac{1}{2}(x+y)^2.$$

Thus

$$\frac{\partial^2}{\partial x\,\partial y}(\log H_0)=\frac{\partial^2}{\partial x\,\partial y}(\log a)=-\frac{2}{(x+y)^2};$$

the equation is

$$\frac{\partial^2 Z}{\partial x\,\partial y}=\frac{2}{(x+y)^2}Z,$$

and its integral is

$$Z=X'-\frac{2}{x+y}X+Y'-\frac{2}{x+y}Y.$$

Moutard's Theorem on Equations with Equal Invariants.

218. The process devised by Moutard* depends upon a theorem which facilitates the construction of the equations of successively increasing rank and, at the same time, puts the increase of rank in evidence.

Let ω denote any integral of the equation

$$\frac{\partial^2\omega}{\partial x\partial y}=\lambda\omega,$$

where λ is a function of x and y; then

$$\omega\left(\frac{\partial^2 z}{\partial x\partial y}-\lambda z\right)=\omega\frac{\partial^2 z}{\partial x\partial y}-z\frac{\partial^2\omega}{\partial x\partial y}$$
$$=\tfrac{1}{2}\frac{\partial}{\partial x}\left(\omega\frac{\partial z}{\partial y}-z\frac{\partial\omega}{\partial y}\right)+\tfrac{1}{2}\frac{\partial}{\partial y}\left(\omega\frac{\partial z}{\partial x}-z\frac{\partial\omega}{\partial x}\right),$$

* It is contained in a memoir, *Journ. de l'École Polyt.*, t. XXVIII (1878), pp. 1—11, which originally was merely the third or last section of a memoir presented to the Académie des Sciences in 1870; see *Comptes Rendus*, t. LXX (1870), pp. 834, 1068. The latter was never published: it seems to have disappeared in 1871 (Darboux, *Théorie générale des surfaces*, t. II, p. 53) during the fires of the Commune, which also caused the destruction of all the materials prepared by Bertrand for his work on differential equations that was to be the third volume of his *Traité de Calcul différentiel et de Calcul intégral*. (Darboux, "Éloge de Bertrand," being the preface to Bertrand's *Éloges académiques*, *Nouvelle Série*, Hachette, 1902.)

so that, when z also is an integral of the same equation, a function ϕ must exist such that

$$\frac{\partial \phi}{\partial x} = \omega \frac{\partial z}{\partial x} - z \frac{\partial \omega}{\partial x} = \omega^2 \frac{\partial}{\partial x}\left(\frac{z}{\omega}\right),$$

$$-\frac{\partial \phi}{\partial y} = \omega \frac{\partial z}{\partial y} - z \frac{\partial \omega}{\partial y} = \omega^2 \frac{\partial}{\partial y}\left(\frac{z}{\omega}\right).$$

Hence

$$\frac{\partial}{\partial y}\left(\frac{1}{\omega^2}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\omega^2}\frac{\partial \phi}{\partial y}\right) = 0,$$

and therefore

$$\omega \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial \omega}{\partial x}\frac{\partial \phi}{\partial y} = 0.$$

To express this equation in a canonical form, we take

$$\phi = \omega\theta,$$

and we easily find

$$\begin{aligned}\frac{1}{\theta}\frac{\partial^2 \theta}{\partial x \partial y} &= -\frac{1}{\omega}\frac{\partial^2 \omega}{\partial x \partial y} + \frac{2}{\omega^2}\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y} \\ &= \omega \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right) \\ &= \frac{1}{\frac{1}{\omega}}\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right).\end{aligned}$$

The relations between z and θ are simple: on the one hand, we have

$$\begin{aligned}\theta &= \frac{1}{\omega}\phi \\ &= \frac{1}{\omega}\int \omega^2 \left\{\frac{\partial}{\partial x}\left(\frac{z}{\omega}\right) dx - \frac{\partial}{\partial y}\left(\frac{z}{\omega}\right) dy\right\};\end{aligned}$$

and, on the other hand, we have

$$\begin{aligned}z &= \omega \int \frac{1}{\omega^2}\left(\frac{\partial \phi}{\partial x} dx - \frac{\partial \phi}{\partial y} dy\right) \\ &= \omega \int \frac{1}{\omega^2}\left\{\frac{\partial (\omega\theta)}{\partial x} dx - \frac{\partial (\omega\theta)}{\partial y} dy\right\}.\end{aligned}$$

Thus we have Moutard's theorem :—

Writing

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \Delta(z),$$

where Δ *is regarded as a symbol of operation, then any integral of either of the equations*

$$\Delta(z) = \Delta(\omega), \quad \Delta(z) = \Delta\left(\frac{1}{\omega}\right),$$

leads, by a definite simple quadrature, to an integral of the other equation.

The importance of the theorem lies in its application to the construction of the equations of successive rank with equal invariants; it obviously leads to the following method of proceeding:—

Let the general integral ζ *of the equation*

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \Delta(z) = \lambda$$

be supposed known: and let ω *denote a value obtained from* ζ *by assigning any general* values to the arbitrary functions which occur in* ζ*: then the general integral of the equation*

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \Delta(z) = \Delta\left(\frac{1}{\omega}\right) = \lambda_1,$$

is given by

$$z\omega = \int\left\{\left(\omega\frac{\partial \zeta}{\partial x} - \zeta\frac{\partial \omega}{\partial x}\right)dx - \left(\omega\frac{\partial \zeta}{\partial y} - \zeta\frac{\partial \omega}{\partial y}\right)dy\right\};$$

and the rank of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda_1$$

in each of the variables is greater by unity than the rank of the original equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda.$$

The first part of this statement is merely a repetition of Moutard's theorem in a slightly modified form. As regards the second part relating to the rank of the equation, the quantity $\dfrac{\partial \zeta}{\partial x}$ under the quadrature is of higher rank in x than ζ by one unit; and it is there multiplied by ω, so that this increase of

* The significance of this limitation will be illustrated later (Ex. 7): at present, it can be regarded merely as a direction not to take exceedingly special values of the arbitrary functions that occur in ζ.

rank is maintained after the quadrature; and similarly for $\frac{\partial \zeta}{\partial y}$. Thus the rank of the new equation in each of the variables is greater by one unit than the rank of the original equation.

Before proceeding to a general proposition dealing with the quadrature, some examples of the process will indicate its working.

Ex. 1. The general integral of

$$\Delta(z)=0$$

is

$$\zeta = X + Y.$$

Let ω denote the particular value of ζ represented by $X_1 + Y_1$: then

$$\Delta\left(\frac{1}{\omega}\right) = \omega \frac{\partial^2}{\partial x \, \partial y}\left(\frac{1}{\omega}\right)$$

$$= \frac{2X_1' Y_1'}{(X_1 + Y_1)^2},$$

where X_1' and Y_1' as usual are the derivatives of X_1 and Y_1. It follows, from Moutard's theorem, that the general integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \, \partial y} = \frac{2X_1' Y_1'}{(X_1 + Y_1)^2}$$

is given by

$$\begin{aligned}(X_1 + Y_1) z = z\omega &= \int \left\{ \left(\omega \frac{\partial \zeta}{\partial x} - \zeta \frac{\partial \omega}{\partial x}\right) dx - \left(\omega \frac{\partial \zeta}{\partial y} - \zeta \frac{\partial \omega}{\partial y}\right) dy \right\} \\ &= \int [\{(X_1 + Y_1) X' - (X + Y) X_1'\} \, dx \\ &\qquad - \{(X_1 + Y_1) Y' - (X + Y) Y_1'\} \, dy].\end{aligned}$$

Now

$$(X_1 + Y_1) X' - (X + Y) X_1' = \frac{\partial}{\partial x} \{(X_1 + Y_1)(X - Y)\} - 2XX_1',$$

$$-(X_1 + Y_1) Y' + (X + Y) Y_1' = \frac{\partial}{\partial y} \{(X_1 + Y_1)(X - Y)\} + 2YY_1',$$

and therefore

$$\begin{aligned}(X_1 + Y_1) z &= -2 \int XX_1' \, dx + 2 \int YY_1' \, dy \\ &\qquad + \int \left[\frac{\partial}{\partial x} \{(X_1 + Y_1)(X - Y)\} \, dx + \frac{\partial}{\partial y} \{(X_1 + Y_1)(X - Y)\} \, dy\right] \\ &= -2 \int XX_1' \, dx + 2 \int YY_1' \, dy + (X_1 + Y_1)(X - Y).\end{aligned}$$

Let two new arbitrary functions $f(x)$ and $g(y)$ be introduced by the defining relations

$$XX_1' = \frac{df}{dx}, \quad YY_1' = -\frac{dg}{dy};$$

then

$$X=\frac{1}{X_1'}\frac{df}{dx}, \quad Y=-\frac{1}{Y_1'}\frac{dg}{dy},$$

and so

$$z=-2\frac{f+g}{X_1+Y_1}+\frac{1}{X_1'}\frac{df}{dx}+\frac{1}{Y_1'}\frac{dg}{dy};$$

or now writing X and Y for f and g respectively, we have the general integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2X_1'Y_1'}{(X_1+Y_1)^2}$$

given by

$$z=\frac{X'}{X_1'}+\frac{Y'}{Y_1'}-2\frac{X+Y}{X_1+Y_1}.$$

Note. The invariant h_1 connected with this equation vanishes: but

$$h_1=h-\frac{\partial^2 \log h}{\partial x\,\partial y},$$

and

$$h=\lambda=\frac{2X_1'Y_1'}{(X_1+Y_1)^2}.$$

Hence, if we take $h=e^\zeta$, we have

$$\frac{\partial^2\zeta}{\partial x\,\partial y}=e^\zeta,$$

and

$$e^\zeta=\frac{2\phi'(x)\,\psi'(y)}{\{\phi(x)+\psi(y)\}^2}.$$

The latter is the general integral of the equation for ζ, a result first given by J. Liouville.

Ex. 2. Without any loss of generality, and merely by changing the independent variables, the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=2\frac{X_1'Y_1'}{(X_1+Y_1)^2}$$

can be changed into the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2};$$

and the general integral of the latter is (after the preceding example) given by

$$\zeta=X'+Y'-2\frac{X+Y}{x+y}.$$

Let ω denote the particular value that arises by taking

$$X=x^3, \quad Y=y^3,$$

in ζ: thus

$$\begin{aligned}\omega&=3x^2+3y^2-2(x^2-xy+y^2)\\&=(x+y)^2.\end{aligned}$$

Also

$$\Delta\left(\frac{1}{\omega}\right)=\frac{6}{(x+y)^2};$$

and therefore the integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{6}{(x+y)^2}$$

is given by

$$z(x+y)^2=\int\Big[\Big\{(x+y)^2\frac{\partial\zeta}{\partial x}-2(x+y)\zeta\Big\}\,dx$$
$$-\Big\{(x+y)^2\frac{\partial\zeta}{\partial y}-2(x+y)\zeta\Big\}\,dy\Big]=\Omega.$$

The quantity upon which a quadrature is to be effected is

$$d\Omega=(x+y)^2(X''dx-Y''dy)-4(x+y)(X'dx-Y'dy)$$
$$-2(x+y)(Y'dx-X'dy)+6(Ydx-Xdy)+6Xdx-6Ydy.$$

Now

$$d\{(x+y)^2(X'-Y')\}=(x+y)^2(X''dx-Y''dy)+2(x+y)(X'-Y')(dx+dy),$$

and therefore

$$d\{\Omega-(x+y)^2(X'-Y')\}$$
$$=-6(x+y)(X'dx-Y'dy)+6(Ydx-Xdy)+6Xdx-6Ydy$$
$$=-6d\{(x+y)(X-Y)\}+12(Xdx-Ydy):$$

hence

$$\Omega=(x+y)^2(X'-Y')-6(x+y)(X-Y)+12\int(Xdx-Ydy).$$

Writing

$$\xi=\int Xdx,\quad \eta=-\int Ydy,$$

we have

$$\Omega=(x+y)^2(\xi''+\eta'')-6(x+y)(\xi'+\eta')+12(\xi+\eta);$$

and therefore the integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{6}{(x+y)^2}$$

is given by

$$z=\xi''+\eta''-6\frac{\xi'+\eta'}{x+y}+12\frac{\xi+\eta}{(x+y)^2},$$

where ξ and η are arbitrary functions of x and of y respectively.

Ex. 3. Deduce the integral of

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{6X_1'Y_1'}{(X_1+Y_1)^2}.$$

Ex. 4. Comparing the integral of

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2},$$

which is

$$\zeta = X' + Y' - 2\,\frac{X+Y}{x+y},$$

with the integral of

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2X_1'Y_1'}{(X_1+Y_1)^2},$$

which is

$$\zeta = \frac{X'}{X_1'} + \frac{Y'}{Y_1'} - 2\,\frac{X+Y}{X_1+Y_1},$$

though (owing to a transformation of the independent variables) the two equations are essentially the same, we have an illustration of the remark (§ 217) that, according to the form of the equation, the derivative of highest order of the arbitrary function X has either unity or a function of x for its coefficient, and likewise for the derivative of highest order of the arbitrary function Y.

The same holds good of the two equations

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{6}{(x+y)^2}, \quad \frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{6X_1'Y_1'}{(X_1+Y_1)^2},$$

which are essentially the same: and so in other cases.

Ex. 5. Shew that the equation

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\mu}{(x+y)^2},$$

where μ is constant, is of finite rank in each of the variables when μ is of the form

$$\mu = n(n+1),$$

where n is an integer (which manifestly can be taken to be positive).

Assuming that μ has this value, prove that the general integral of the equation is

$$z = \sum_{r=0}^{n} \left\{ (-1)^r \frac{(n+r)!}{(n-r)!\,r!} (x+y)^{-r} \left(\frac{d^{n-r} X}{dx^{n-r}} + \frac{d^{n-r} Y}{dy^{n-r}} \right) \right\},$$

where X and Y are arbitrary functions of x and of y respectively.

Ex. 6. The equation

$$\frac{1}{z}\,\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\mu}{(x+y)^2}$$

remains unaltered when x and y are changed into $\frac{1}{x}$ and $\frac{1}{y}$: discuss the effect of these changes upon the general integral in the cases

$$\mu = 2, \quad \mu = 6.$$

Ex. 7. As an illustration of the remark (in § 218, p. 141) that an equation, of rank next greater than a given equation, may not necessarily arise when any special form ω of the general integral ζ of the given equation is chosen, consider once more the equation

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2}{(x+y)^2}\,z,$$

the general integral of which is

$$\zeta = X' + Y' - 2\frac{X+Y}{x+y}.$$

If we take ω as the form of ζ defined by

$$X = \tfrac{1}{6}x^3, \quad Y = \tfrac{1}{6}y^3,$$

we have

$$\omega = \tfrac{1}{2}(x+y) - \tfrac{1}{3}\frac{x^3+y^3}{x+y}$$

$$= \frac{xy}{x+y};$$

and then

$$\omega \frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right) = 0.$$

The resulting equation for the quantity θ, being

$$\frac{1}{\theta}\frac{\partial^2\theta}{\partial x\,\partial y} = \omega\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right) = 0,$$

is actually of lower rank than the equation from which it is derived.

On the other hand, the assumption

$$X = -\frac{1}{x}, \quad Y = -\frac{1}{y},$$

does lead to an equation of higher rank.

219. The law by which equations, of finite rank in both variables and having equal invariants, can be constructed in succession, may be expressed in a different form. Let the equation of rank n be

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda_n,$$

and let its general integral be

$$z = \zeta_n,$$

where ζ_n contains two arbitrary functions; and let ω_n be a form of ζ_n. Then, by Moutard's theorem, we form the expression

$$\lambda_{n+1} = \omega_n \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega_n}\right);$$

and then the equation of rank $n+1$ is

$$\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda_{n+1}.$$

Moreover,

$$\lambda_n = \frac{1}{\omega_n}\frac{\partial^2 \omega_n}{\partial x \partial y},$$

because ω_n is an integral of the equation of rank n; hence

$$\lambda_{n+1} - \lambda_n = -\frac{1}{\omega_n}\frac{\partial^2\omega_n}{\partial x\partial y} + \frac{2}{\omega_n^2}\frac{\partial\omega_n}{\partial x}\frac{\partial\omega_n}{\partial y} - \frac{1}{\omega_n}\frac{\partial^2\omega_n}{\partial x\partial y}$$

$$= -2\frac{\partial^2}{\partial x\partial y}(\log \omega_n).$$

Consequently

$$\lambda_n = \lambda_1 - 2\frac{\partial^2}{\partial x\partial y}\{\log(\omega_1\omega_2\ldots\omega_{n-1})\}$$

$$= -2\frac{\partial^2}{\partial x\partial y}\{\log(\omega_1\omega_2\ldots\omega_{n-1})\},$$

because λ_1 is zero, the equation of rank unity being

$$\frac{\partial^2 z}{\partial x\partial y} = 0.$$

Each of the quantities $\omega_1, \omega_2, \ldots, \omega_{n-1}$ contains two functions that can be regarded as arbitrary: hence this expression contains $2n-2$ arbitrary functions, $n-1$ of them being functions of x, and $n-1$ of them functions of y. Now it was seen (in § 217) that the relation, satisfied by a quantity λ which belongs to an equation of rank $m+1$ in each of the variables with equal invariants, is a partial differential equation of order $2m$; hence our quantity λ_n satisfies a partial differential equation of order $2n-2$. The expression obtained for λ_n contains $2n-2$ arbitrary functions: hence so long as they are kept arbitrary, the expression provides the general integral of that partial differential equation. What, however, is of greater importance for the present purpose is that the value of λ_n is given explicitly, when we know the general integral of each equation of lower rank in the series.

Integrals of Equations having Equal Invariants.

220. The result just established renders it possible to form the equation of any finite rank: and, as is known from Moutard's theorem, its integral can be obtained from the general integral of the equation of next lower rank by a process of quadrature. The process of quadrature can be effected in general through the following simplifications.

Let the equation, supposed to be of finite rank $n+1$ in each of the variables, be

$$\frac{\partial^2 z}{\partial x \partial y} = \lambda z;$$

and let the part of the general integral ζ, which involves the arbitrary function X and its derivatives linearly, be denoted by

$$\zeta = AX^{(n)} + A_1 X^{(n-1)} + \ldots + A_n X.$$

This part of ζ must of course satisfy the differential equation identically: hence substituting and taking account of the highest derivative of X after the substitution, we have

$$\frac{\partial A}{\partial y} = 0,$$

and therefore

$$\begin{aligned} A &= \text{function of } x \text{ alone} \\ &= \xi, \end{aligned}$$

say. Let another arbitrary function X_1 be introduced by the relation

$$X_1 = X\xi;$$

then the new form of ζ is

$$\zeta = X_1{}^{(n)} + B_1 X_1{}^{(n-1)} + \ldots + B_n X_1;$$

in other words, we can take A as equal to unity without loss of generality. Accordingly, we take

$$\zeta = X^{(n)} + A_1 X^{(n-1)} + \ldots + A_n X.$$

Thus

$$\begin{aligned} \frac{\partial^2 \zeta}{\partial x \partial y} &= \frac{\partial A_1}{\partial y} X^{(n)} + \sum_{r=1}^{n-1} \left\{ \left(\frac{\partial^2 A_r}{\partial x \partial y} + \frac{\partial A_{r+1}}{\partial y} \right) X^{(n-r)} \right\} + \frac{\partial^2 A_n}{\partial x \partial y} X \\ &= \lambda \left\{ X^{(n)} + A_1 X^{(n-1)} + \ldots + A_n X \right\}; \end{aligned}$$

and denoting by ω the particular value of ζ, to be used in constructing the integral of the equation next higher in rank, we have

$$\frac{\partial^2 \omega}{\partial x \partial y} = \lambda \omega.$$

Hence

$$\begin{aligned} \frac{\partial A_1}{\partial y} = \frac{1}{A_r} \left\{ \frac{\partial^2 A_r}{\partial x \partial y} + \frac{\partial A_{r+1}}{\partial y} \right\} &= \frac{1}{A_n} \frac{\partial^2 A_n}{\partial x \partial y} \\ &= \frac{1}{\omega} \frac{\partial^2 \omega}{\partial x \partial y}, \end{aligned}$$

for $r = 1, \ldots, n-1$.

Now we had

$$\frac{\partial(\omega\theta)}{\partial x} = \omega\frac{\partial\zeta}{\partial x} - \zeta\frac{\partial\omega}{\partial x}, \qquad -\frac{\partial(\omega\theta)}{\partial y} = \omega\frac{\partial\zeta}{\partial y} - \zeta\frac{\partial\omega}{\partial y};$$

hence

$$\begin{aligned}\frac{\partial(\omega\theta)}{\partial x} &= \omega\left\{X^{(n+1)} + A_1X^{(n)} + \sum_{r=1}^{n-1}\left(\frac{\partial A_r}{\partial x} + A_{r+1}\right)X^{(n-r)} + \frac{\partial A_n}{\partial x}X\right\}\\ &\quad - \frac{\partial\omega}{\partial x}\left\{X^{(n)} + \sum_{r=1}^{n-1} A_rX^{(n-r)} + A_nX\right\}\\ &= \omega X^{(n+1)} + \rho_1X^{(n)} + \sum_{r=1}^{n-1}\rho_{r+1}X^{(n-r)} + \rho_{n+1}X,\end{aligned}$$

where

$$\rho_1 = \omega A_1 - \frac{\partial\omega}{\partial x},$$

$$\rho_{n+1} = \omega\frac{\partial A_n}{\partial x} - A_n\frac{\partial\omega}{\partial x},$$

and

$$\rho_{r+1} = \omega A_{r+1} + \omega\frac{\partial A_r}{\partial x} - A_r\frac{\partial\omega}{\partial x},$$

the last holding for $r = 1, \ldots, n-1$. Similarly,

$$\frac{\partial(\omega\theta)}{\partial y} = \sigma_1X^{(n)} + \sum_{r=1}^{n-1}\sigma_{r+1}X^{(n-r)} + \sigma_{n+1}X,$$

where

$$\sigma_1 = \frac{\partial\omega}{\partial y},$$

$$\sigma_{r+1} = A_r\frac{\partial\omega}{\partial y} - \omega\frac{\partial A_r}{\partial y},$$

the last holding for $r = 1, \ldots, n$.

As regards these quantities $\rho_1, \ldots, \rho_{n+1}, \sigma_1, \ldots, \sigma_{n+1}$, we have

$$\begin{aligned}\frac{\partial\rho_r}{\partial y} - \frac{\partial\sigma_r}{\partial x} &= \frac{\partial}{\partial y}\left(\omega A_r + \omega\frac{\partial A_{r-1}}{\partial x} - A_{r-1}\frac{\partial\omega}{\partial x}\right) - \frac{\partial}{\partial x}\left(A_{r-1}\frac{\partial\omega}{\partial y} - \omega\frac{\partial A_{r-1}}{\partial y}\right)\\ &= A_r\frac{\partial\omega}{\partial y} + \omega\frac{\partial A_r}{\partial y} + 2\left(\omega\frac{\partial^2A_{r-1}}{\partial x\partial y} - A_{r-1}\frac{\partial^2\omega}{\partial x\partial y}\right)\\ &= A_r\frac{\partial\omega}{\partial y} + \omega\frac{\partial A_r}{\partial y} - 2\omega\frac{\partial A_r}{\partial y}\\ &= \sigma_{r+1},\end{aligned}$$

on using the relations between the coefficients A given by the condition that ζ satisfies the original equation. This equation holds for $r = 1, \ldots, n$; also

$$\frac{\partial \rho_{n+1}}{\partial y} - \frac{\partial \sigma_{n+1}}{\partial x} = 2\left(\omega \frac{\partial^2 A_n}{\partial x \partial y} - A_n \frac{\partial^2 \omega}{\partial x \partial y}\right)$$
$$= 0.$$

Hence

$$\sigma_{r+1} = -\frac{\partial \sigma_r}{\partial x} + \frac{\partial \rho_r}{\partial y},$$

$$-\frac{\partial \sigma_r}{\partial x} = \frac{\partial^2 \sigma_{r-1}}{\partial x^2} - \frac{\partial^2 \rho_{r-1}}{\partial x \partial y},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(-1)^{r-1}\frac{\partial^{r-1}\sigma_2}{\partial x^{r-1}} = (-1)^r \frac{\partial^r \sigma_1}{\partial x^r} - (-1)^r \frac{\partial^r \rho_1}{\partial x^{r-1}\partial y},$$

$$= (-1)^r \frac{\partial^{r+1}\omega}{\partial x^r \partial y} + (-1)^{r-1}\frac{\partial^r \rho_1}{\partial x^{r-1}\partial y};$$

and therefore, on adding, we have

$$\sigma_{r+1} = \frac{\partial}{\partial y}\left\{\rho_r - \frac{\partial \rho_{r-1}}{\partial x} + \ldots + (-1)^{r-1}\frac{\partial^{r-1}\rho_1}{\partial x^{r-1}} + (-1)^r \frac{\partial^r \omega}{\partial x^r}\right\},$$

for the values $r = 1, \ldots, n$. Similarly, if

$$\Omega = \rho_{n+1} - \frac{\partial \rho_n}{\partial x} + \ldots + (-1)^n \frac{\partial^n \rho_1}{\partial x^n} + (-1)^{n+1}\frac{\partial^{n+1}\omega}{\partial x^{n+1}},$$

then

$$\frac{\partial \Omega}{\partial y} = 0,$$

so that Ω is a function of x alone. Let

$$\pi_r = \rho_r - \frac{\partial \rho_{r-1}}{\partial x} + \ldots + (-1)^{r-1}\frac{\partial^{r-1}\rho_1}{\partial x^{r-1}} + (-1)^r \frac{\partial^r \omega}{\partial x^r},$$

for $r = 1, \ldots, n$, so that

$$\frac{\partial \pi_r}{\partial y} = \sigma_{r+1}, \qquad \pi_{r+1} + \frac{\partial \pi_r}{\partial x} = \rho_{r+1},$$

with the convention $\pi_{n+1} = \Omega$; and let ψ denote the function

$$\psi = \omega X^{(n)} + \pi_1 X^{(n-1)} + \pi_2 X^{(n-2)} + \ldots + \pi_n X.$$

Then

$$\frac{\partial(\omega\theta - \psi)}{\partial y} = 0,$$

$$\frac{\partial(\omega\theta - \psi)}{\partial x} = \Omega X.$$

Now Ω is a function of x only; hence we may take

$$\omega\theta - \psi = \int^x \Omega X\,dx,$$

that is,

$$\omega\theta = \omega X^{(n)} + \pi_1 X^{(n-1)} + \ldots + \pi_n X + \int^x \Omega X\,dx.$$

Introduce a new arbitrary function X_1 such that

$$\int^x \Omega X\,dx = \Omega X_1,$$

so that

$$X = X_1' + X_1 \frac{\Omega'}{\Omega} = X_1' + X_1 \frac{d \log \Omega}{dx};$$

then

$$X^{(r)} = X_1^{(r+1)} + X_1^{(r)} \frac{d \log \Omega}{dx} + r X_1^{(r-1)} \frac{d^2 \log \Omega}{dx^2} + \ldots,$$

for $r = 1, \ldots, n+1$; substituting these in the expression for $\omega\theta$, and dividing by ω, we have

$$\theta = X_1^{(n+1)} + B_1 X_1^{(n)} + \ldots + B_{n+1} X_1,$$

so that θ is of grade $n+2$ in x.

Similar calculations, applied to the part of ζ which involves Y and its derivatives, lead to a quantity

$$\vartheta = Y_1^{(n+1)} + C_1 Y_1^{(n)} + \ldots + C_{n+1} Y_1,$$

where Y_1 is an arbitrary function of y: and then the quantity Θ, where

$$\Theta = \theta + \vartheta,$$

is the general integral of the equation

$$\frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial x \partial y} = \omega \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{\omega}\right),$$

which thus is seen to be of finite rank $n+2$ in both variables.

Note. In stating this conclusion, two assumptions have been made. One of them is that the equation

$$\Omega = 0$$

is not satisfied: if it were satisfied, then θ would only be of rank $n+1$ in the variable x instead of being of rank $n+2$. The other is that the corresponding equation, say $\Upsilon = 0$, arising in connection

with the part of $\omega\theta$ which involves Y and its derivatives, also is not satisfied.

Now Ω, which has been proved to be independent of y, is really a linear combination of the derivatives of ω with regard to x; and Υ, which is independent of x, is really a linear combination of the derivatives of ω with regard to y. If ω preserves some of the generality of ζ, then neither Ω nor Υ will vanish; and the equation for θ will then be of rank $n+2$. Even if ω is made exceedingly special, *the new equation for θ will be of increased rank, provided neither Ω nor Υ vanishes: and these conditions are sufficient as well as necessary.*

Ex. 1. The last proposition can be illustrated by one more reference to the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2},$$

having

$$\zeta=X'+Y'-2\frac{X+Y}{x+y},$$

for its general integral.

For the calculation of Ω in this case, we have

$$\begin{aligned}
n&=1,\\
A_1&=-\frac{2}{x+y},\\
\rho_1&=\omega A_1-\frac{\partial\omega}{\partial x},\\
\rho_2&=\omega\frac{\partial A_1}{\partial x}-A_1\frac{\partial\omega}{\partial x},\\
\Omega&=\rho_2-\frac{\partial\rho_1}{\partial x}+\frac{\partial^2\omega}{\partial x^2}\\
&=2\left(\frac{\partial^2\omega}{\partial x^2}+\frac{2}{x+y}\frac{\partial\omega}{\partial x}\right);
\end{aligned}$$

and similarly

$$\Upsilon=2\left(\frac{\partial^2\omega}{\partial y^2}+\frac{2}{x+y}\frac{\partial\omega}{\partial y}\right).$$

Hence the particular form ω of ζ, obtained by specialising X and Y, must be such that neither of the quantities

$$\frac{\partial^2\omega}{\partial x^2}+\frac{2}{x+y}\frac{\partial\omega}{\partial x},\quad \frac{\partial^2\omega}{\partial y^2}+\frac{2}{x+y}\frac{\partial\omega}{\partial y}$$

shall vanish.

It is easy to verify that both the quantities vanish if $X=\frac{1}{4}x^2$, $Y=\frac{1}{4}y^2$: for then

$$\omega=\frac{xy}{x+y}.$$

(See Ex. 7, § 218).

Ex. 2. As a particular example of the general method of proceeding, consider the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2}{(x+y)^2},$$

having the general integral

$$\zeta=X'+Y'-2\frac{X+Y}{x+y}.$$

To obtain ω, let

$$X=x^3,\quad Y=y^3,$$

and then

$$\omega=(x+y)^2,$$

so that (see the last example)

$$\Omega=2\left(\frac{\partial^2\omega}{\partial x^2}+\frac{2}{x+y}\frac{\partial\omega}{\partial x}\right)=12,$$

and similarly

$$\Upsilon=12;$$

thus ω will lead to an equation of higher rank, and the equation is

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y}=\omega\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right)=\frac{6}{(x+y)^2}.$$

Now, in the present case,

$$n=1,\quad A_1=\frac{-2}{x+y},$$

so that

$$\rho_1=\omega A_1-\frac{\partial\omega}{\partial x}=-4(x+y),$$

$$\rho_2=\omega\frac{\partial A_1}{\partial x}-A\frac{\partial\omega}{\partial x}=6.$$

Hence

$$\pi_1=\rho_1-\frac{\partial\omega}{\partial x}=-6(x+y),$$

and therefore

$$\begin{aligned}\psi&=\omega X'+\pi_1 X\\&=(x+y)^2X'-6(x+y)X.\end{aligned}$$

The required part θ is given by

$$\begin{aligned}(x+y)^2\theta-\psi&=\omega\theta-\psi\\&=12\int X dx:\end{aligned}$$

or if

$$X_1=\int X dx,$$

we have

$$\theta=X_1''-\frac{6}{x+y}X_1'+\frac{12}{(x+y)^2}X_1.$$

Similarly, the other part ϑ is given by

$$\vartheta=Y_1''-\frac{6}{x+y}Y_1'+\frac{12}{(x+y)^2}Y_1.$$

The general integral of the equation

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y}=\frac{6}{(x+y)^2}$$

is given by

$$\Theta=\theta+\vartheta.$$

Ex. 3. We proceed to construct the equation of rank 3 having equal invariants which, subject to change of the independent variables, is the most general.

The equation of rank 1 is

$$s=0:$$

its most general integral is (say)

$$\zeta_1=X+Y,$$

and (in the notation of § 219) we can take

$$\omega_1=x+y,$$

this assumption effectively fixing the independent variables.

The equation of rank 2 is

$$s=\frac{2}{(x+y)^2}z:$$

its most general integral is

$$\zeta_2=X'+Y'-2\frac{X+Y}{x+y},$$

and (in the notation of § 219) we can take

$$\omega_2=\xi'+\eta'-2\frac{\xi+\eta}{x+y},$$

where ξ and η are functions of x and of y respectively.

Let the equation of rank 3 be

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y}=\lambda;$$

then, by § 219,

$$\begin{aligned}\lambda&=-2\frac{\partial^2}{\partial x\,\partial y}\{\log(\omega_1\omega_2)\}\\&=-2\frac{\partial^2}{\partial x\,\partial y}[\log\{(x+y)(\xi'+\eta')-2\xi-2\eta\}].\end{aligned}$$

We proceed to construct Θ, by the use of the preceding analysis. We have

$$X'-\frac{2}{x+y}X$$

as the part of ζ_2 involving the arbitrary function of X: thus

$$n=1,\quad A_1=\frac{-2}{x+y},$$

$$\rho_1=-\frac{2\omega_2}{x+y}-\frac{\partial\omega_2}{\partial x},\quad \rho_2=\frac{2}{(x+y)^2}\omega_2+\frac{2}{x+y}\frac{\partial\omega_2}{\partial x},$$

$$\sigma_1=\frac{\partial\omega_2}{\partial y}\qquad,\quad \sigma_2=\frac{-2}{x+y}\frac{\partial\omega_2}{\partial y}-\frac{2}{(x+y)^2}\omega_2.$$

Consequently,

$$\pi_1 = \rho_1 - \frac{\partial \omega_2}{\partial x}$$

$$= -2\frac{\omega_2}{x+y} - 2\frac{\partial \omega_2}{\partial x}$$

$$= 2\left(\frac{\xi' - \eta'}{x+y} - \xi''\right),$$

on reduction: also

$$\Omega = \rho_2 - \frac{\partial \rho_1}{\partial x} + \frac{\partial^2 \omega_2}{\partial x^2}$$

$$= 2\xi''',$$

on reduction. Hence

$$\omega_2\theta = \omega_2 X' + \pi_1 X + 2\int X\xi''' dx,$$

which gives the part of θ depending upon the arbitrary function of x. To exhibit the rank more clearly, we take

$$\int X\xi''' dx = X_1 \xi''',$$

and we have

$$\theta = X_1'' + a_1 X_1' + a_2 X_1,$$

where

$$a_1 = \frac{\pi_1}{\omega_2} + \frac{d \log \xi'''}{dx}$$

$$= \frac{2}{\omega_2}\left(\frac{\xi' - \eta'}{x+y} - \xi''\right) + \frac{d \log \xi'''}{dx},$$

$$a_2 = \frac{2\xi'''}{\omega_2} + \frac{\pi_1}{\omega_2}\frac{d \log \xi'''}{dx} + \frac{d^2 \log \xi'''}{dx^2}.$$

Similarly, the part of Θ depending upon the arbitrary function of Y is

$$\vartheta = Y_1'' + \beta_1 Y_1' + \beta_2 Y_1,$$

where

$$\beta_1 = \frac{\tau_1}{\omega_2} + \frac{d \log \eta'''}{dy}$$

$$= \frac{2}{\omega_2}\left(\frac{\eta' - \xi'}{x+y} - \eta''\right) + \frac{d \log \eta'''}{dy},$$

$$\beta_2 = \frac{2\eta'''}{\omega_2} + \frac{\tau_1}{\omega_2}\frac{d \log \eta'''}{dy} + \frac{d^2 \log \eta'''}{dy^2}.$$

The value of Θ is

$$\Theta = \theta + \vartheta$$

$$= X_1'' + a_1 X_1' + a_2 X_1 + Y_1'' + \beta_1 Y_1' + \beta_2 Y_1;$$

and it is actually of rank three in each of the variables, provided neither ξ''' nor η''' vanishes, that is, provided neither ξ nor η is a quadratic polynomial in the respective variables.

Ex. 4. Prove that, on taking

$$\xi = x^3, \quad \eta = y^3$$

in the preceding example, the equation for Θ is

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y} = \frac{6}{(x+y)^2};$$

and that, on taking

$$\xi = x^4, \quad \eta = y^4,$$

in the preceding example, the equation for Θ is

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial x\,\partial y} = \frac{2}{(x^3+y^3)^2}(x^4 - 2x^3y + 12x^2y^2 - 2xy^3 + y^4).$$

Deduce the integrals of each of these equations.

Ex. 5. Verify that

$$\omega = (x-y)^m (x+y)^n$$

satisfies the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{m(m-1)}{(x-y)^2} + \frac{n(n-1)}{(x+y)^2};$$

and apply Moutard's theorem to prove that the equation can be integrated in finite terms when m and n are integers. (The integers evidently can be taken positive.)

Obtain the integral in the form

$$z = \frac{\partial^{m+n}}{\partial u^m \partial v^n}\{\phi(u^{\frac{1}{2}} + v^{\frac{1}{2}}) + \psi(u^{\frac{1}{2}} - v^{\frac{1}{2}})\}$$

where $u = (x-y)^2$, $v = (x+y)^2$, and ϕ, ψ are arbitrary functions. (Darboux.)

Ex. 6. Prove that, if an integral

$$\omega = \xi'' + \eta'' - 6\frac{\xi' + \eta'}{x+y} + 12\frac{\xi + \eta}{(x+y)^2}$$

of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y} = \frac{6}{(x+y)^2}$$

is used, by Moutard's theorem, to construct another equation, the equation so constructed will not be of rank 4 in both variables unless the quantities

$$\frac{d^5\xi}{dx^5}, \quad \frac{d^5\eta}{dy^5}$$

are different from zero.

Assuming that ξ and η are not quartic polynomials in their respective variables, form the equation of rank 4 and obtain its general integral.

Ex. 7. Integrate the equations:—

(i) $r - t = 2n\dfrac{p}{x}$;

(ii) $c^2 x^{\frac{4}{3}} r - t = 0$;

(iii) $(2n+1)^2 x^{\frac{4n}{2n+1}} r - t = 0$;

where n is an integer in (i) and in (iii). (Sersawy, Winckler.)

Ex. 8. Shew that the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\phi(x+y)-\psi(x-y)$$

possesses an infinitude of integrals of the form*

$$\omega=f(x+y)\,g(x-y),$$

where the forms of the functions f and g are determined by the ordinary linear equations

$$\frac{f''(t)}{f(t)}=\phi(t)+a,\quad \frac{g''(t)}{g(t)}=\psi(t)+a,$$

the quantity a being a constant.

Denoting any other integral of the equation by

$$z=F(x+y)\,G(x-y),$$

prove that the integral of the equation

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial x\,\partial y}=\omega\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{\omega}\right),$$

which is derived from z by Moutard's theorem, is

$$Z=\left(F'-F\frac{f'}{f}\right)\left(G'-G\frac{g'}{g}\right).$$

(Darboux.)

Ex. 9. With the same notation as in the last example, shew that the general integral Z of the equation

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial x\,\partial y}=\frac{2f'^2-ff''}{f^2}-\psi(x-y)-a$$

is given by

$$Z=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}-2z\frac{f'}{f},$$

where z is the general integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\phi(x+y)-\psi(x-y),$$

the quantity a being a constant, and the form f of $f(x+y)$ being determined by

$$\frac{f''(t)}{f(t)}=\phi(t)+a.$$

(Darboux.)

Ex. 10. Prove that, if ω be any integral of the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\phi(x+y)-\psi(x-y),$$

then

$$\Omega=\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}-2\omega\,\{\phi(x+y)+\psi(x-y)\}$$

is another integral of the equation. (Darboux.)

* The equation is called a *harmonic* equation: an integral, such as ω, is called a *harmonic* integral.

Ex. 11. Shew that, if the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\lambda(x,y)=\lambda$$

can be a harmonic equation, so that it is expressible in the form

$$\frac{1}{z}\frac{\partial^2 z}{\partial x'\,\partial y'}=\phi(x'+y')-\psi(x'-y'),$$

then the independent variables are given by

$$x'=\int X^{-\frac{1}{2}}dx,\quad y'=\int Y^{-\frac{1}{2}}dy,$$

where the equation

$$2X\frac{\partial^2\lambda}{\partial x^2}+3X'\frac{\partial\lambda}{\partial x}+\lambda X''=2Y\frac{\partial^2\lambda}{\partial y^2}+3Y'\frac{\partial\lambda}{\partial y}+\lambda Y''$$

must be satisfied.

When the equation in question is Euler's equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{m(1-m)}{(x-y)^2},$$

where m is a constant, prove that X is a quartic polynomial in x and that Y is the same quartic polynomial in y; and obtain the various forms of equation according to the equalities of the roots of X.

(Darboux.)

Ex. 12. Shew that the equation

$$\frac{1}{z}\frac{\partial^2 z}{\partial x\,\partial y}=\frac{m(m-1)}{(x+y)^2}-\frac{n(n-1)}{(x-y)^2}+\frac{m'(m'-1)}{(1-xy)^2}-\frac{n'(n'-1)}{(1+xy)^2},$$

where m, n, m', n' are constants, can be made harmonic. (Darboux.)

CHAPTER XV.

FORMS OF EQUATIONS OF THE SECOND ORDER IN TWO INDEPENDENT VARIABLES HAVING THEIR GENERAL INTEGRALS IN EXPLICIT FINITE FORM.

THE substance of this chapter is based entirely on Cosserat's proof of the theorem enunciated by Moutard: the proof is contained in Note III at the end (pp. 405—422) of the fourth volume of Darboux's *Théorie générale des surfaces*, published in 1896.

It should be added that these investigations are the matter of a paper* by Tanner whose results, expressed in a slightly different form, are a clear anticipation of many of Cosserat's results. There is a difference in notation, but it is unessential to the main properties: Tanner uses the successive integrals of an arbitrary function of x or of y, instead of the successive derivatives. When the corresponding changes are made in the notation, comparison of the two sets of results is easy and immediate.

Reference may also be made to a memoir by Goursat †.

221. In the two preceding chapters, we have been occupied mainly with the discussion of linear equations of the second order the integrals of which, whether half general or completely general, are expressible in finite form; and in each instance, it is the equation that is given while it is the integral that has to be determined.

But a different point of view may be adopted: as was the case with early investigations of classical analysts such as Euler and Lagrange, we may consider equations as determined by their integrals. In particular, we shall consider those equations of the second order the integral of which is composed of a single relation

* *Proc. L. M. S.*, t. VIII (1876), pp. 159—174.

† *Ann. de Toulouse*, 2^{me} Sér., t. I (1899), pp. 31—78.

between x, y, z; this relation is to involve a couple of arbitrary functions, as well as derivatives of these arbitrary functions up to specified orders which are finite for each of them; the arguments of the arbitrary functions are to be given explicitly in terms of the variables, and are to be distinct from one another; and all expressions in the relation are to be free from partial quadratures. What is required is the aggregate of equations possessing integrals of this type: they are subject to the following theorem, first enunciated* by Moutard:—

Equations of the second order, which have an integral of the type indicated and which, by a transformation of the variables, cannot be expressed in Laplace's linear form

$$s + ap + bq + cz = 0,$$

or in Liouville's form

$$s = e^z,$$

are, with two simple exceptions, reducible to the form

$$s = \frac{\partial}{\partial x}(Ae^z) - \frac{\partial}{\partial y}(Be^{-z}),$$

where A and B are functions of the independent variables alone, satisfying certain conditions; and the integration of this equation can be made to depend uniquely upon the integration of

$$\frac{\partial^2 \alpha}{\partial x \partial y} = \frac{1}{A}\frac{\partial A}{\partial x}\frac{\partial \alpha}{\partial y} + AB\alpha,$$

which is of the linear type considered by Laplace.

In the preceding statement of the type of integral relation which is to lead to a partial differential equation of the second order, it has been laid down as a condition that the arguments of the two arbitrary functions are distinct from one another. This condition is effectively bound up with the finiteness of the form of the integral relation. For if the two arguments be the same, let a change in the independent variables be made whereby this common argument becomes the variable y; then the only deriva-

* The proof of the theorem formed the first part of the memoir, mentioned on p. 189, note, which was destroyed in 1871. The author did not rewrite this part; the statement of the theorem is taken from the abstract, as given in the *Comptes Rendus*, t. LXX (1870), pp. 834—838, and as reproduced in the *Journ. de l'Éc. Poly.*, t. XXXVII (1886), pp. 1—5.

tive of the second order that can occur is r. Suppose the differential equation resolved for r, and let the resolved form be

$$r = f(x, y, z, p, q).$$

If z is finite and explicit in form, so that derivatives of η and Y (the two arbitrary functions of y that occur in z) of only finite order occur in z, then (as in § 186) q contains derivatives of one order higher than those that occur in r, z, p: thus, if the equation is to be satisfied, q cannot occur in f. An equation

$$r = f(x, y, z, p)$$

is effectively an ordinary equation, not a partial equation*.

Accordingly, we adhere to the condition that, for the present purpose, the arguments of the two arbitrary functions are to be different from one another.

COSSERAT'S PROOF OF MOUTARD'S THEOREM.

222. The following process of establishing the theorem is due† to Cosserat.

Let x' and y' denote the arguments of the arbitrary functions in the integral relation; they are to be definite and explicit quantities involving x, y and (it may be) z; and they will be assumed different from one another. Let $\phi(x')$ and $\psi(y')$ be the arbitrary functions; and suppose that $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n$ are all the derivatives of ϕ and ψ that occur, the suffix in each case being the order of derivation with regard to the argument. The integral relation can be expressed in a form

$$F(x, y, z, \phi, \phi_1, \ldots, \phi_m, \psi, \psi_1, \ldots, \psi_n) = 0.$$

Change the independent variables so that they become x' and y'; denote ϕ, a function of one of the independent variables alone, by ξ and denote ψ, a function of the other independent variable alone, by η; and suppose that the equation, after the transforma-

* See also, on this argument, § 208.

† In Note III, pp. 405—422, added at the end of the fourth volume of Darboux's *Théorie générale des surfaces*.

tions have been effected, is resolved with regard to z. Then it can be taken in a form

$$\begin{aligned} z &= f(x', y', \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n) \\ &= f(x, y, \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n), \end{aligned}$$

on dropping the dashes; and this relation is to be the general integral of an equation of the second order. As the arguments of the arbitrary functions in the integral are x and y respectively, the only derivative of the second order that can be expected (§ 186) to occur in the equation is s.

We have

$$p = \frac{\partial f}{\partial x} + \sum_{i=0}^{m} \frac{\partial f}{\partial \xi_i} \xi_{i+1}, \qquad q = \frac{\partial f}{\partial y} + \sum_{j=0}^{n} \frac{\partial f}{\partial \eta_j} \eta_{j+1},$$

$$s = \frac{\partial^2 f}{\partial x \partial y} + \sum_{j=0}^{n} \frac{\partial^2 f}{\partial x \partial \eta_j} \eta_{j+1} + \sum_{i=0}^{m} \frac{\partial^2 f}{\partial y \partial \xi_i} \xi_{i+1} + \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{\partial^2 f}{\partial \xi_i \partial \eta_j} \xi_{i+1} \eta_{j+1} :$$

as already explained, r and t will not occur, all the more obviously because r alone contains ξ_{m+2} and t alone contains η_{n+2}, neither of which quantities occurs in z, p, q, s, and neither of which could be eliminated among these derivatives. To construct the differential equation, the two arbitrary functions and their derivatives have to be eliminated: eliminating ξ_{m+1} and η_{n+1}, which do not occur in z, we have an equation of the form

$$s + \zeta pq + ap + bq + c = 0,$$

where ζ, a, b, c are functions of x and y. When the other derivatives of ξ and η are eliminated by means of z, this equation being the required differential equation of the second order, ζ, a, b, c are functions of x, y, and z. Evidently,

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} + \zeta \frac{\partial f}{\partial \xi_m} \frac{\partial f}{\partial \eta_n} = 0,$$

so that, in its first form, ζ is a function of x and y only and involves no quantities that do not occur in z, and in its final form ζ is a function of x, y, z.

Let the value of $\int \zeta dz$ be obtained, on the hypothesis that x and y are constant through the quadrature: and writing

$$z_0 = e^{\int \zeta dz},$$

ɜt a new variable Z be introduced, defined by the relation

$$Z = \int z_0 dz,$$

ʼith the same hypothesis for the quadrature as before, so that ' is a function of x, y, z, which clearly will be of the same general ype as the original quantity z. Taking Z as a new dependent ariable, we have

$$P = pz_0 + \int \frac{\partial z_0}{\partial x} dz,$$

$$Q = qz_0 + \int \frac{\partial z_0}{\partial y} dz,$$

$$\begin{aligned} S &= sz_0 + q\left(p\frac{\partial z_0}{\partial z} + \frac{\partial z_0}{\partial x}\right) + p\frac{\partial z_0}{\partial y} + \int \frac{\partial^2 z_0}{\partial x \partial y} dz, \\ &= z_0(s + pq\zeta) + q\frac{\partial z_0}{\partial x} + p\frac{\partial z_0}{\partial y} + \int \frac{\partial^2 z_0}{\partial x \partial y} dz; \end{aligned}$$

nd the quantities, for which quadrature with regard to z is ɜquired, are functions of x, y, z. Thus the differential equation ›r Z is of the form

$$S + AP + BQ + C = 0,$$

ʼhen z is eliminated from the coefficients by the relation

$$Z = \int z_0 dz.$$

'he integral Z of the equation is of the same type as before. We ıay therefore make $\zeta = 0$; and thus we have to determine a, b, c ɜ functions of x, y, z alone, such that the equation

$$s + ap + bq + c = 0$$

nd the relation

$$z = f(x, y, \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n)$$

re equivalent to one another. But, as ζ now is zero, there is the mitation

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$$

pon the form of the integral relation.

THREE TYPES OF COEFFICIENTS a AND b.

223. Let the deduced expressions for s, p, q be substituted in the equation

$$s + ap + bq + c = 0,$$

which is to be satisfied identically. The term, which involves $\xi_{m+1}\eta_{n+1}$, arises through s alone: it disappears, owing to the foregoing limitation upon the form of f. The term, which involves ξ_{m+1}, must disappear: thus

$$\frac{\partial^2 f}{\partial y \partial \xi_m} + \sum_{\beta=0}^{n-1} \frac{\partial^2 f}{\partial \xi_m \partial \eta_\beta} \eta_{\beta+1} + a \frac{\partial f}{\partial \xi_m} = 0,$$

and this must be satisfied identically when the value of z is substituted in a. The term, which involves η_{n+1}, must disappear: thus

$$\frac{\partial^2 f}{\partial x \partial \eta_n} + \sum_{\alpha=0}^{m-1} \frac{\partial^2 f}{\partial \xi_\alpha \partial \eta_n} \xi_{\alpha+1} + b \frac{\partial f}{\partial \eta_n} = 0,$$

and this must be satisfied identically when the value of z is substituted in b.

It follows, from the condition $\dfrac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$, that $\dfrac{\partial f}{\partial \xi_m}$ does not contain η_n; hence* $\dfrac{\partial^2 f}{\partial y \partial \xi_m}$ does not contain η_n. The first of the two equations, which become identities when the value of z is substituted in a and in b, contains a term

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} \eta_n;$$

consequently a involves η_n only in the first power after the value of z has been substituted. The equation is then satisfied identically; hence, differentiating with regard to η_n, we have

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} + \frac{\partial f}{\partial \xi_m} \frac{\partial a}{\partial z} \frac{\partial f}{\partial \eta_n} = 0,$$

also satisfied identically. Now $\dfrac{\partial f}{\partial \xi_m}$ does not contain η_n, and

* The quantity $\dfrac{\partial^2 f}{\partial y \partial \xi_m}$ is only the partial, not the complete, derivative of $\dfrac{\partial f}{\partial \xi_m}$ with regard to y.

therefore $\dfrac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}}$ does not contain it: hence, differentiating the last identity with regard to η_n, we have

$$\frac{\partial}{\partial \eta_n}\left(\frac{\partial a}{\partial z}\frac{\partial f}{\partial \eta_n}\right) = 0,$$

that is,

$$\frac{\partial a}{\partial z}\frac{\partial^2 f}{\partial \eta_n^2} + \frac{\partial^2 a}{\partial z^2}\left(\frac{\partial f}{\partial \eta_n}\right)^2 = 0,$$

which is to be satisfied identically when the value of z is substituted in $\dfrac{\partial a}{\partial z}$ and $\dfrac{\partial^2 a}{\partial z^2}$.

This relation clearly will be satisfied identically, if a does not involve z: we thus have one possible case. For other cases, we have

$$\frac{\dfrac{\partial^2 a}{\partial z^2}}{\dfrac{\partial a}{\partial z}} = -\frac{\dfrac{\partial^2 f}{\partial \eta_n^2}}{\left(\dfrac{\partial f}{\partial \eta_n}\right)^2}.$$

Now, $\dfrac{\partial f}{\partial \eta_n}$ does not involve ξ_m because of the relation $\dfrac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$; hence the right-hand fraction, when expressed in terms of x, y, z, cannot involve z. If the fraction is zero, then

$$a = \mu + \lambda z,$$

where μ and λ can be functions of x and y; and then

$$\frac{\partial^2 f}{\partial \eta_n^2} = 0.$$

If the fraction is not zero, then

$$a = \mu + \lambda e^{z\rho},$$

where μ, λ, ρ can be functions of x and y; and then

$$\frac{\partial^2 f}{\partial \eta_n^2} + \rho\left(\frac{\partial f}{\partial \eta_n}\right)^2 = 0.$$

Consequently, *there are three possible forms for* a, viz.

$$\mu, \quad \mu + \lambda z, \quad \mu + \lambda e^{z\rho},$$

with corresponding limitations upon the form of f for the second and the third; and the quantities λ, μ, ρ can be functions of the independent variables.

Proceeding in the same way from the other of the equations which become identities when the value of z is substituted, we find that *there are three possible forms for* b, of the same type as those for a, and accompanied by the corresponding limitations on the form of f for the second and the third.

Ex. 1. Given an integral relation

$$F(u_1, \ldots, u_r, x, y, z)=0,$$

where $u_1, \ldots, u_r$ are r arbitrary functions, each having one definite argument, and where F is a definite function, shew that, if differential relations equivalent to $F=0$ are formed, the lowest aggregate will generally consist of r equations of order $2r-1$. (Falk.)

Ex. 2. Shew that, if

$$z=F(u, v)$$

be an integral equation in which F is a determinate function, and u, v are arbitrary functions of definite arguments, then z satisfies a differential equation of the second order if (and only if)

$$\frac{\dfrac{\partial^2 F}{\partial u\,\partial v}}{\dfrac{\partial F}{\partial u}\,\dfrac{\partial F}{\partial v}}$$

is independent of u and v or can be made independent of u and v by means of the primitive relation. (Falk.)

224. We proceed to consider the possible combinations. In the first place, let

$$a=\mu+\lambda e^{z\rho};$$

we then have

$$\frac{\partial^2 f}{\partial \eta_n^2}+\rho\left(\frac{\partial f}{\partial \eta_n}\right)^2=0,$$

so that, as ρ is a function of x and y only and as $\dfrac{\partial f}{\partial \eta_n}$ does not involve ξ_m, we have

$$\frac{\partial f}{\partial \eta_n}=\frac{1}{\rho(\eta_n+f_2)},$$

where f_2 is a quantity that does not involve ξ_m or η_n; and therefore

$$e^{z\rho}=e^{f\rho}=(\eta_n+f_2)\,F,$$

where F is a quantity that does not involve η_n. Now the value of b is to satisfy the equation

$$\frac{\partial^2 f}{\partial x\partial \eta_n}+\sum_{a=0}^{m-1}\frac{\partial^2 f}{\partial \xi_a\partial \eta_n}\xi_{a+1}+b\frac{\partial f}{\partial \eta_n}=0,$$

so that, on substituting the obtained value of $\dfrac{\partial f}{\partial \eta_n}$, we have

$$\left(b - \frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)(\eta_n + f_2) = \frac{\partial f_2}{\partial x} + \sum_{a=0}^{m-1} \frac{\partial f_2}{\partial \xi_a} \xi_{a+1},$$

and therefore

$$\left(b - \frac{1}{\rho}\frac{\partial \rho}{\partial x}\right) e^{z\rho} = \left(\frac{\partial f_2}{\partial x} + \sum_{a=0}^{m-1} \frac{\partial f_2}{\partial \xi_a} \xi_{a+1}\right) F.$$

The right-hand side does not involve η_n: hence the left-hand side, when expressed in terms of the variables, must be explicitly free from z. It is therefore either zero or, if not zero, a function of x and y at the utmost.

If the right-hand side is not zero, let its value be denoted by σ, where σ can be a function of x and y; then

$$b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-z\rho}.$$

If the right-hand side is zero, then

$$b = \frac{1}{\rho}\frac{\partial \rho}{\partial x}.$$

These are the two forms for b which can be associated with $\mu + \lambda e^{z\rho}$ as the value for a: the combination

$$a = \mu + \lambda e^{z\rho}, \quad b = \mu' + \lambda' z,$$

is not possible. We take the two possible forms in turn.

First Combination of Coefficients.

225. Suppose that

$$a = \mu + \lambda e^{z\rho}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-z\rho},$$

the form of b being derived from the assumed form of a. A precisely similar argument can be used to deduce the form of a from an assumed form of b: we merely need to change the sign of ρ and to interchange the independent variables, and we have

$$\mu = \frac{1}{-\rho}\frac{\partial(-\rho)}{\partial y} = \frac{1}{\rho}\frac{\partial \rho}{\partial y}.$$

Moreover, as regards z, on taking account of these interchanges and changes, we have

$$e^{-z\rho} = e^{-f\rho} = (\xi_m + f_1)\, G,$$

where f_1 is a quantity that does not involve ξ_m or η_n, and G is a quantity that does not involve ξ_m; and also

$$\left(a - \frac{1}{\rho}\frac{\partial\rho}{\partial y}\right)(\xi_m + f_1) = \frac{\partial f_1}{\partial y} + \sum_{\beta=0}^{n-1} \frac{\partial f_1}{\partial \eta_\beta}\, \eta_{\beta+1}.$$

The two expressions for z, given by

$$e^{z\rho} = (\eta_n + f_2)\, F, \quad e^{-z\rho} = (\xi_m + f_1)\, G,$$

can be combined into the single expression

$$e^{z\rho} = \frac{\eta_n + f_2}{\xi_m + f_1}\, e^{f_3},$$

where now the quantities f_1, f_2, f_3 do not involve either ξ_m or η_n. Also, from the equation

$$\begin{aligned}\left(b - \frac{1}{\rho}\frac{\partial\rho}{\partial x}\right)(\eta_n + f_2) &= \frac{\partial f_2}{\partial x} + \sum_{\alpha=0}^{m-1} \frac{\partial f_2}{\partial \xi_\alpha}\, \xi_{\alpha+1} \\ &= \frac{d}{dx}(\eta_n + f_2),\end{aligned}$$

we have

$$\sigma e^{-z\rho} = \frac{d}{dx}\{\log(\eta_n + f_2)\};$$

and, similarly,

$$\lambda e^{z\rho} = \frac{d}{dy}\{\log(\xi_m + f_1)\}.$$

The differential equation becomes, on substitution for a and b,

$$\frac{\partial^2 z}{\partial x \partial y} + \left(\lambda e^{z\rho} + \frac{1}{\rho}\frac{\partial\rho}{\partial y}\right)\frac{\partial z}{\partial x} + \left(\sigma e^{-z\rho} + \frac{1}{\rho}\frac{\partial\rho}{\partial x}\right)\frac{\partial z}{\partial y} + c = 0;$$

and so, introducing a new dependent variable Z such that

$$Z = z\rho,$$

we have

$$\frac{\partial^2 Z}{\partial x \partial y} + \frac{\partial}{\partial x}(\lambda e^{Z}) - \frac{\partial}{\partial y}(\sigma e^{-Z}) + C = 0,$$

where C is a function of x, y and (possibly) Z; and the integral of this equation is

$$Z = \log\left(\frac{\eta_n + f_2}{\xi_m + f_1}\right) + f_3.$$

Substituting this value of Z in the deduced equation, and taking account of the earlier values of $\lambda e^{z\rho}$ and $\sigma e^{-z\rho}$, we have

$$C = -\frac{\partial^2 f_3}{\partial x \partial y}.$$

Now, as f_3 does not explicitly involve either ξ_m or η_n, the right-hand side can involve both of them in a term

$$-\frac{\partial^2 f_3}{\partial \xi_{m-1} \partial \eta_{n-1}} \xi_m \eta_n,$$

and into no other term do both ξ_m and η_n enter. If C should involve Z, the quantities ξ_m and η_n enter into its expression (when substitution takes place) only through the fraction

$$\frac{\eta_n + f_2}{\xi_m + f_1}.$$

The two forms are incompatible; hence C is a function of x and y only. Consequently $\dfrac{\partial^2 f_3}{\partial x \partial y}$ also is a function of x and y only; and therefore we may take

$$f_3 = g_3 + X + Y,$$

where g_3 is a specific function of x and y only, free from the arbitrary functions ξ and η, while X and Y are any functions we please of ξ and of η respectively, subject to the limitation that f_3 does not involve ξ_m nor η_n. Let z' be a new variable, where

$$z' = Z - g_3 - X - Y,$$

and let the factors $e^{g_3 + X + Y}$ and $e^{-g_3 - X - Y}$ be absorbed into λ and σ respectively, making them A and B respectively. Then the equation takes the form

$$\frac{\partial^2 z'}{\partial x \partial y} + \frac{\partial}{\partial x}(A e^{z'}) - \frac{\partial}{\partial y}(B e^{-z'}) = 0;$$

and its integral is

$$z' = \log\left(\frac{\eta_n + f_2}{\xi_m + f_1}\right),$$

that is,

$$e^{z'} = \frac{\eta_n + f_2}{\xi_m + f_1},$$

where f_1 and f_2 do not involve either ξ_m or η_n. The quantities A and B are functions of x and y only; and

$$A e^{z'} = \frac{d}{dy}\{\log(\xi_m + f_1)\},$$

that is,

$$A(\eta_n+f_2)=\frac{d}{dy}(\xi_m+f_1);$$

and, similarly,

$$B(\xi_m+f_1)=\frac{d}{dx}(\eta_n+f_2).$$

The differential equation is to be free from any expression of the arbitrary functions, so that A and B are functions of x and y only; these two equations limit the forms of f_1 and f_2, and they may impose conditions upon A and B. If with given coefficients A and B, satisfying the conditions (if any), the values of f_1 and f_2 can be obtained, then the integral of the original equation can be regarded as known.

Assuming that the conditions affecting A and B are satisfied, we evidently have

$$\frac{d}{dx}\left\{\frac{1}{A}\frac{d}{dy}(\xi_m+f_1)\right\}=B(\xi_m+f_1);$$

so that, if

$$u=\xi_m+f_1,$$

u is an integral of the equation

$$\frac{\partial^2 u}{\partial x\partial y}-\frac{1}{A}\frac{\partial A}{\partial x}\frac{\partial u}{\partial y}=ABu,$$

which is of Laplace's linear form. Similarly, if

$$v=\eta_n+f_2,$$

v is an integral of the equation

$$\frac{\partial^2 v}{\partial x\partial y}-\frac{1}{B}\frac{\partial B}{\partial y}\frac{\partial v}{\partial x}=ABv,$$

also of Laplace's linear form. Moreover,

$$Av=\frac{\partial u}{\partial y},\qquad Bu=\frac{\partial v}{\partial x},$$

so that, if the integral of either equation can be obtained, the integral of the other will be known; and therefore the integration of the equation

$$\frac{\partial^2 z'}{\partial x\partial y}+\frac{\partial}{\partial x}(Ae^{z'})-\frac{\partial}{\partial y}(Be^{-z'})=0,$$

where A and B satisfy the appropriate conditions, can be made to depend upon the integration of an equation of Laplace's linear form.

Thus the combination

$$a=\frac{1}{\rho}\frac{\partial\rho}{\partial y}+\lambda e^{\rho z},\quad b=\frac{1}{\rho}\frac{\partial\rho}{\partial x}+\sigma e^{-\rho z},$$

leads to the establishment of part of the theorem enunciated.

Note. The relation between the differential equation and the two linear relations can be exhibited in a different form. The differential equation is

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial z'}{\partial y}+Ae^{z'}\right)&=\frac{\partial}{\partial y}(Be^{-z'})\\&=\frac{d}{dy}\left\{\frac{d}{dx}\log(\eta_n+f_2)\right\}\\&=\frac{\partial}{\partial y}\left(\frac{1}{v}\frac{\partial v}{\partial x}\right),\end{aligned}$$

so that we may take

$$\frac{\partial z'}{\partial y}+Ae^{z'}=\frac{1}{v}\frac{\partial v}{\partial y},$$

that is,

$$\frac{\partial z'}{\partial y}+\frac{1}{u}\frac{\partial u}{\partial y}=\frac{1}{v}\frac{\partial v}{\partial y}.$$

Similarly, we have

$$\frac{\partial z'}{\partial x}-Be^{-z'}=-\frac{1}{u}\frac{\partial u}{\partial x},$$

that is,

$$\frac{\partial z'}{\partial x}-\frac{1}{v}\frac{\partial v}{\partial x}=-\frac{1}{u}\frac{\partial u}{\partial x};$$

both of these are immediate inferences from the integral of the differential equation, which is

$$z'=\log\left(\frac{v}{u}\right).$$

Ex. 1. The simplest case of all is provided by $m=0$, $n=0$: but it is trivial, for we easily see that

$$A=0,\quad B=0,$$

and the equation is merely

$$s=0.$$

Ex. 2. The case, next in simplicity, is provided by taking

$$m=0,\quad n=1:$$

the case $m=1$, $n=0$, can be derived from it, by an interchange of the independent variables and a change in the sign of z. Then

$$e^{z}=\frac{\eta'+f_2}{\xi+f_1},$$

where f_1 and f_2 may involve η but not ξ. Writing

$$u=\xi+f_1, \quad v=\eta'+f_2,$$

we have

$$\beta(\xi+f_1)=\frac{\partial v}{\partial x}=\frac{\partial f_2}{\partial x},$$

and f_2 does not involve ξ: hence β is zero, and f_2 does not involve x. Also

$$a(\eta'+f_2)=\frac{\partial u}{\partial y}=\frac{\partial f_1}{\partial \eta}\eta'+\frac{\partial f_1}{\partial y},$$

so that

$$f_1=a\eta+\theta,$$

and then

$$af_2=\eta\frac{\partial a}{\partial y}+\frac{\partial \theta}{\partial y},$$

that is,

$$f_2=\eta\frac{1}{a}\frac{\partial a}{\partial y}+\frac{1}{a}\frac{\partial \theta}{\partial y}.$$

Now f_2 is to be independent of x: thus we must have

$$a=g(x)\,h'(y),$$
$$\theta=g(x)\,k(y),$$

and then

$$f_2=\eta\frac{h''(y)}{h'(y)}+\frac{k'(y)}{h'(y)},$$
$$f_1=\{\eta h'(y)+k(y)\}\,g(x);$$

the equation is

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\{e^z g(x)\,h'(y)\}=0.$$

Without any loss of generality, we may take

$$h'(y)=1,$$

and $k(y)$ can be absorbed into η: hence the type of equation is

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\{e^z g(x)\}=0,$$

and its general integral is

$$e^z=\frac{\eta'}{\xi+\eta g(x)},$$

where ξ and η are arbitrary functions of x and of y respectively.

All other equations of this type, for which $m=0$, $n=1$, are deducible from the foregoing by transformations of the variables

$$x'=\phi(x), \quad y'=\psi(y).$$

Ex. 3. Next, consider the case

$$m=0, \quad n=2;$$

the case $m=2$, $n=0$ can be deduced from it by changing the sign of z and interchanging the independent variables. For the present, we have

$$e^z=\frac{v}{u},$$

where

$$v=\eta''+f_2, \quad u=\xi+f_1\ ;$$

in this case, f_1 and f_2 may contain η and η' but not ξ. Also

$$av=\frac{\partial u}{\partial y}, \quad \beta u=\frac{\partial v}{\partial x}.$$

As f_2 does not contain ξ, the last equation shews that

$$\beta=0, \quad \frac{\partial f_2}{\partial x}=0,$$

so that f_2 is a function of y only. Again,

$$\begin{aligned} a(\eta''+f_2)=av &= \frac{\partial u}{\partial y} \\ &= \frac{\partial f_1}{\partial \eta'}\eta''+\frac{\partial f_1}{\partial \eta}\eta'+\frac{\partial f_1}{\partial y}\ ; \end{aligned}$$

and therefore, as f_1 does not contain η'', we have

$$f_1=a\eta'+g_1,$$

where now g_1 may contain η but does not contain η'. Also, on substituting this value of f_1, we have

$$af_2=\eta'\left(\frac{\partial a}{\partial y}+\frac{\partial g_1}{\partial \eta}\right)+\frac{\partial g_1}{\partial y}.$$

Hence, as f_2 is a function of y only, we must have

$$\left.\begin{aligned} \frac{\partial a}{\partial y}+\frac{\partial g_1}{\partial \eta}&=ay_1 \\ \frac{\partial g_1}{\partial y}&=ay_2 \end{aligned}\right\},$$

where y_1 and y_2 are functions of y only. Hence

$$\begin{aligned} \frac{\partial^2 a}{\partial y^2} &= \frac{\partial}{\partial y}(ay_1)-\frac{\partial}{\partial \eta}(ay_2) \\ &= \frac{\partial}{\partial y}(ay_1)-a\frac{\partial y_2}{\partial \eta}\ ; \end{aligned}$$

hence, as a is independent of η, we may take y_2 and g_1 as linear in η, and y_1 as independent of η. Consequently, if

$$\begin{aligned} v &= \eta''+y_1\eta'+P\eta+Q, \\ u &= \xi\ +a\eta'\ +R\eta+S, \end{aligned}$$

where P and Q involve y alone, the relation

$$av=\frac{\partial u}{\partial y}$$

leads to

$$ay_1=\frac{\partial a}{\partial y}+R, \quad aP=\frac{\partial R}{\partial y}, \quad aQ=\frac{\partial S}{\partial y}.$$

We may make Q and S zero ; for let

$$\eta=H-\bar{\eta},$$

where H is a new arbitrary function of y, and $\bar{\eta}$ is a function of y at ou disposal. The new values of v and u are

$$v = H'' + y_1 H' + PH + (Q - \bar{\eta}'' - y_1\bar{\eta}' - P\bar{\eta}),$$
$$u = \xi \quad + aH' + RH + (S - a\bar{\eta}' - R\bar{\eta}).$$

Now

$$a(Q - \bar{\eta}'' - y_1\bar{\eta}' - P\bar{\eta}) = \frac{\partial S}{\partial y} - a\bar{\eta}'' - \frac{\partial a}{\partial y} a\bar{\eta}' - R\eta' - \bar{\eta}\frac{\partial R}{\partial y}$$
$$= \frac{\partial}{\partial y}(S - a\bar{\eta}' - R\bar{\eta}).$$

Choose $\bar{\eta}$, so that

$$Q - \bar{\eta}'' - y_1\bar{\eta}' - P\bar{\eta} = 0\,;$$

then

$$S - a\bar{\eta}' - R\bar{\eta}$$

is a function of x only, and it can be absorbed into ξ. Thus we may take

$$\left.\begin{aligned} v &= \eta'' + y_1\eta' + P\eta \\ u &= \xi \quad + a\eta' + R\eta \end{aligned}\right\},$$

with the relations

$$ay_1 = \frac{\partial a}{\partial y} + R, \quad aP = \frac{\partial R}{\partial y}\,;$$

the equation for a is

$$\frac{\partial^2 a}{\partial y^2} = \frac{\partial}{\partial y}(ay_1) - aP,$$

so that, as y_1 and P are functions of x only, we have

$$a = g(x)\,a(y) + h(x)\,b(y),$$

where g and h are functions of x only, a and b are functions of y only. Then

$$a'' = y_1 a' + (y_1' - P)\,a,$$
$$b'' = y_1 b' + (y_1' - P)\,b,$$

and therefore

$$y_1 = \frac{a''b - ab''}{a'b - ab'} = \frac{\partial}{\partial y}\log(a'b - ab').$$

Hence

$$R = ay_1 - \frac{\partial a}{\partial y}$$
$$= a\frac{\partial}{\partial y}\left\{\log\left(\frac{a'b - ab'}{a}\right)\right\} = a\rho,$$

say; and therefore

$$u = \xi + a\eta' + R\eta$$
$$= \xi + a(\eta' + \rho\eta),$$
$$v = \eta'' + y_1\eta' + P\eta$$
$$= \frac{1}{a}\frac{\partial u}{\partial y}$$
$$= \frac{1}{a}\frac{\partial}{\partial y}\{a(\eta' + \rho\eta)\}.$$

Hence the differential equation

$$\frac{\partial^2 z}{\partial x \, \partial y} + \frac{\partial}{\partial x}(ae^z) = 0,$$

where

$$a = g(x)\, a(y) + h(x)\, b(y),$$

has its general integral given by

$$e^z = \frac{\dfrac{\partial}{\partial y}\{a(\eta' + \rho\eta)\}}{a\xi + a^2(\eta' + \rho\eta)},$$

where

$$\rho = \frac{\partial}{\partial y}\left\{\log\left(\frac{a'b - ab'}{a}\right)\right\}.$$

Note. In the particular case, when

$$b(y) = Aa(y),$$

where A is a pure constant, we have

$$y_1 = 0, \quad \rho = 0;$$

we can take $a(y) = 1$ without loss of generality, and then

$$e^z = \frac{a\eta''}{a\xi + a^2\eta'}, = \frac{\eta''}{\xi + a\eta'},$$

in effect, the result in Ex. 2, above.

Hence, *for the present case, where* $m = 0$ *and* $n = 2$, *we must have* $ab' - a'b$ *different from zero.*

Ex. 4. Consider the case provided by taking

$$m = 1, \quad n = 1;$$

the differential equation

$$\frac{\partial^2 z}{\partial x \, \partial y} + \frac{\partial}{\partial x}(ae^z) - \frac{\partial}{\partial y}(\beta e^{-z}) = 0$$

has

$$e^z = \frac{\eta' + f_2}{\xi' + f_1}$$

for its integral, where f_1 and f_2 involve only ξ, η, x, y. Also, if

$$u = \xi' + f_1, \quad v = \eta' + f_2,$$

we have

$$av = \frac{\partial u}{\partial y} = \frac{\partial f_1}{\partial \eta}\eta' + \frac{\partial f_1}{\partial y},$$

$$\beta u = \frac{\partial v}{\partial x} = \frac{\partial f_2}{\partial \xi}\xi' + \frac{\partial f_2}{\partial x};$$

and therefore

$$a = \frac{\partial f_1}{\partial \eta}, \quad \beta = \frac{\partial f_2}{\partial \xi}.$$

Consequently

$$f_1 = a\eta + g_1, \quad f_2 = \beta\xi + g_2,$$

where g_1 may involve ξ but not η, and g_2 may involve η but not ξ: also

$$\frac{\partial f_1}{\partial y}=af_2=a\beta\xi+ag_2,$$

$$\frac{\partial f_2}{\partial x}=\beta f_1=a\beta\eta+\beta g_1,$$

that is,

$$\eta\frac{\partial a}{\partial y}+\frac{\partial g_1}{\partial y}=a\beta\xi+ag_2,$$

$$\xi\frac{\partial \beta}{\partial x}+\frac{\partial g_2}{\partial x}=a\beta\eta+\beta g_1.$$

We therefore take

$$g_1=\theta_1\xi+\phi_1,\quad g_2=\theta_2\eta+\phi_2,$$

where ϕ_1 and ϕ_2 do not involve ξ or η. Substituting these values and equating coefficients, we find

$$\frac{\partial a}{\partial y}=a\theta_2,\quad \frac{\partial\theta_1}{\partial y}=a\beta,\quad \frac{\partial\phi_1}{\partial y}=a\phi_2,$$

$$\frac{\partial\beta}{\partial x}=\beta\theta_1,\quad \frac{\partial\theta_2}{\partial x}=a\beta,\quad \frac{\partial\phi_2}{\partial x}=\beta\phi_1;$$

and therefore

$$u=\xi'+f_1=\xi'+\frac{1}{\beta}\frac{\partial\beta}{\partial x}\xi+a\eta+\phi_1,$$

$$v=\eta'+f_2=\eta'+\beta\xi+\frac{1}{a}\frac{\partial a}{\partial y}\eta+\phi_2.$$

The two equations determining ϕ_1 and ϕ_2 are the same as those determining u and v: also, ϕ_1 and ϕ_2 do not involve ξ or η, so that, taking particular functions X of x and Y of y, we have

$$\phi_1=X'+\frac{1}{\beta}\frac{\partial\beta}{\partial x}X+aY,$$

$$\phi_2=Y'+\beta X+\frac{1}{a}\frac{\partial a}{\partial y}Y.$$

Obviously ϕ_1 and ϕ_2 can be absorbed into the other parts of u and v respectively by taking new arbitrary functions $\xi+X$, $\eta+Y$: hence, keeping ξ and η perfectly general, we have

$$\left.\begin{aligned}u&=\xi'+\frac{1}{\beta}\frac{\partial\beta}{\partial x}\xi+a\eta\\ v&=\eta'+\beta\xi+\frac{1}{a}\frac{\partial a}{\partial y}\eta\end{aligned}\right\}.$$

We still have to satisfy implicit limitations on a and β. Now

$$\theta_2=\frac{\partial\log a}{\partial y},\quad \theta_1=\frac{\partial\log\beta}{\partial x}:$$

and therefore

$$\frac{\partial^2\log a}{\partial x\,\partial y}=\frac{\partial\theta_2}{\partial x}=a\beta=\frac{\partial\theta_1}{\partial y}=\frac{\partial^2\log\beta}{\partial x\,\partial y}.$$

From the equality of the first and the last of these expressions, we have

$$\frac{a}{X_1}=\frac{\beta}{Y_1}=\theta,$$

say ; and then

$$\frac{\partial^2 \log \theta}{\partial x\, \partial y}=\theta^2 X_1 Y_1.$$

When we take

$$\theta^2=e^{\mu}, \quad \sqrt{2 X_1}\, dx=dx', \quad \sqrt{2 Y_1}\, dy=dy',$$

this equation becomes

$$\frac{\partial^2 \mu}{\partial x'\, \partial y'}=e^{\mu},$$

and therefore (§ 218, Ex. 1, *Note*)

$$e^{\mu}=2\, \frac{\phi'(x')\, \psi'(y')}{\{\phi(x')+\psi(y')\}^2}.$$

Let

$$\phi(x')=\rho(x), \quad \psi(y')=\sigma(y);$$

then

$$\phi'(x')\, X_1 \sqrt{2}=\rho'(x), \quad \psi'(y')\, Y_1 \sqrt{2}=\sigma'(y),$$

so that

$$\theta^2=e^{\mu}=\frac{1}{X_1 Y_1}\, \frac{\rho'(x)\, \sigma'(y)}{\{\rho(x)+\sigma(y)\}^2},$$

and therefore

$$a=\left(\frac{X_1}{Y_1}\right)^{\frac{1}{2}} \frac{\{\rho'(x)\, \sigma'(y)\}^{\frac{1}{2}}}{\rho(x)+\sigma(y)},$$

$$\beta=\left(\frac{Y_1}{X_1}\right)^{\frac{1}{2}} \frac{\{\rho'(x)\, \sigma'(y)\}^{\frac{1}{2}}}{\rho(x)+\sigma(y)}.$$

The quantities X_1 and Y_1 are at our disposal; we introduce new functions $h(x)$ and $k(y)$, such that

$$X_1=\rho'(x)\, h^2(x), \quad Y_1=\sigma'(y)\, k^2(y),$$

and then

$$a=\frac{\rho'(x)}{\rho(x)+\sigma(y)}\, \frac{h(x)}{k(y)},$$

$$\beta=\frac{\sigma'(y)}{\rho(x)+\sigma(y)}\, \frac{k(y)}{h(x)}.$$

With these values of a and β, the general integral of the differential equation

$$\frac{\partial^2 z}{\partial x\, \partial y}+\frac{\partial}{\partial x}(a e^{z})-\frac{\partial}{\partial y}(\beta e^{-z})=0$$

is given by

$$e^{z}=\frac{\eta'+\beta\xi+\dfrac{1}{a}\dfrac{\partial a}{\partial y}\eta}{\xi'+\dfrac{1}{\beta}\dfrac{\partial \beta}{\partial x}\xi+a\eta}.$$

The functions h, k, ρ, σ are at our disposal: special forms assumed for them will lead to special equations with corresponding integrals

Ex. 5. Integrate the equations :—

(i) $$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left(\frac{e^z}{x+y}\right)-\frac{\partial}{\partial y}\left(\frac{e^{-z}}{x+y}\right)=0;$$

(ii) $$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left\{e^z\left(\frac{1}{x+y}-\frac{1}{x}\right)\right\}-\frac{\partial}{\partial y}\left\{e^{-z}\left(\frac{1}{x+y}-\frac{1}{y}\right)\right\}=0;$$

verifying in each case that α and β conform to the general conditions.

Ex. 6. Shew that, if the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}(e^{\alpha+z})-\frac{\partial}{\partial y}(e^{\beta-z})=0$$

possesses a general integral of the form

$$e^z=\frac{\eta''+f_2}{\xi'+f_1},$$

where f_1 and f_2 do not involve η'' or ξ', then α and β must satisfy the equations

$$\frac{\partial^2\beta}{\partial x\,\partial y}=e^{\alpha+\beta},$$

$$\frac{\partial^3\alpha}{\partial x\,\partial y^2}-\frac{\partial^2\alpha}{\partial x\,\partial y}\left(2\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial y}\right)=3e^{\alpha+\beta}\frac{\partial\alpha}{\partial y};$$

and prove that

$$f_2=\frac{\partial\beta}{\partial y}\eta'+\left\{\frac{\partial^2\beta}{\partial y^2}+\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}-\frac{\partial^2\alpha}{\partial y^2}-\left(\frac{\partial\alpha}{\partial y}\right)^2\right\}\eta+\xi e^\beta,$$

$$f_1=\eta' e^\alpha+e^\alpha\left(\frac{\partial\beta}{\partial y}-\frac{\partial\alpha}{\partial y}\right)\eta+\xi\frac{\partial\beta}{\partial y}.$$

Ex. 7. Obtain the equations which must be satisfied by α and β, in order that the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial}{\partial x}(\alpha e^z)-\frac{\partial}{\partial y}(\beta e^{-z})=0$$

may possess a general integral of the form

$$e^z=\frac{\eta''+f_2}{\xi''+f_1},$$

where f_1 and f_2 do not involve ξ'' or η''.

SECOND COMBINATION OF COEFFICIENTS a AND b.

226. In the next place, we consider the alternative form of b that can be associated with the value of a given by

$$a=\mu+\lambda e^{z\rho}:$$

we have

$$b=\frac{1}{\rho}\frac{\partial\rho}{\partial x}.$$

On reference to the analysis in § 224, it appears that

$$\frac{\partial f_2}{\partial x} + \sum_{a=0}^{m-1} \frac{\partial f_2}{\partial \xi_a} \xi_{a+1} = 0,$$

that is,

$$\frac{df_2}{dx} = 0.$$

Hence f_2 is independent of x; and therefore, as it does not involve η_n, it involves only $\eta_{n-1}, \ldots, \eta_1, \eta, y$. The differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + (\mu + \lambda e^{\rho z}) \frac{\partial z}{\partial x} + \frac{1}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial z}{\partial y} + c = 0.$$

Let a new dependent variable Z be taken such that

$$Z = \rho z + \log \lambda,$$

so that

$$e^Z = \lambda e^{\rho z};$$

the differential equation becomes

$$\frac{\partial Z}{\partial x \partial y} + (e^Z + \phi) \frac{\partial Z}{\partial x} + \psi = 0,$$

where ϕ is a function involving x and y but not Z, and ψ is a function involving x, y, and Z, in general.

Repeating the application of the conditions, at present under consideration, to the equation in this form, we see that their effect will be guaranteed by taking $\rho = 1$, $\lambda = 1$, in the value of a. Thus

$$\frac{\partial f}{\partial \eta_n} = \frac{1}{\eta_n + f_2},$$

while we still have

$$\frac{df_2}{dx} = 0,$$

so that f_2 involves only $\eta_{n-1}, \ldots, \eta_1, \eta, y$; and therefore

$$f = \log (\eta_n + f_2) + f_1,$$

where f_1 does not involve η_n but may involve all the lower derivatives of η, and does involve all the derivatives of ξ of all orders up to ξ_m. Now, with the value $\lambda = 1$, we have

$$\begin{aligned} e^Z = e^z &= e^f \\ &= (\eta_n + f_2)\, e^{f_1}; \end{aligned}$$

and we had, in general,

$$\frac{\partial^2 f}{\partial y \partial \xi_m} + \sum_{\beta=0}^{n-1} \frac{\partial^2 f}{\partial \xi_m \partial \eta_\beta} \eta_{\beta+1} + a \frac{\partial f}{\partial \xi_m} = 0,$$

so that, as $a = e^Z + \phi$, we have

$$\frac{d}{dy}\left(\log \frac{\partial f_1}{\partial \xi_m}\right) + (\eta_n + f_2)\, e^{f_1} + \phi = 0:$$

an equation which also results from direct substitution, of the relation giving Z, in the differential equation satisfied by Z.

Again, we have (as before)

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} + \frac{\partial f}{\partial \xi_m} \frac{\partial f}{\partial \eta_n} \frac{\partial}{\partial Z}(e^Z + \phi) = 0,$$

that is,

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_{n-1}} + \frac{\partial f}{\partial \xi_m} e^Z \frac{\partial f}{\partial \eta_n} = 0:$$

also

$$e^Z \frac{\partial f}{\partial \eta_n} = e^f \frac{\partial f}{\partial \eta_n} = e^{f_1},$$

$$\frac{\partial f}{\partial \xi_m} = \frac{\partial f_1}{\partial \xi_m};$$

and therefore

$$\frac{\partial^2 f_1}{\partial \xi_m \partial \eta_{n-1}} + e^{f_1} \frac{\partial f_1}{\partial \xi_m} = 0.$$

Consequently

$$\frac{\partial f_1}{\partial \eta_{n-1}} + e^{f_1} = \text{quantity independent of } \xi_m$$

$$= \frac{\dfrac{\partial^2 f_4}{\partial \eta^2_{n-1}}}{\dfrac{\partial f_4}{\partial \eta_{n-1}}},$$

say, where f_4 is independent of ξ_m; and therefore, after a single integration,

$$e^{-f_1} \frac{\partial f_4}{\partial \eta_{n-1}} - f_4 = \text{quantity independent of } \eta_{n-1}$$

$$= f_3,$$

say, where f_3 involves no derivative of η of order higher than η_{n-2} and does involve ξ_m, while f_4 does involve η_{n-1}. Thus

$$e^{f_1} = \frac{1}{f_3 + f_4} \frac{\partial f_4}{\partial \eta_{n-1}}.$$

Further, the equation

$$\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right) + (\eta_n + f_2)\, e^{f_1} + \phi = 0$$

is satisfied identically, f_2 not involving η_n and being entirely independent of x, and ϕ being a function of x and y only; hence

$$\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} + (\eta_n + f_2)\, e^{f_1}\frac{\partial f_1}{\partial \xi_m} = 0,$$

and therefore

$$\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} - \left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right) + \phi\right\}\frac{\partial f_1}{\partial \xi_m} = 0,$$

that is,

$$\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} - \frac{d}{dy}\left(\frac{\partial f_1}{\partial \xi_m}\right) = \phi\,\frac{\partial f_1}{\partial \xi_m}.$$

Now

$$e^{f_1} = \frac{1}{f_3 + f_4}\,\frac{\partial f_4}{\partial \eta_{n-1}},$$

and f_4 is independent of ξ_m: hence

$$\frac{\partial f_1}{\partial \xi_m} = -\frac{1}{f_3 + f_4}\,\frac{\partial f_3}{\partial \xi_m}.$$

This equation does not involve η_n: but it can involve η_{n-1}, which (though not occurring in f_3) can be introduced by f_4.

Again,

$$\begin{aligned}\frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\log(f_3 + f_4)\right\} &= \frac{\partial}{\partial \xi_m}\left\{\frac{1}{f_3 + f_4}\left(\frac{df_3}{dy} + \frac{df_4}{dy}\right)\right\}\\ &= \frac{1}{f_3 + f_4}\,\frac{\partial}{\partial \xi_m}\left(\frac{df_3}{dy}\right) - \frac{1}{(f_3 + f_4)^2}\left(\frac{df_3}{dy} + \frac{df_4}{dy}\right)\frac{\partial f_3}{\partial \xi_m}\\ &= -\frac{d}{dy}\left(\frac{\partial f_1}{\partial \xi_m}\right);\end{aligned}$$

and consequently

$$\begin{aligned}\phi\,\frac{\partial f_1}{\partial \xi_m} &= \frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_1}{\partial \xi_m}\right)\right\} - \frac{d}{dy}\left(\frac{\partial f_1}{\partial \xi_m}\right)\\ &= \frac{\partial}{\partial \xi_m}\left\{\frac{d}{dy}\left(\log\frac{\partial f_3}{\partial \xi_m}\right)\right\}.\end{aligned}$$

Now f_3 involves no derivative of η higher than η_{n-2}, so that the right-hand side can, at the utmost, only be linear in η_{n-1}; also f_4

does involve η_{n-1} and $\dfrac{\partial f_3}{\partial \xi_m}$ is not zero, so that $\dfrac{\partial f_1}{\partial \xi_m}$ cannot be only linear in η_{n-1}. Hence ϕ, which is to involve none of the derivatives of the arbitrary quantities ξ and η, must be zero. The differential equation now becomes

$$\frac{\partial^2 Z}{\partial x \partial y} + e^Z \frac{\partial Z}{\partial x} + \psi = 0,$$

where ψ can involve x, y, Z, and where Z is given by

$$Z = f = \log(\eta_n + f_2) + f_1,$$

$$e^{f_1} = \frac{1}{f_3 + f_4} \frac{\partial f_4}{\partial \eta_{n-1}};$$

the quantity f_2 does not involve η_n and is a function of $\eta_{n-1}, \ldots, \eta, y$ only; the quantity f_3 does not involve η_n or η_{n-1}, but it does involve ξ_m in such a way as to have

$$\frac{d}{dy}\left(\log \frac{\partial f_3}{\partial \xi_m}\right)$$

independent of ξ_m; and the quantity f_4 does not involve ξ_m or η_n, but does involve η_{n-1}.

Let the value of Z be substituted in the equation: then as

$$\begin{aligned}\frac{\partial Z}{\partial y} &= \frac{d}{dy}\{\log(\eta_n + f_2)\} + \frac{df_1}{dy}\\ &= \frac{d}{dy}\{\log(\eta_n + f_2)\} + \frac{\partial f_1}{\partial \eta_{n-1}}\eta_n + \ldots,\end{aligned}$$

and therefore

$$\frac{\partial^2 Z}{\partial x \partial y} = \eta_n \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial \eta_{n-1}}\right) + \ldots,$$

$$\frac{\partial Z}{\partial x} = \frac{df_1}{dx},$$

we have

$$\psi + \eta_n \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial \eta_{n-1}}\right) + \ldots + (\eta_n + f_2) e^{f_1} \frac{df_1}{dx} = 0.$$

Thus ψ is of the form

$$\alpha\eta_n + \beta,$$

where α and β are independent of η_n; and ψ is known to be a function of x, y, and Z alone. Hence

$$\psi = me^Z + n,$$

where m and n are functions of x and y only; and therefore the equation in Z is

$$\frac{\partial^2 Z}{\partial x \partial y} + e^Z \frac{\partial Z}{\partial x} + me^Z + n = 0.$$

Let a new dependent variable u be introduced by the relation

$$\frac{\partial Z}{\partial x} + m = e^{-u},$$

so that the differential equation for Z then gives

$$e^Z = \frac{\partial u}{\partial y} - ke^u,$$

where

$$k = n - \frac{\partial m}{\partial y};$$

and therefore, on the elimination of Z, we have

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial}{\partial x}(ke^u) + (m - e^{-u})\frac{\partial u}{\partial y} - mke^u + k = 0.$$

When the equation in Z has an integral of the specified type, the preceding relations prescribe that type also for the value of u, and conversely.

If k is not zero, this equation is of the form

$$\frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c = 0,$$

where

$$a = -ke^u, \quad b = m - e^{-u};$$

that is, the equation belongs to the first alternative of the former type. In this case, $\rho = 1$; and therefore

$$m = \frac{1}{\rho}\frac{\partial \rho}{\partial x} = 0.$$

If k is zero, the equation for u is

$$\frac{\partial^2 u}{\partial x \partial y} + (m - e^{-u})\frac{\partial u}{\partial y} = 0,$$

or changing the variables from x and y to $-y$ and x, and changing the sign of u, we have

$$\frac{\partial^2 u}{\partial x \partial y} + (e^u - m)\frac{\partial u}{\partial x} = 0,$$

which is a particular case of

$$\frac{\partial^2 Z}{\partial x \partial y} + (e^Z + \phi)\frac{\partial Z}{\partial x} + \psi = 0.$$

For the latter, it was proved that ϕ could be made zero: hence

$$m = 0.$$

Thus, whether k is zero or is not zero, our equation for Z can be taken in the form

$$\frac{\partial^2 Z}{\partial x \partial y} + e^Z \frac{\partial Z}{\partial x} + n = 0,$$

where n is a function of x and y only, not involving z. Let a quantity θ be taken such that

$$n = \frac{\partial^2 \theta}{\partial x \partial y},$$

and take a new variable ζ such that

$$\zeta = Z + \theta.$$

Then

$$\frac{\partial^2 \zeta}{\partial x \partial y} = -e^Z \frac{\partial Z}{\partial x} = -\frac{\partial}{\partial x}(e^{\zeta - \theta}),$$

that is,

$$\frac{\partial^2 \zeta}{\partial x \partial y} + \frac{\partial}{\partial x}(Ae^{\zeta}) = 0,$$

where

$$A = e^{-\theta}.$$

We thus have a particular case of the former equation in § 225, now given by making $B = 0$ in that equation.

The general integral can be at once obtained in the form

$$e^{\zeta} = \frac{Y}{X + \int AY dy},$$

which involves partial quadratures; there must be limitations upon the form of A which allow this expression to take a finite explicit form free from partial quadratures. If we write the integral

$$e^{\zeta} = \frac{v}{u},$$

where

$$v = \eta^{(n)} + y_1\eta^{(n-1)} + y_2\eta^{(n-2)} + \dots + y_n\eta,$$
$$u = \xi_1 + P_1\eta^{(n-1)} + P_2\eta^{(n-2)} + \dots + P_n\eta,$$

we have

$$P_1 = A,$$
$$Av = \frac{\partial u}{\partial y},$$

where $y_1, y_2, \dots, y_n$ are functions of y only; and then

$$Ay_r = \frac{\partial P_r}{\partial y} + P_{r+1},$$

for $r = 1, \dots, n-1$, with

$$A = P_1, \quad Ay_n = \frac{\partial P_n}{\partial y}.$$

Thus A is easily seen to satisfy the equation

$$\frac{\partial^n A}{\partial y^n} - \frac{\partial^{n-1}(Ay_1)}{\partial y^{n-1}} + \frac{\partial^{n-2}(Ay_2)}{\partial y^{n-2}} - \dots + (-1)^n Ay_n = 0,$$

so that it is of the form

$$A = \xi_1 Y_1 + \dots + \xi_n Y_n,$$

where $Y_1, \dots, Y_n$ are linearly independent integrals of this ordinary equation of order n, and $\xi_1, \dots, \xi_n$ are functions of x at our disposal.

On the whole, it appears that the equation determined by the combination

$$a = \mu + \lambda e^{z\rho}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x},$$

is only a particular form of the equation determined by the combination

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y} + \lambda e^{z\rho}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-z\rho}.$$

Examples of the particular form have already been given.

Ex. Work out the detailed form of A, and of the integral

$$e^z = \frac{\eta''' + y_1\eta'' + y_2\eta' + y_3\eta}{\xi + A\eta'' + P\eta' + Q\eta},$$

of the differential equation

$$\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial}{\partial x}(Ae^z) = 0.$$

THIRD COMBINATION OF COEFFICIENTS a AND b.

227. In the next place, after having considered all the cases that arise when a has the form $\mu + \lambda e^{\rho z}$, we proceed to consider the cases that arise when

$$a = \mu + \lambda z.$$

As the equation is unaltered when x and y are interchanged as well as A and B, provided the sign of z is changed, it follows (from the knowledge of the forms of b which can be associated with $\mu + \lambda e^{\rho z}$ as the form of a) that b cannot have the form $\mu' + \lambda' e^{\rho' z}$; it can only be $\mu' + \lambda' z$ (which will be seen not to be possible) or σ, where μ', λ', σ are functions of x and y alone.

In the initial investigation, the equation

$$\frac{\partial a}{\partial z}\frac{\partial^2 f}{\partial \eta_n{}^2} + \frac{\partial^2 a}{\partial z^2}\left(\frac{\partial f}{\partial \eta_n}\right)^2 = 0$$

occurred, as holding in general: hence, for the present case, on the assumption that λ is different from zero, we have

$$\frac{\partial^2 f}{\partial \eta_n{}^2} = 0.$$

But $\dfrac{\partial f}{\partial \eta_n}$ cannot be zero, for z would then not involve η_n. Also, the equation

$$\frac{\partial^2 f}{\partial x \partial \eta_n} + \sum_{a=0}^{m-1} \frac{\partial^2 f}{\partial \xi_a \partial \eta_n}\xi_{a+1} + b\frac{\partial f}{\partial \eta_n} = 0$$

holds in general: and it is satisfied identically when substitution takes place for z. Taking derivatives with regard to η_n, and using the property that $\dfrac{\partial^2 f}{\partial \eta_n{}^2}$ vanishes for the present case, we have

$$\frac{\partial b}{\partial z}\left(\frac{\partial f}{\partial \eta_n}\right)^2 = 0.$$

Thus b does not explicitly involve z: consequently

$$b = \sigma,$$

where σ is a function of x and y only.

The differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + (\mu + \lambda z)\frac{\partial z}{\partial x} + \sigma\frac{\partial z}{\partial y} + c = 0.$$

Let

$$z = \theta Z + \phi,$$

choosing θ and ϕ to be functions of x and y such that

$$\frac{\partial \theta}{\partial x} + \sigma\theta = 0,$$

$$\frac{\partial \theta}{\partial y} + \mu\theta = -\lambda\theta\phi;$$

then the equation for Z is

$$\frac{\partial^2 Z}{\partial x \partial y} + \lambda\theta Z \frac{\partial Z}{\partial x} + \gamma = 0,$$

where $\lambda\theta$ is a function of x and y only, and γ involves x, y, Z explicitly. Thus the equation

$$\frac{\partial^2 z}{\partial x \partial y} + Mz\frac{\partial z}{\partial x} + c = 0$$

has an integral of the specified type, when M involves only x and y, and c involves x, y, z.

For this form of equation $b = 0$, and therefore

$$\frac{\partial^2 f}{\partial x \partial \eta_n} + \sum_{a=0}^{m-1} \frac{\partial^2 f}{\partial \xi_a \partial \eta_n} \xi_{a+1} = 0;$$

also

$$\frac{\partial^2 f}{\partial \xi_m \partial \eta_n} = 0$$

in general: hence, for the equation in question,

$$\frac{\partial}{\partial \eta_n}\left(\frac{df}{dx}\right) = 0,$$

that is, $\dfrac{df}{dx}$ does not contain η_n. Now, as $\dfrac{\partial^2 f}{\partial {\eta_n}^2}$ is zero, f (that is, z) is only linear in η_n; and therefore $Mz\dfrac{\partial z}{\partial x}$ is only linear in η_n. Also, as $\dfrac{\partial z}{\partial x}$ does not contain η_n, it follows that $\dfrac{\partial^2 z}{\partial x \partial y}$ can contain η_n only linearly at the utmost; so that, as the differential equation is to be satisfied identically when substitution is made for z, c can only be linear in η_n, that is, quà function of the variables x, y, z alone, c can only be linear in z. Let

$$c = \alpha z + \beta,$$

where α and β are functions of x and y only; the differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + Mz\frac{\partial z}{\partial x} + \alpha z + \beta = 0.$$

Now take a new dependent variable u, such that

$$u = \frac{\partial z}{\partial y} + \tfrac{1}{2}Mz^2;$$

forming the derivatives of u, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial^2 z}{\partial x \partial y} + Mz\frac{\partial z}{\partial x} + \tfrac{1}{2}z^2\frac{\partial M}{\partial x} \\ &= \tfrac{1}{2}z^2\frac{\partial M}{\partial x} - \alpha z - \beta,\end{aligned}$$

so that z and $\frac{\partial z}{\partial y}$ can be expressed in terms of u and $\frac{\partial u}{\partial x}$; and radicals involving $\frac{\partial u}{\partial x}$ occur unless $\frac{\partial M}{\partial x}$ vanishes. Again,

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial M}{\partial x}z - \alpha\right)\frac{\partial z}{\partial y} + \tfrac{1}{2}z^2\frac{\partial^2 M}{\partial x \partial y} - z\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial y},$$

the right-hand side of which, on substitution for z and $\frac{\partial z}{\partial y}$, becomes a function of u, $\frac{\partial u}{\partial x}$, and of x, y. Owing to the explicit value of u, which is

$$u = \frac{\partial z}{\partial y} + \tfrac{1}{2}Mz^2,$$

the form of u is of the same type as that of z: and it has just been seen that u satisfies an equation of the second order. But, at the earliest stage of the investigation, it was seen that such an equation must be of the form

$$\frac{\partial^2 u}{\partial x \partial y} + \rho\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c = 0,$$

where ρ, a, b, c are functions of x, y, u. The preceding equation is certainly not of this form, being irrational in $\frac{\partial u}{\partial x}$ unless $\frac{\partial M}{\partial x}$ vanishes: consequently

$$\frac{\partial M}{\partial x} = 0,$$

that is, M is a function of y only. Taking

$$Mdy = dy',$$

and dividing the equation in z by M, we effectively make $M=1$: and thus the equation is

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} + \alpha z + \beta = 0,$$

and the new variable u is

$$u = \frac{\partial z}{\partial y} + \tfrac{1}{2}z^2.$$

Proceeding again to the formation of the equation of the second order satisfied by u, we have

$$\frac{\partial u}{\partial x} = -\alpha z - \beta,$$

so that, unless α is zero, z and $\frac{\partial z}{\partial y}$ are expressible in terms of u and $\frac{\partial u}{\partial x}$. Also

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= -\alpha\frac{\partial z}{\partial y} - \frac{\partial \beta}{\partial y}\\ &= -\frac{\partial \beta}{\partial y} - \alpha\left(u - \tfrac{1}{2}z^2\right)\\ &= -\frac{\partial \beta}{\partial y} - \alpha u + \frac{1}{2\alpha}\left(\frac{\partial u}{\partial x} + \beta\right)^2,\end{aligned}$$

which, though a differential equation of the second order, is certainly not of the required form. Hence we must have $\alpha = 0$, which alone will prevent this forbidden form from occurring; and therefore the differential equation is

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} + \beta = 0,$$

where β is a function of x and y only.

228. We now proceed to prove that β must be zero. Let

$$e^{-v} = \frac{\partial z}{\partial x},$$

so that

$$z = \frac{\partial v}{\partial y} - \beta e^{v},$$

and therefore

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(\beta e^v) - e^{-v} = 0.$$

Obviously v is of the same type as z; and v is seen to satisfy a differential equation of the second order. This equation belongs to the form

$$s + ap + bq + c = 0,$$

where

$$a = -\beta e^v, \quad b = 0,$$

that is, a is of the form $\mu + \lambda e^{\rho z}$. We have seen that the type of equation is then

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(\alpha e^z) = 0,$$

to which the foregoing equation does not conform: hence, so long as β is different from zero, we have a form of a which leads to an equation that cannot arise in connection with such a form. It is therefore necessary that $\beta = 0$: and our equation is

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} = 0.$$

We can obtain the general integral. Integrating with regard to x, we have

$$\frac{\partial z}{\partial y} + \tfrac{1}{2}z^2 = \text{function of } y \text{ only.}$$

The integral of this is obviously of the form

$$z = \frac{v}{\phi(x) + u} + w,$$

where u, v, w are functions of y only: we find, on substituting, that

$$v = 2\frac{du}{dy}, \quad w = -\frac{\dfrac{d^2u}{dy^2}}{\dfrac{du}{dy}},$$

and then

$$-\{u, y\} = \text{above function of } y,$$

where $\{u, y\}$ is the Schwarzian derivative of u. As this function of y is arbitrary, we may take u as arbitrary, say $u = \psi(y)$; and then

$$z = \frac{2\psi'(y)}{\phi(x) + \psi(y)} - \frac{\psi''(y)}{\psi'(y)},$$

where ϕ and ψ are arbitrary functions.

FOURTH COMBINATION OF COEFFICIENTS a AND b.

229. Lastly, suppose that a has the third of its forms, so that we take

$$a = \mu.$$

Owing to the investigation in § 226, we may dispense with the consideration of the value $\mu' + \sigma e^{\rho z}$ of b; and owing to that in § 227, we may dispense with the consideration of the value $\mu' + \sigma z$ of b. There is thus only one form left: we take

$$b = \sigma,$$

that is, we may regard a and b in the equation

$$s + ap + bq + c = 0$$

as functions of x and y alone which do not involve z explicitly.

The two equations (§ 223) connecting a and b with the value of z are expressible in the form

$$\frac{d}{dy}\left(\log \frac{\partial f}{\partial \xi_m}\right) + a = 0, \quad \frac{d}{dx}\left(\log \frac{\partial f}{\partial \eta_n}\right) + b = 0.$$

From the former, we have

$$\begin{aligned}\log\left(\frac{\partial f}{\partial \xi_m}\right) &= -\int a\,dy + \text{a function of } x \text{ only}\\ &= \log u_1 + \text{a function of } x \text{ only,}\end{aligned}$$

say; and from the latter we have

$$\begin{aligned}\log\left(\frac{\partial f}{\partial \eta_n}\right) &= -\int b\,dx + \text{a function of } y \text{ only}\\ &= \log u_2 + \text{a function of } y \text{ only,}\end{aligned}$$

say: the quantities u_1 and u_2 being such that

$$\frac{\partial u_1}{\partial y} + au_1 = 0, \quad \frac{\partial u_2}{\partial x} + bu_2 = 0.$$

Hence we may take

$$\begin{aligned}z &= u_1 f_1(x, \xi, \xi_1, \ldots, \xi_m) + u_2 f_2(y, \eta, \eta_1, \ldots, \eta_n) + f_3\\ &= u_1 f_1 + u_2 f_2 + f_3,\end{aligned}$$

where ξ_m occurs only in f_1, and η_n occurs only in f_2, while f_3 does not contain either ξ_m or η_n. Substituting this value of z in the

differential equation, and taking account of the relations which define u_1 and u_2, we find

$$-u_1 f_1\left(\frac{\partial a}{\partial x}+ab\right)-u_2 f_2\left(\frac{\partial b}{\partial y}+ab\right)+\frac{\partial^2 f_3}{\partial x\partial y}+a\frac{\partial f_3}{\partial x}+b\frac{\partial f_3}{\partial y}+c=0:$$

and this relation must be satisfied identically. It is to be noted that u_1 is not zero, for otherwise z would not involve ξ_m; and similarly u_2 is not zero, for otherwise z would not involve η_n; also ξ_m does not occur in f_2 or in any of its derivatives, while η_n does not occur in f_1 or in any of its derivatives. But ξ_m and η_n do occur in c, after substitution of the value of z, unless c is free from z; and they occur, in lineo-linear fashion, in the combination of the derivatives of f_3, their form being

$$\frac{\partial^2 f_3}{\partial\xi_{m-1}\partial\eta_{n-1}}\xi_m\eta_n+a\frac{\partial f_3}{\partial\xi_{m-1}}\xi_m+b\frac{\partial f_3}{\partial\eta_{n-1}}\eta_n.$$

It therefore follows that, when the preceding relation is differentiated with regard to ξ_m twice, and with regard to η_n twice, derivatives of f_3 will not occur: the results, on dropping the non-vanishing factors u_1 and u_2 respectively, are

$$\left(\frac{\partial c}{\partial z}-\frac{\partial a}{\partial x}-ab\right)\frac{\partial^2 f_1}{\partial\xi_m{}^2}+u_1\frac{\partial^2 c}{\partial z^2}\left(\frac{\partial f_1}{\partial\xi_m}\right)^2=0,$$

$$\left(\frac{\partial c}{\partial z}-\frac{\partial b}{\partial y}-ab\right)\frac{\partial^2 f_2}{\partial\eta_n{}^2}+u_2\frac{\partial^2 c}{\partial z^2}\left(\frac{\partial f_2}{\partial\eta_n}\right)^2=0.$$

Assuming in the first place that $\dfrac{\partial^2 c}{\partial z^2}$ is not zero (the alternative assumption will come later), we have

$$\frac{1}{\dfrac{\partial c}{\partial z}-\dfrac{\partial a}{\partial x}-ab}\frac{\partial^2 c}{\partial z^2}=-\frac{1}{u_1}\left(\frac{\partial f_1}{\partial\xi_m}\right)^{-2}\frac{\partial^2 f_1}{\partial\xi_m{}^2},$$

$$\frac{1}{\dfrac{\partial c}{\partial z}-\dfrac{\partial b}{\partial y}-ab}\frac{\partial^2 c}{\partial z^2}=-\frac{1}{u_2}\left(\frac{\partial f_2}{\partial\eta_n}\right)^{-2}\frac{\partial^2 f_2}{\partial\eta_n{}^2}.$$

The quantities on the right-hand sides are independent of η_n and of ξ_m respectively: hence neither of them, when expressed in terms of the variables alone, can involve z; and therefore they are functions of x and y alone.

If the values of the two fractions on the left-hand sides, thus expressed in terms of x and y alone, are unequal, we take the quotients of the respective members of the two relations. The result is to give $\frac{\partial c}{\partial z}$ as a function of x and y alone: consequently c is a linear function of z, a contingency provisionally excluded. Accordingly, we may assume that the fractions are equal to one another: so that, as $\frac{\partial^2 c}{\partial z^2}$ is supposed not to vanish, we have

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}.$$

There thus exists a quantity θ, which is a function of x and y, such that

$$a = \frac{1}{\theta}\frac{\partial \theta}{\partial y}, \quad b = \frac{1}{\theta}\frac{\partial \theta}{\partial x};$$

and then, when we take a new dependent variable Z defined by the relation

$$Z = z\theta,$$

our differential equation becomes

$$\begin{aligned}\frac{\partial^2 Z}{\partial x \partial y} &= -c\theta + Z\frac{1}{\theta}\frac{\partial^2 \theta}{\partial x \partial y}\\ &= -C,\end{aligned}$$

where C is a function of x, y, and Z. Hence, in the present case, we may take our equation in the form

$$\frac{\partial^2 z}{\partial x \partial y} + c = 0,$$

where c is a function of x, y, and z.

But now as a is zero, u_1 is a function of x only, so that the quantity

$$\frac{1}{u_1}\left(\frac{\partial f_1}{\partial \xi_m}\right)^{-2}\frac{\partial^2 f_1}{\partial \xi_m^2}$$

is a function of x alone; also, as b is zero, u_2 is a function of y only, the quantity

$$\frac{1}{u_2}\left(\frac{\partial f_2}{\partial \eta_n}\right)^{-2}\frac{\partial^2 f_2}{\partial \eta_n^2}$$

is a function of y alone. On the present assumption, the two quantities are equal; consequently, they are equal to one and the

same constant, say $-k$, so that (as a and b now are zero) we have

$$\frac{\partial^2 c}{\partial z^2} = k\frac{\partial c}{\partial z},$$

and therefore

$$kc = e^{k(z+\mu)} + \rho,$$

where μ and ρ are functions of x and y only. Taking

$$v = k(z+\mu),$$

the differential equation for v is

$$\frac{\partial^2 v}{\partial x \partial y} + e^v + \rho = 0.$$

Let

$$\frac{\partial v}{\partial y} = -V:$$

then

$$\frac{\partial V}{\partial x} = e^v + \rho,$$

and therefore

$$\frac{\partial^2 V}{\partial x \partial y} + V\frac{\partial V}{\partial x} + \rho V - \frac{\partial \rho}{\partial y} = 0.$$

Now

$$V = -k\frac{\partial(z+\mu)}{\partial y},$$

and therefore V is of the same type as z: moreover, it satisfies an equation of the second order. Comparing the form of this equation with the admissible forms, we have

$$\rho = 0,$$

and so the differential equation is expressible in the form

$$\frac{\partial^2 v}{\partial x \partial y} + e^v = 0.$$

This is Liouville's equation: its general integral (§ 218, Ex. 1, *Note*) is

$$e^v = -2\frac{\xi'\eta'}{(\xi+\eta)^2},$$

where ξ and η are arbitrary functions of x and y respectively.

Ex. Obtain the primitive of the equation

$$r + t = e^{az},$$

where a is a constant, in the form

$$e^{az} = \frac{2}{au^2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\},$$

the quantity u denoting

$$f(x+iy)+g(x-iy),$$

and f and g being arbitrary functions.

230. Lastly, we have to consider the omitted case set on one side in the preceding assumption, viz. when c is a linear function of z: let

$$c = \lambda z + \mu.$$

Then the differential equation is

$$s + ap + bq + \lambda z + \mu = 0.$$

If ζ is any particular integral, then, taking a new dependent variable Z such that

$$Z = z - \zeta,$$

we have

$$S + aP + bQ + \lambda Z = 0,$$

where a, b, λ are functions of x and y alone. This equation coincides, in form, with Laplace's linear equation which has already been discussed.

SUMMARY OF RESULTS.

231. The results of the investigation can be summarised as follows.

I. When an integral relation is given in a form

$$z = f(x, y, \xi, \xi_1, \ldots, \xi_m, \eta, \eta_1, \ldots, \eta_n),$$

where ξ is an arbitrary function of x and η is an arbitrary function of y, and when it satisfies an equation of the second order, then, either directly or after a transformation of the dependent variable which does not affect the specific character of the integral relation, the equation of the second order can be made to acquire the form

$$s + ap + bq + c = 0,$$

where a, b, c are functions of x, y, z alone. The functions a and b can have any one of three possible forms, viz.

$$\mu + \lambda e^{\rho z}, \quad \mu + \lambda z, \quad \mu,$$

where μ, λ, ρ are functions of x and y only.

II. When both a and b are of the form $\mu + \lambda e^{\rho z}$, then

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y} + \lambda e^{\rho z}, \quad b = \frac{1}{\rho}\frac{\partial \rho}{\partial x} + \sigma e^{-\rho z};$$

by transformation of the dependent variable which does not affect the character of the integral, the equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(Ae^z) - \frac{\partial}{\partial y}(Be^{-z}) = 0,$$

and its general integral is

$$e^z = \frac{\eta_n + f_2}{\xi_m + f_1} = \frac{v}{u},$$

where f_1 and f_2 do not involve ξ_m or η_n, and A and B are functions of x and y only such that

$$Av = \frac{\partial u}{\partial y}, \quad Bu = \frac{\partial v}{\partial x}.$$

III. If a is of the form $\mu + \lambda e^{\rho z}$ and if b is not of this form, then

$$b = \frac{1}{\rho}\frac{\partial \rho}{\partial x};$$

while, if b is of the form $\mu + \lambda e^{\rho z}$ and if a is not of this form, then

$$a = \frac{1}{\rho}\frac{\partial \rho}{\partial y}.$$

In the former case, the equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial}{\partial x}(Ae^z) = 0,$$

having an integral

$$e^z = \frac{v}{u},$$

where

$$v = \eta^{(n)} + y_1 \eta^{(n-1)} + \dots + y_n \eta,$$
$$u = \xi + A\eta^{(n-1)} + P_2 \eta^{(n-2)} + \dots + P_n \eta,$$

and

$$\frac{\partial u}{\partial y} = Av;$$

in the latter case, the equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial}{\partial y}(Be^{-z}) = 0,$$

having an integral of corresponding form.

IV. If one of the two quantities a and b has the form $\mu+\lambda z$, the other has the form σ, where μ, λ, σ are functions of x and y alone. Then, by transformations of the variables which do not affect the character of the integral relation, the differential equation can be made to acquire the form

$$\frac{\partial^2 z}{\partial x \partial y} + z\frac{\partial z}{\partial x} = 0,$$

and the value of z is

$$z = \frac{2\eta'}{\xi+\eta} - \frac{\eta''}{\eta'}.$$

V. When a and b are functions of x and y alone, either the equation can be changed to Liouville's form

$$\frac{\partial^2 z}{\partial x \partial y} + e^z = 0,$$

and then

$$e^z = -\frac{2\xi'\eta'}{(\xi+\eta)^2};$$

or it can be changed so as to acquire the form of Laplace's linear equation

$$s + ap + bq + cz = 0,$$

where a, b, c are functions of x and y alone.

The two forms in (III) are obviously derivable from one another, by changing the sign of z and interchanging the variables x and y. All equations of the second order having their general integral of the specified type can, by transformations of the variables which do not affect the character of the integral, be expressed in one or other of the foregoing forms.

Ex. 1. Integrate the equations :—

(i) $s+\dfrac{e^z}{(x+y)^2}p+e^{-z}q=\dfrac{2e^z}{(x+y)^3}-\dfrac{1}{(x+y)^2}$;

(ii) $s+e^zp=0$;

(iii) $s+e^zp+\dfrac{1}{(x+y)^2}=0$;

(iv) $s+e^zp+\dfrac{n}{(x+y)^2}=0$,

n being a positive integer. (Tanner.)

Ex. 2. Shew that the integral of the equation

$$s^2 = 4pq\lambda,$$

when λ is a function of x and y, is given by

$$z = \int \left\{ u^2\, dx + \frac{1}{\lambda}\left(\frac{\partial u}{\partial y}\right)^2 dy \right\},$$

where u is an integral of the equation

$$\frac{\partial^2 u}{\partial x\, \partial y} - \frac{1}{2\lambda}\frac{\partial \lambda}{\partial x}\frac{\partial u}{\partial y} - \lambda u = 0.$$

Apply this property to integrate the equations

$$s^2 = \frac{4pq}{(x+y)^2},$$

$$s^2 = \frac{16pq}{(x+y)^2}.$$ (Goursat.)

Ex. 3. Obtain the general integral of the equation

$$sz = \{(1+p^2)(1+q^2)\}^{\frac{1}{2}}$$

in the form

$$z^2 = \left\{\int X\, dx - \int \frac{dy}{Y}\right\}\left\{\int Y\, dy - \int \frac{dx}{X}\right\},$$

where X and Y are arbitrary functions of x and y respectively. (Goursat.)

Ex. 4. Integrate the equations:—

(i) $s \sin z = \{(1+p^2)(1+q^2)\}^{\frac{1}{2}}$;

(ii) $sz + \phi(x, p)\,\psi(y, q) = 0$,

where, in the latter equation, ϕ and ψ satisfy the conditions

$$\frac{\partial \phi}{\partial p} = \frac{p}{\phi} + a, \quad \frac{\partial \psi}{\partial q} = \frac{q}{\psi} + a,$$

and a is a constant. (Goursat.)

CHAPTER XVI.

Equations of the Second Order in Two Independent Variables, having an Intermediate Integral.

The present chapter is devoted mainly to the consideration of equations of the second order which are compatible with an equation of the first order or which (to use the customary phrase) possess an intermediate integral. A brief outline of the methods of Monge and of Boole is prefixed, those methods depending, for their proof, upon the assumption that an intermediate integral of a specified type does exist. Later, a method is given which does not depend upon that assumption and which leads to an intermediate integral, if it exists: in the process, the conditions under which the preceding integral exists are obtained.

In the preparation of the chapter, I have frequently used Boole's *Supplementary Volume*, quoted in § 236, and Imschenetsky's valuable memoir, quoted in § 180; other references are given in their appropriate connections. Some historical notes are given in Chapter III of Imschenetsky's memoir.

232. When we pass from strictly linear equations of the second order that are amenable to the Laplace process as developed by Darboux, and from the wider range of equations of the second order the general integral of which is expressible in finite and explicit form, the processes of integration that prove practicable are somewhat limited in range unless the results are allowed to be of a form that is not finite in expression. We always have the possibility of the application of Cauchy's theorem; the expression takes the form of a series which is at least singly infinite and may become doubly infinite, and there is no obvious mode of obtaining the integral in more compact form, even if there were certain knowledge that such compact form exists.

Other methods of proceeding to an integral have therefore to be devised. Among these, two are of prime importance—the methods devised by Ampère and by Darboux respectively: but

even these are not of universal application, and they lead to results that are completely definite only when the equations to which they are applied are characterised by more or less well-defined properties. It is therefore worth while considering equations of classes which, though undoubtedly specialised, do possess something of a comprehensive character. Limiting ourselves still to equations of the second order in a single dependent variable and two independent variables, we shall here consider equations of the form

$$F(x, y, z, p, q, r, s, t) = 0,$$

which, in some form or other, possess integrals that are amenable to some finite processes of integration.

Among such equations, one of the most important classes (judged either from the historical development of the subject or from their occurrence in applications to subjects such as geometry or physics) is that class usually associated with the name of Monge*. The implicitly assumed property of these equations of the second order is that they possess an intermediate integral which involves derivatives of the first order and contains an arbitrary function in its expression: and the equation of the second order is assumed to be the unique equivalent, in that order, of the intermediate integral. When u and v denote two functions of x, y, z, p, q, which are distinct from one another, an intermediate integral of this type is represented by

$$f(u, v) = 0,$$

where f is any arbitrary function: then, as is well known†, the equation of the second order, which is the equivalent of this equation of the first order on the elimination of the arbitrary function, is

$$Rr + 2Ss + Tt + U(rt - s^2) = V,$$

where R, S, T, U, V are definite functions of x, y, z, p, q.

But the general converse is not valid: that is to say, if an equation of this form is propounded in which R, S, T, U, V are

* *Hist. de l'Acad. des Sciences*, 1784, pp. 118—192.

† In the following discussion, and for the sake of brevity, the customary method due to Monge will be assumed as belonging to the elements of the subject: it is expounded in the author's *Treatise on Differential Equations*, (third edition, 1903), §§ 229—241. Monge discussed only equations for which $U=0$: his method is applicable to equations without this restriction.

definite functions of x, y, z, p, q, there does not necessarily (and there certainly does not unconditionally) exist an intermediate integral of an equation of the first order equivalent to the propounded equation. In point of fact, the four quantities given by the ratios of R, S, T, U, V to one another are functions of the derivatives of u and v, when the equation is derived from the intermediate integral; and therefore, if the process is to be regarded as reversible, these four quantities must be expressible in terms of the derivatives of the two functions u and v. We should therefore expect that at least two conditions would be satisfied by the four quantities in question.

Assuming for the moment that the necessary conditions (whatever their number) are satisfied, so that the intermediate integral exists, there are various ways of proceeding to the construction of that intermediate integral.

One of these ways is Monge's method: it is actually comprised in Ampère's general method for the integration of partial equations which is applicable even when no intermediate integral exists. To give effect to the method, it is necessary to construct integrable combinations of certain ordinary equations which are homogeneous and linear in differential elements of the variables x, y, z, p, q.

Another method is that which customarily is associated with the name of Boole, though in effect it was given (at least partially) in earlier memoirs by De Morgan and Bour: it is actually comprised in Darboux's general method for the integration of partial equations which is applicable even when no intermediate integral exists. To give effect to the method, it is necessary to obtain the most general integral of a number of homogeneous linear partial differential equations of the first order which constitute a complete Jacobian system.

As is usual in such cases*, the conditions that the equations in the differential elements shall possess a number of integrable combinations are the same as the conditions that the simultaneous partial equations of the first order shall possess the same number of algebraically distinct integrals. The two methods, in so far as they are applicable to the equation in question, repose upon the

* See Part I of this work, §§ 26, 38.

same assumptions as to the fulfilment of implicit conditions: they are, in effect, equivalent to one another and are (so far as concerns the equation) merely different modes of arranging the analysis that contributes to the integration.

MONGE'S METHOD FOR THE EQUATION $Rr+2Ss+Tt+U(rt-s^2)=V$.

233. Monge's method is as follows, in outline*. Let the equation be

$$Rr + 2Ss + Tt + U(rt - s^2) = V.$$

Two forms arise according as U is not zero, or is zero. We shall deal first with the form when U is not zero: we then divide the equation throughout by U, so that, without loss of generality in this case, we may take U as equal to unity, and the equation is

$$rt - s^2 + Rr + 2Ss + Tt = V.$$

The equations

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

are used to eliminate two of the three derivatives of the second order from the equation: when r and t are eliminated, the result is

$$\begin{aligned} dp\,dq + R\,dp\,dy + T\,dq\,dx - V\,dx\,dy \\ = s\,(R\,dy\,dy - 2S\,dx\,dy + T\,dx\,dx + dp\,dx + dq\,dy), \end{aligned}$$

and the equation will be satisfied if the equations

$$\begin{aligned} A &= \qquad dp\,dq + R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0, \\ B &= dp\,dx + dq\,dy + R\,(dy)^2 - 2S\,dx\,dy + T\,(dx)^2 = 0, \end{aligned}$$

are satisfied.

If r and s be eliminated, the result can be expressed in the form

$$tB + \frac{A\,dx - B\,dq}{dy} = 0:$$

while, if s and t be eliminated, the result can be expressed in the form

$$rB + \frac{A\,dy - B\,dp}{dx} = 0:$$

in each instance, it is sufficient to take $A = 0$, $B = 0$.

* The establishment of the various propositions, on the assumption that the necessary conditions are satisfied, is made in the various sections of the work quoted on p. 200, *note*.

Accordingly, the equations

$$\left.\begin{aligned} A &= dp\,dq + R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \\ B &= dp\,dx + dq\,dy + R\,(dy)^2 - 2S\,dx\,dy + T\,(dx)^2 = 0 \\ C &= dz - p\,dx - q\,dy = 0 \end{aligned}\right\}$$

are taken as a simultaneous set. Let

$$u = a, \quad v = b,$$

be two integrals of this set, the quantities a and b being constants: then it is proved* that the relation

$$u = f(v),$$

where f is an arbitrary function to be eliminated, leads to the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

the proper relations between R, S, T, V being satisfied. As either u or v or both u and v will involve the derivatives of the first order, this equation

$$u = f(v)$$

is an intermediate integral. The method thus depends, for its effectiveness, upon the construction of the quantities u and v.

Now the equations $A = 0$ and $B = 0$ give

$$(dp + T\,dx)(dq + R\,dy) = (RT + V)\,dx\,dy,$$

$$(dp + T\,dx)\,dx + (dq + R\,dy)\,dy = 2S\,dx\,dy,$$

and therefore

$$dp + T\,dx + m\,dy = 0,$$

$$dq + R\,dy + n\,dx = 0,$$

where

$$mn = RT + V, \quad m + n = -2S.$$

Hence m and n are the roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0.$$

Two cases arise.

When the quadratic has equal roots, so that the condition

$$S^2 = RT + V$$

* *L.c.*, § 232.

is satisfied, then $m = n = -S$; the set of equations in the differential elements can be uniquely represented by

$$\left.\begin{aligned} dp + Tdx - Sdy &= 0 \\ dq \; - Sdx + Rdy &= 0 \\ dz \; - pdx - \; qdy &= 0 \end{aligned}\right\}.$$

The conditions for the existence of an intermediate integral being supposed to be satisfied, it will be obtained in a form

$$u = f(v),$$

where $u = a$, $v = b$, are integral equations of the differential relations linear in the differential elements. It may be noticed that there are three differential relations and that therefore, if all the appropriate conditions are satisfied, there could be three integral relations

$$u = a, \quad v = b, \quad w = c,$$

equivalent to them, where a, b, c are constants. We shall return later to the consideration of this last possibility: meanwhile, an intermediate integral is obtainable on the supposition that the general conditions are satisfied.

When the quadratic has unequal roots, let

$$S^2 - RT - V = \theta^2,$$

where θ is not zero. Then m is not equal to n: we have

$$m, n = -S \pm \theta;$$

and therefore, taking

$$\rho = -S + \theta, \quad \sigma = -S - \theta,$$

the equations $A = 0$, $B = 0$ give either

$$\begin{aligned} dp + Tdx + \rho dy &= 0, \\ dq + Rdy + \sigma dx &= 0; \end{aligned}$$

or

$$\begin{aligned} dp + Tdx + \sigma dy &= 0, \\ dq + Rdy + \rho dx &= 0; \end{aligned}$$

and the equations $A = 0$, $B = 0$, may give both of these systems, though this is not necessarily a fact. Thus the set of equations

in the differential elements can be replaced by one or other of the systems

$$\left.\begin{aligned} dp + Tdx + \rho dy &= 0 \\ dq + \sigma dx + Rdy &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dp + Tdx + \sigma dy &= 0 \\ dq + \rho dx + Rdy &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\},$$

where

$$\rho = -S + (S^2 - RT - V)^{\frac{1}{2}} = -S + \theta,$$

$$\sigma = -S - (S^2 - RT - V)^{\frac{1}{2}} = -S - \theta;$$

and so far, there is nothing to exclude the possibility of both systems (under proper conditions) being admissible. The original equations in the differential elements possessed integrals which because of the conditions that were satisfied, led to an intermediate integral; consequently, one or other of the two systems, linear in the differential elements and replacing the original equations, must possess these integrals. Let them belong to the first set in a form

$$u_1 = a, \quad v_1 = b,$$

so that the conditions are satisfied for the first set; the intermediate integral is

$$u_1 = f(v_1),$$

where f is an arbitrary function.

It may happen that the conditions are satisfied for both sets, and that the second set possess integrals in a form

$$u_2 = a', \quad v_2 = b',$$

where a' and b' are constants; an intermediate integral is

$$u_2 = g(v_2),$$

where g is an arbitrary function. In these circumstances, there are two distinct intermediate integrals; it is part of the theory, and it is proved*, that these two distinct intermediate integrals coexist, so that they can be used as simultaneous equations to express p and q in terms of x, y, z, the values of p and q given by them being such as to render

$$dz = pdx + qdy$$

* *L.c.*, § 236; also see hereafter, § 239.

an exact equation, quadrature of which gives a primitive of the original equation. More generally, however, if the conditions are satisfied, they are satisfied for only one of the two linear sets: and then there is only one intermediate integral.

234. The corresponding equations for the case when U vanishes, so that the equation has the form

$$Rr + 2Ss + Tt = V,$$

can be stated similarly: the assumption being made that the equation* possesses an intermediate integral. The equation $A = 0$ is now replaced by

$$A' = Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0,$$

and the equation $B = 0$ is now replaced by

$$B' = R\,(dy)^2 - 2S\,dx\,dy + T\,(dx)^2 = 0.$$

Let ρ_1 and σ_1 be the roots of

$$R\mu^2 - 2S\mu + T = 0,$$

so that

$$R\rho_1 = S + \alpha, \quad R\sigma_1 = S - \alpha,$$

where

$$\alpha^2 = S^2 - RT.$$

Then, if the quadratic has equal roots, so that $\alpha = 0$, the equations

$$A' = 0, \quad B' = 0, \quad dz - p\,dx - q\,dy = 0,$$

can uniquely be replaced by the system

$$\left.\begin{aligned} R\,dy - S\,dx &= 0 \\ Rdp + Sdq - V\,dx &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\};$$

if $u = a$, $v = b$, where a and b are constants, be integrals of this linear system, then

$$u = f(v),$$

where f is an arbitrary function, is the single intermediate integral that can be obtained in this way.

* If R, S, T involve only x and y, and if V is homogeneous and linear in p, q, z, having functions of x and y for coefficients, the equation belongs to the linear form already discussed in Chapter XIII. In general, however, even for integrable equations of the type now under consideration, these limitations are not observed.

If the quadratic has unequal roots, it can be replaced by one or other of the two systems

$$\left.\begin{aligned} R\rho_1 dp + T dq - V\rho_1 dx &= 0 \\ dy - \rho_1 dx &= 0 \\ dz - (p + \rho_1 q)\, dx &= 0 \end{aligned}\right\},\qquad \left.\begin{aligned} R\sigma_1 dp + T dq - V\sigma_1 dx &= 0 \\ dy - \sigma_1 dx &= 0 \\ dz - (p + \sigma_1 q)\, dx &= 0 \end{aligned}\right\}:$$

or what is the equivalent, by one or other of the systems

$$\left.\begin{aligned} dp + \sigma_1 dq - \frac{V}{R}\, dx &= 0 \\ dy - \rho_1 dx &= 0 \\ dz - (p + \rho_1 q)\, dx &= 0 \end{aligned}\right\},\qquad \left.\begin{aligned} dp + \rho_1 dq - \frac{V}{R}\, dx &= 0 \\ dy - \sigma_1 dx &= 0 \\ dz - (p + \sigma_1 q)\, dx &= 0 \end{aligned}\right\}.$$

The equation is supposed to possess an intermediate integral, so that the necessary conditions are satisfied; they are therefore satisfied in connection with one or other of the systems, say, with the first. If integrals of that first system are obtained in a form

$$u_1 = a, \quad v_1 = b,$$

where a and b are constants, an intermediate integral is given by the equation

$$u_1 = f(v_1),$$

where f is an arbitrary function.

It may happen that the conditions are satisfied also for the other linear system, so that it possesses integrals of the form

$$u_2 = a', \quad v_2 = b',$$

where a' and b' are constants: then an intermediate integral is given by the equation

$$u_2 = g(v_2),$$

where g is an arbitrary function. As before, these two distinct intermediate integrals coexist: when they are resolved, so as to express p and q in terms of x, y, z, the values of p and q so provided make

$$dz - p\,dx - q\,dy = 0$$

an exact equation, quadrature of which gives a primitive of the original equation. More generally, however, when the conditions are satisfied, they are satisfied in connection with only one of the

two linear sets: and then only one intermediate integral can be obtained.

The form adopted for the linear system implies that R is not zero: the appropriate modifications when $R=0$ can easily be made.

235. To complete the integration, whether the original equation be

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

or be

$$Rr + 2Ss + Tt = V,$$

we integrate the intermediate integral, if only one has been obtainable, by the methods which apply to equations of the first order; according to the form of the intermediate integral, we may have one or more forms for the final primitive. We have seen that, when two intermediate integrals have been obtained, the final primitive is obtained by resolving the two equations for p and q and effecting a quadrature.

Such, in brief outline, is Monge's method of integrating the equations. It is effective only if the appropriate conditions are satisfied; and the explicit expression of these conditions must be obtained. We shall first, however, in similar brevity, give an outline of Boole's method of integrating the equations.

Boole's Method for the Equations.

236. Boole's method*, like Monge's, is based upon an assumption that an intermediate integral of the form

$$u = f(v),$$

where f is an arbitrary function, and u, v are definite functions of x, y, z, p, q, is possessed by the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

or by the equation

$$Rr + 2Ss + Tt = V,$$

* It is contained in chapters XXVIII and XXIX (in the latter more particularly) of the *Supplementary Volume* of his *Treatise on Differential Equations*: this volume was published in 1865, the year after his death. See also the memoir, *Crelle*, t. LXI (1863), pp. 309—333.

in the respective instances subjected to Monge's method. The difference from Monge's method lies in the fact that Boole obtains u and v as integrals of simultaneous partial equations of the first order, whereas Monge obtains them as integrals of equations in differential elements. Boole's procedure is as follows.

Denoting by $\frac{du}{dx}, \frac{du}{dy}, \frac{dv}{dx}, \frac{dv}{dy}$ the complete derivatives of u and v with regard to x and y, so that

$$\frac{d\theta}{dx} = \frac{\partial\theta}{\partial x} + p\frac{\partial\theta}{\partial z}, \quad \frac{d\theta}{dy} = \frac{\partial\theta}{\partial y} + q\frac{\partial\theta}{\partial z},$$

for any quantity θ, the assumed intermediate integral leads to the relations

$$\frac{du}{dx} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} = \left(\frac{dv}{dx} + r\frac{\partial v}{\partial p} + s\frac{\partial v}{\partial q}\right) f'(v),$$

$$\frac{du}{dy} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} = \left(\frac{dv}{dy} + s\frac{\partial v}{\partial p} + t\frac{\partial v}{\partial q}\right) f'(v).$$

When $f'(v)$ is eliminated between these two relations, and the resulting equation is arranged with reference to the combinations of r, s, t, we have

$$U_1(rt - s^2) + R_1 r + S_1 s + T_1 t = V_1,$$

where

$$R_1 = \frac{\partial u}{\partial p}\frac{dv}{dy} - \frac{\partial v}{\partial p}\frac{du}{dy},$$

$$S_1 = \frac{\partial u}{\partial q}\frac{dv}{dy} - \frac{\partial v}{\partial q}\frac{du}{dy} + \frac{\partial v}{\partial p}\frac{du}{dx} - \frac{\partial u}{\partial p}\frac{dv}{dx},$$

$$T_1 = \frac{\partial v}{\partial q}\frac{du}{dx} - \frac{\partial u}{\partial q}\frac{dv}{dx},$$

$$U_1 = \frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p},$$

$$V_1 = \frac{du}{dy}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dy}.$$

If this derived equation is the same as the original differential equation in either of the forms propounded for integration (with the appropriate conditions satisfied), then

$$\frac{R_1}{R} = \frac{S_1}{2S} = \frac{T_1}{T} = \frac{V_1}{V} = U_1,$$

for the form

$$rt - s^2 + Rr + 2Ss + Tt = V;$$

and

$$\frac{R_1}{R} = \frac{S_1}{2S} = \frac{T_1}{T} = \frac{V_1}{V}, \quad U_1 = 0,$$

for the form

$$Rr + 2Ss + Tt = V.$$

We take these in turn.

We have

$$R_1 \left(\frac{\partial u}{\partial q}\right)^2 - S_1 \frac{\partial u}{\partial q}\frac{\partial u}{\partial p} + T_1 \left(\frac{\partial u}{\partial p}\right)^2$$

$$= \left(\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q}\right)\left(\frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}\right)$$

$$= U_1 \left(\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q}\right);$$

hence

$$\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q} - R\left(\frac{\partial u}{\partial q}\right)^2 + 2S\frac{\partial u}{\partial q}\frac{\partial u}{\partial p} - T\left(\frac{\partial u}{\partial p}\right)^2 = 0$$

for the first form of the equation, and

$$R\left(\frac{\partial u}{\partial q}\right)^2 - 2S\frac{\partial u}{\partial q}\frac{\partial u}{\partial p} + T\left(\frac{\partial u}{\partial p}\right)^2 = 0$$

for the second form of the equation. Similarly,

$$R_1 \frac{du}{dx}\frac{\partial u}{\partial q} + T_1 \frac{du}{dy}\frac{\partial u}{\partial p} + V_1 \frac{\partial u}{\partial p}\frac{\partial u}{\partial q}$$

$$= \left(\frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}\right)\frac{du}{dx}\frac{du}{dy}$$

$$= U_1 \frac{du}{dx}\frac{du}{dy};$$

and therefore

$$\frac{du}{dx}\frac{du}{dy} - R\frac{du}{dx}\frac{\partial u}{\partial q} - T\frac{du}{dy}\frac{\partial u}{\partial p} - V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0$$

for the first form of the equation, and

$$R\frac{du}{dx}\frac{\partial u}{\partial q} + T\frac{du}{dy}\frac{\partial u}{\partial p} + V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0$$

for the second form of the equation.

As the quantities R_1, S_1, T_1, U_1, V_1 are skew symmetric in u and v, it is the fact (as may easily be verified) that the same equations, for the respective forms, are satisfied by v also.

For the first form of the equation, we have

$$\left(\frac{du}{dx} - T\frac{\partial u}{\partial p}\right)\left(\frac{du}{dy} - R\frac{\partial u}{\partial q}\right) = (RT + V)\frac{\partial u}{\partial p}\frac{\partial u}{\partial q},$$

and

$$\left(\frac{du}{dx} - T\frac{\partial u}{\partial p}\right)\frac{\partial u}{\partial p} + \left(\frac{du}{dy} - R\frac{\partial u}{\partial q}\right)\frac{\partial u}{\partial q} = -2S\frac{\partial u}{\partial p}\frac{\partial u}{\partial q};$$

and therefore, either

$$\left.\begin{aligned}\frac{du}{dx} - T\frac{\partial u}{\partial p} - \sigma\frac{\partial u}{\partial q} = 0\\ \frac{du}{dy} - \rho\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0\end{aligned}\right\},$$

or

$$\left.\begin{aligned}\frac{du}{dx} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0\\ \frac{du}{dy} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0\end{aligned}\right\},$$

where

$$\rho = -S + (S^2 - RT - V)^{\frac{1}{2}} = -S + \theta,$$
$$\sigma = -S - (S^2 - RT - V)^{\frac{1}{2}} = -S - \theta,$$

so that ρ and σ are the roots of the equation

$$\mu^2 + 2\mu S + RT + V = 0.$$

Thus the equations satisfied by u (and by v also) can be replaced by one or other of the above pairs of homogeneous linear equations: in particular cases, both the pairs may be valid.

When the roots of the quadratic are equal, so that $\rho = \sigma = -S$, the equations for u (and for v also) are the single pair

$$\left.\begin{aligned}\frac{du}{dx} - T\frac{\partial u}{\partial p} + S\frac{\partial u}{\partial q} = 0\\ \frac{du}{dy} + S\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0\end{aligned}\right\}.$$

For the second form of the equation, we have

$$\left(\frac{\partial u}{\partial q} - \rho_1\frac{\partial u}{\partial p}\right)\left(\frac{\partial u}{\partial q} - \sigma_1\frac{\partial u}{\partial p}\right) = 0,$$

where, as before, ρ_1 and σ_1 are the roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0,$$

so that

$$R\rho_1 = S + \alpha, \quad R\sigma_1 = S - \alpha,$$

and

$$\alpha^2 = S^2 - RT.$$

Thus either

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \rho_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \sigma_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\},$$

or

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \sigma_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \rho_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

The equations satisfied by u (and by v also) can be replaced by one or other of these pairs of homogeneous linear equations: in particular cases, both the pairs may be valid.

When the roots of the quadratic are equal, so that $\rho_1 = \sigma_1 = \dfrac{S}{R}$, the equations for u (and for v also) are the single pair

$$\left.\begin{aligned} R\frac{\partial u}{\partial q} - S\frac{\partial u}{\partial p} = 0 \\ R\frac{du}{dx} + S\frac{du}{dy} + V\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

237. The sets of equations which have been constructed, being (in Monge's method) homogeneous and linear in the differential elements, and (in Boole's method) homogeneous and linear in the first derivatives of an unknown dependent variable, have been obtained on the hypothesis that the differential equation possesses an intermediate integral involving an arbitrary function in its expression. It is important to observe that the equations in Monge's method are equivalent to those in Boole's method, so that the problem of obtaining the integral equivalent of one aggregate is effectively the same as that of obtaining the integral equivalent of the other aggregate.

This remark admits of simple verification for each of the two forms of equation.

When the original equation, supposed to possess an intermediate integral involving an arbitrary function, is of the form

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

the equations, subsidiary to the construction of that integral in Monge's method, are the respective aggregates comprised in one or other of the sets

$$\left.\begin{aligned} dp + T dx + \rho dy &= 0 \\ dq + \sigma dx + R dy &= 0 \\ dz - p dx - q dy &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dp + T dx + \sigma dy &= 0 \\ dq + \rho dx + R dy &= 0 \\ dz - p dx - q dy &= 0 \end{aligned}\right\},$$

where ρ and σ are the roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0.$$

To construct the intermediate integral of the original equation, we need two integrals

$$u = a, \quad v = b,$$

of one or other of the systems. Let

$$\theta = \text{constant},$$

where θ is a function of x, y, z, p, q, be an integral of the first system; then the relation

$$\frac{\partial\theta}{\partial x} dx + \frac{\partial\theta}{\partial y} dy + \frac{\partial\theta}{\partial z} dz + \frac{\partial\theta}{\partial p} dp + \frac{\partial\theta}{\partial q} dq = 0$$

is satisfied identically in virtue of the equations in that system. Substituting in that relation the values of dz, dp, dq as given by the system in terms of dx and dy, we have

$$\left(\frac{\partial\theta}{\partial x} + p\frac{\partial\theta}{\partial z} - T\frac{\partial\theta}{\partial p} - \sigma\frac{\partial\theta}{\partial q}\right) dx + \left(\frac{\partial\theta}{\partial y} + q\frac{\partial\theta}{\partial z} - \rho\frac{\partial\theta}{\partial p} - R\frac{\partial\theta}{\partial q}\right) dy = 0,$$

that is,

$$\left(\frac{d\theta}{dx} - T\frac{\partial\theta}{\partial p} - \sigma\frac{\partial\theta}{\partial q}\right) dx + \left(\frac{d\theta}{dy} - \rho\frac{\partial\theta}{\partial p} - R\frac{\partial\theta}{\partial q}\right) dy = 0.$$

In the absence of any relation between dx and dy, we have

$$\left.\begin{aligned} \frac{d\theta}{dx} - T\frac{\partial\theta}{\partial p} - \sigma\frac{\partial\theta}{\partial q} &= 0 \\ \frac{d\theta}{dy} - \rho\frac{\partial\theta}{\partial p} - R\frac{\partial\theta}{\partial q} &= 0 \end{aligned}\right\},$$

which are the equations of the first system in Boole's method to be satisfied by the quantities u and v that are needed for the construction of the intermediate integral

$$u = f(v).$$

Similarly, the equations of the second system in Monge's method lead to the equations of the second system in Boole's method.

When the two systems in Monge's method merge into a single system owing to the condition

$$S^2 = RT + V,$$

which gives equal roots for the quadratic in μ, the two systems in Boole's method merge into a single system owing to the same condition; and the single system in Monge's method then leads to the single system in Boole's method.

Again, when the original equation, supposed to possess an intermediate integral involving an arbitrary function, is of the form

$$Rr + 2Ss + Tt = V,$$

the equations, subsidiary to the construction of that integral in Monge's method, are the respective aggregates comprised in one or other of the sets

$$\left.\begin{aligned} dp + \sigma_1 dq - \frac{V}{R}\,dx &= 0 \\ dy - \rho_1 dx &= 0 \\ dz - (p + \rho_1 q)\,dx &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dp + \rho_1 dq - \frac{V}{R}\,dx &= 0 \\ dy - \sigma_1 dx &= 0 \\ dz - (p + \sigma_1 q)\,dx &= 0 \end{aligned}\right\},$$

where ρ_1 and σ_1 are the roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0.$$

To construct the intermediate integral of the original equation, we need two integrals

$$u = a, \quad v = b,$$

of one or other of the two systems. Let

$$\phi = \text{constant},$$

where ϕ is a function of x, y, z, p, q, be an integral of the first system; then the relation

$$\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz + \frac{\partial\phi}{\partial p}\,dp + \frac{\partial\phi}{\partial q}\,dq = 0$$

is satisfied identically in virtue of the equations in that system. Substituting in that relation the value of dp in terms of dq and dx, as well as the values of dy and dz as given by the system, we have

$$\left\{\frac{\partial\phi}{\partial x}+\rho_1\frac{\partial\phi}{\partial y}+(p+\rho_1 q)\frac{\partial\phi}{\partial z}+\frac{V}{R}\frac{\partial\phi}{\partial p}\right\}dx+\left(\frac{\partial\phi}{\partial q}-\sigma_1\frac{\partial\phi}{\partial p}\right)dp=0,$$

that is,

$$\left(\frac{d\phi}{dx}+\rho_1\frac{d\phi}{dy}+\frac{V}{R}\frac{\partial\phi}{\partial p}\right)dx+\left(\frac{\partial\phi}{\partial q}-\sigma_1\frac{\partial\phi}{\partial p}\right)dp=0.$$

In the absence of any relation between dx and dp alone, we have

$$\left.\begin{aligned}\frac{\partial\phi}{\partial q}-\sigma_1\frac{\partial\phi}{\partial p}&=0\\ \frac{d\phi}{dx}+\rho_1\frac{d\phi}{dy}+\frac{V}{R}\frac{\partial\phi}{\partial p}&=0\end{aligned}\right\},$$

which are the equations in one of the systems in Boole's method to be satisfied by the quantities u and v that are needed for the construction of the intermediate integral

$$u=f(v).$$

Similarly, the equations of the second system in Monge's method lead to the equations of the alternative system in Boole's method.

When the two systems in Monge's method merge into a single system owing to the condition

$$S^2=RT,$$

which gives equal roots for the quadratic in μ, the two systems in Boole's method also merge into a single system owing to the same condition; and the single system in Monge's method then leads to the single system in Boole's method.

If for either form of equation, there are two distinct intermediate integrals derivable by Monge's method, they are derivable also by Boole's method: for, in each method, both subsidiary systems are then valid.

The primitive of the original equation is derived from the intermediate integral or integrals, whether obtained by the one method or the other.

Ex. 1. Integrate the equation

$$2pqyr+(p^2y+qx)s+xpt=p^2q(rt-s^2)+xy,$$

by obtaining a general intermediate integral.

Comparing the form with

$$rt-s^2+Rr+2Ss+Tt=V,$$

we have

$$R=-\frac{2y}{p},\quad 2S=-\frac{y}{q}-\frac{x}{p^2},\quad T=-\frac{x}{pq},\quad V=-\frac{xy}{p^2q};$$

thus the critical quadratic is

$$\mu^2-\mu\left(\frac{y}{q}+\frac{x}{p^2}\right)+\frac{xy}{p^2q}=0,$$

so that

$$\rho,\ \sigma=\frac{x}{p^2},\ \frac{y}{q},$$

in either arrangement.

The two systems of equations in Monge's method are, firstly,

$$\left.\begin{aligned} dp-\frac{x}{pq}\,dx+\frac{x}{p^2}\,dy&=0\\ dq+\frac{y}{q}\,dx-\frac{2y}{p}\,dy&=0\\ dz-p\,dx-q\,dy&=0 \end{aligned}\right\},$$

which do not possess an integrable combination: and, secondly,

$$\left.\begin{aligned} dp-\frac{x}{pq}\,dx+\frac{y}{q}\,dy&=0\\ dq+\frac{x}{p^2}\,dx-2\frac{y}{p}\,dy&=0\\ dz-p\,dx-q\,dy&=0 \end{aligned}\right\}.$$

Two integrals of the latter are easily seen to be

$$p^2q-\tfrac{1}{2}x^2=a,\quad pq-\tfrac{1}{2}y^2=b;$$

and therefore an intermediate integral is

$$p^2q-\tfrac{1}{2}x^2=f(pq-\tfrac{1}{2}y^2),$$

where f is an arbitrary function. It is the only intermediate integral of the original equation.

When we proceed by Boole's method, one set of equations is

$$\left.\begin{aligned} \frac{d\theta}{dx}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{y}{q}\frac{\partial\theta}{\partial q}&=0\\ \frac{d\theta}{dy}-\frac{x}{p^2}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}&=0 \end{aligned}\right\},$$

which do not possess a common integral; and the other set of subsidiary equations is

$$\left.\begin{aligned}\frac{d\theta}{dx}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{x}{p^2}\frac{\partial\theta}{\partial q}=0\\ \frac{d\theta}{dy}-\frac{y}{q}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}=0\end{aligned}\right\},$$

which do possess common integrals. When expressed in full, the latter equations are

$$\theta_1=\frac{\partial\theta}{\partial x}+p\frac{\partial\theta}{\partial z}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{x}{p^2}\frac{\partial\theta}{\partial q}=0,$$

$$\theta_2=\frac{\partial\theta}{\partial y}+q\frac{\partial\theta}{\partial z}-\frac{y}{q}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}=0;$$

in order that these may coexist, we must have

$$(\theta_1,\ \theta_2)=0,$$

that is

$$\left(\frac{y}{q}-\frac{x}{p^2}\right)\frac{\partial\theta}{\partial z}=0,$$

and therefore the equations are

$$\vartheta_1=\frac{\partial\theta}{\partial x}+\frac{x}{pq}\frac{\partial\theta}{\partial p}-\frac{x}{p^2}\frac{\partial\theta}{\partial q}=0,$$

$$\vartheta_2=\frac{\partial\theta}{\partial y}-\frac{y}{q}\frac{\partial\theta}{\partial p}+\frac{2y}{p}\frac{\partial\theta}{\partial q}=0,$$

$$\vartheta_3=\frac{\partial\theta}{\partial z}=0.$$

We now have

$$(\vartheta_1,\ \vartheta_2)=0,\quad(\vartheta_1,\ \vartheta_3)=0,\quad(\vartheta_2,\ \vartheta_3)=0;$$

thus the set of these three equations of the first order is a complete Jacobian system in involution. As five variables x, y, z, p, q occur in this Jacobian system, it possesses two integrals algebraically independent of one another: by the ordinary processes explained in Chapter IV in the preceding volume, these are found to be

$$p^2q-\tfrac{1}{2}x^2,\quad pq-\tfrac{1}{2}y^2.$$

Thus, as before, the intermediate integral of the original equation is

$$p^2q-\tfrac{1}{2}x^2=f(pq-\tfrac{1}{2}y^2),$$

where f is an arbitrary function.

When this equation, as an equation of the first order, is integrated, it will give a primitive of the original equation. The subsidiary system in Charpit's method does not appear to offer integrable combinations in finite terms, when f remains a quite arbitrary function. Evidently

$$p^2q-\tfrac{1}{2}x^2=a$$

is a particular intermediate integral, a being constant; it leads to a particular primitive

$$z=a''+\frac{1}{a'}\int(\tfrac{1}{2}x^2+a)^{\frac{1}{2}}\,dx+\frac{a'^2}{y},$$

where a, a', a'' are arbitrary constants. Evidently

$$pq-\tfrac{1}{2}y^2=c$$

is another particular intermediate integral, c being constant, (which, however, is incompatible with the preceding particular intermediate integral); it leads to a particular primitive

$$z=c''+c'x+\frac{1}{c'}(cy+\tfrac{1}{6}y^3),$$

where c, c', c'' are arbitrary constants. Other particular primitives can be obtained by taking other particular forms of f in the general intermediate integral.

In the two particular primitives that have been given, three arbitrary constants occur. We shall hereafter (Chapter XIX) see how, by a method due to Imschenetsky, it is possible to generalise a primitive of such an equation of the second order that contains three arbitrary constants.

Ex. 2. Integrate the equation

$$q(1+q)r-(1+p+q+2pq)s+p(1+p)t=0.$$

Here

$$U=0,\quad V=0,\quad R=q(1+q),\quad 2S=-(1+p+q+2pq),\quad T=p(1+p);$$

the quadratic in μ for this case is

$$q(1+q)\mu^2+(1+p+q+2pq)\mu+p(1+p)=0,$$

and therefore

$$\rho_1,\ \sigma_1=-\frac{p}{1+q},\quad -\frac{1+p}{q}.$$

One subsidiary system is

$$\left.\begin{aligned}\phi_1&=\frac{\partial\phi}{\partial q}+\frac{p}{1+q}\frac{\partial\phi}{\partial p}=0\\ \phi_2&=\frac{d\phi}{dx}-\frac{1+p}{q}\frac{d\phi}{dy}=0\end{aligned}\right\}.$$

The latter, in full, is

$$\phi_2=\frac{\partial\phi}{\partial x}-\frac{1+p}{q}\frac{\partial\phi}{\partial y}-\frac{\partial\phi}{\partial z}=0.$$

In order that $\phi_1=0$ and $\phi_2=0$ may coexist, we must have

$$0=(\phi_1,\ \phi_2)=\frac{1+p+q}{q^2(1+q)}\frac{\partial\phi}{\partial y}=0,$$

that is, we have three equations

$$\phi_1=\frac{\partial\phi}{\partial q}+\frac{p}{1+q}\frac{\partial\phi}{\partial p}=0,$$

$$\phi_2'=\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial z}=0,$$

$$\phi_3=\frac{\partial\phi}{\partial y}=0:$$

and these are easily proved to form a complete Jacobian system. They therefore possess two independent common integrals, which can be taken in the form

$$\frac{1+q}{p}, \quad x+z;$$

hence an intermediate integral is

$$\frac{1+q}{p}=f(x+z),$$

where f denotes an arbitrary function.

Similarly, the other subsidiary system

$$\left.\begin{aligned} \frac{\partial\phi}{\partial q}+\frac{1+p}{q}\frac{\partial\phi}{\partial p}=0 \\ \frac{d\phi}{dx}-\frac{p}{1+q}\frac{d\phi}{dy}=0 \end{aligned}\right\}$$

is found to lead to an intermediate integral

$$\frac{1+p}{q}=g(y+z),$$

where g denotes an arbitrary function.

The primitive can be derived, after a quadrature, by combining the two distinct intermediate integrals. When we take these in the form

$$x+z=\phi(u)=\phi\left(\frac{1+q}{p}\right),$$

$$y+z=\psi(v)=\psi\left(\frac{1+p}{q}\right),$$

we have

$$(1+p+q)\,dz=p\,(dz+dx)+q\,(dz+dy),$$

that is,

$$(1+u)(1+v)\,dz=(v+1)\,\phi'(u)\,du+(u+1)\,\psi'(v)\,dv.$$

Hence

$$\begin{aligned} z&=\int\frac{\phi'(u)}{1+u}\,du+\int\frac{\psi'(v)}{1+v}\,dv \\ &=\Phi(u)+\Psi(v) \\ &=F(x+z)+G(y+z), \end{aligned}$$

where F and G are arbitrary functions.

Ex. 3. Integrate the following equations :—

(i) $q(r+t)+ps=0$;

(ii) $z(r-t)-p^2+q^2=0$;

(iii) $x^2r-y^2t=xp-yq+xy$;

(iv) $(r-t)xy-s(x^2-y^2)=qx-py$;

(v) $(r-s)x=(t-s)y$;

(vi) $x^2r+2xys+y^2t=f(xp+yq)$;

(vii) $q^2r - 2pqs + p^2t = 0$;

(viii) $q^2r - 2pqs + p^2t = \dfrac{(p+q)^2(p-q)}{y-x}$;

(ix) $q(1+q)r + (p+q+2pq)s + p(1+p)t = 0$;

(x) $(b+cq)^2 r - 2(a+cp)(b+cq)s + (a+cp)^2 t = 0$;

(xi) $rt - s^2 = 1$;

(xii) $y(rt-s^2) + qr + (p+x)s + yt + q = 0$;

(xiii) $z(1+q^2)r - 2pqzs + z(1+p^2)t + 1 + p^2 + q^2 + z^2(rt-s^2) = 0$;

(xiv) $xy(rt-s^2) - xqr - ypt + pq = 0$;

(xv) $p^2q^2(rt-s^2) + q^2r + 4pqs + p^2t = 1$;

(xvi) $rt - s^2 + q\left(\dfrac{q}{z} - \dfrac{1}{y}\right)r - 2\dfrac{pq}{z}s + p\left(\dfrac{p}{z} - \dfrac{1}{x}\right)t + \dfrac{pq}{xyz}(z - px - qy) = 0.$

General Method for the Intermediate Integral (if any) of any Equation.

238. Alike in Monge's method and in Boole's method, the differential equation of the second order has been supposed to be constructed by the elimination of an arbitrary function from an intermediate integral of given type. We have seen that an intermediate integral, in the form of an equation of the first order involving two arbitrary constants, leads (on the elimination of these constants) to an equation of the second order (§ 180). It is a characteristic property of the process of elimination that the nature of the eliminated magnitudes is ignored; moreover, the eliminant bears no explicit recognisable trace of the sources from which it came. The process is not reversible; if the effect is to be reversed, definite methods must be devised for the purpose.

Now when an equation of the second order is actually given without any indication of its origin, the preceding methods due to Monge and to Boole respectively may happen to be applicable: but the argument adopted for their construction cannot be used to prove that the methods are applicable, because the mode of construction of the differential equation is not revealed by the equation itself. The tests of applicability must be obtained otherwise: they are provided by assigning the conditions that the subsidiary equations in Monge's method possess two integrable combinations and (what are effectively the same relations) the

conditions that the subsidiary equations in Boole's method possess two independent integrals. If the conditions are not satisfied, neither of the methods as stated leads to an intermediate integral of the type $u = f(v)$, where f is an arbitrary function: such an intermediate integral is not possessed.

In selecting the equations of the type

$$U(rt - s^2) + Rr + 2Ss + Tt = V,$$

regard was paid to the facts, that it was deduced from an intermediate integral of the assumed form and that an equation must be of that type in order to possess such an intermediate integral: but it was not proved (as, indeed, it cannot be proved) either that such an equation unconditionally possesses an intermediate integral of the assigned form or that the intermediate integral of that form is the only kind of intermediate integral which can lead to an equation of the particular type. Accordingly, before proceeding to the discussion of the significance and even of the coexistence of the subsidiary equations, we shall obtain them in a different manner; and the process will shew their organic connection with the original equation.

Consider, more generally, any differential equation of the second order

$$f(x, y, z, p, q, r, s, t) = 0;$$

and suppose that it possesses an intermediate integral of the first order

$$u(x, y, z, p, q) = 0,$$

or (what is the same thing) that it is compatible with such an equation of the first order: no assumption is made as to the character of u. Then the equation $f = 0$ arises from some association of the two equations

$$\frac{du}{dx} + \frac{\partial u}{\partial p} r + \frac{\partial u}{\partial q} s = 0,$$

$$\frac{du}{dy} + \frac{\partial u}{\partial p} s + \frac{\partial u}{\partial q} t = 0,$$

either by the elimination of some arbitrary quantity or by some combination of the two equations made at will: the equation of the second order, being compatible with the equation of the first order, is compatible with the two derivatives of the latter, and therefore the three equations, which involve r, s, t, are not in-

dependent of one another. Consequently, when we proceed to resolve them to obtain expressions for r, s, t, these expressions must be evanescent, that is, on the hypothesis that an equation $u=0$ is compatible with the original equation. When we assign the conditions that the expressions for r, s, t shall be evanescent (which can be secured by substituting for r and t in terms of s from the two derivatives, and making the resulting form of $f=0$ evanescent as an equation for s), we shall have a number of relations that involve the derivatives of u. Thus the quantity u, as to which no assumption has been made save that $u=0$ is an equation of the first order and that therefore both the quantities $\frac{\partial u}{\partial p}$ and $\frac{\partial u}{\partial q}$ do not vanish, satisfies a number of partial differential equations of the first order. *Any common integral of these equations, which involves p or q or both, is an intermediate integral of the original equation.* But, as is known, a simultaneous system of equations of the first order does not unconditionally possess common integrals; in the present instance, the conditions for the possession of common integrals or a common integral are conditions that $f=0$ shall possess an intermediate integral.

Moreover, if the system of equations which must be satisfied by u should possess more than one integral, the relation of these common integrals to one another must be investigated, particularly in connection with the intermediate integral which then is possessed by $f=0$.

Ex. 1. Obtain an intermediate integral, if any, of the equation

$$(sq-tp)^2=(rt-s^2)(sp-rq).$$

Writing

$$u_x=\frac{du}{dx},\quad u_y=\frac{du}{dy},\quad u_p=\frac{\partial u}{\partial p},\quad u_q=\frac{\partial u}{\partial q},$$

we eliminate r and t by means of

$$u_x+ru_p+su_q=0,$$
$$u_y+su_p+tu_q=0,$$

and we make the resulting equation in s evanescent. In order that this may be the fact, we must have

$$\left.\begin{aligned}\frac{u_p^2}{u_q}(pu_p+qu_q)^2&=(pu_p+qu_q)(u_qu_y+u_pu_x)\\ 2pu_y\frac{u_p^2}{u_q}(pu_p+qu_q)&=(pu_p+qu_q)u_xu_y+qu_x(u_qu_y+u_pu_x)\\ p^2u_y^2\frac{u_p^2}{u_q}&=qu_x^2u_y\end{aligned}\right\},$$

which are the simultaneous equations for the determination of u if it exists. We have first to resolve these equations algebraically.

I. We may have

$$pu_p+qu_q=0, \quad u_x=0, \quad u_y=0:$$

that is,

$$p\frac{\partial u}{\partial p}+q\frac{\partial u}{\partial q}=0, \quad \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}=0, \quad \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}=0.$$

The conditions of coexistence require that

$$\frac{\partial u}{\partial z}=0;$$

and then the most general integral is

$$u=\phi\left(\frac{p}{q}\right),$$

where ϕ is arbitrary. Thus there is an intermediate integral

$$\phi\left(\frac{p}{q}\right)=0,$$

that is,

$$p-aq=0,$$

where a is an arbitrary constant: but it is very special, for it satisfies the three equations

$$sq-tp=0, \quad rt-s^2=0, \quad sp-rq=0.$$

II. We may have

$$pu_p+qu_q=0,$$

and u_x, u_y not zero: we find

$$u_x=-qu_p,$$

$$pu_y-qu_x=0:$$

that is,

$$p\frac{\partial u}{\partial p}+q\frac{\partial u}{\partial q}=0, \quad \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}+q\frac{\partial u}{\partial p}=0, \quad p\frac{\partial u}{\partial y}-q\frac{\partial u}{\partial x}=0.$$

The Jacobian conditions of coexistence require that

$$\frac{\partial u}{\partial y}=0,$$

by combining the second and third; the third equation then gives

$$\frac{\partial u}{\partial x}=0.$$

By combining the first and second, we find

$$\frac{\partial u}{\partial z}=0;$$

and the second equation then gives

$$\frac{\partial u}{\partial p}=0,$$

so that the first now is

$$\frac{\partial u}{\partial q}=0.$$

No intermediate integral is thus provided.

III. Assuming that pu_p+qu_q is not zero, as the alternative has been discussed, we find that the three equations are satisfied in virtue of two only, viz.

$$u_x=p\frac{u_p^2}{u_q},\quad u_y=q\frac{u_p^2}{u_q},$$

provided pu_p+2qu_q is not zero. We shall assume this latter condition satisfied for the moment and, later, we shall consider the alternative.

Writing $x, y, z, p, q=x_1, x_2, x_3, x_4, x_5$, we have the two equations in the form

$$f_1=p_1+x_4p_3-x_4\frac{p_4^2}{p_5}=0,$$

$$f_2=p_2+x_5p_3-x_5\frac{p_4^2}{p_5}=0\ ;$$

the Jacobian condition of coexistence is

$$0=(f_1, f_2)=-\left(p_3-\frac{p_4^2}{p_5}\right)\left(x_4\frac{p_4^2}{p_5^2}+2x_5\frac{p_4}{p_5}\right).$$

The second factor on the right-hand side is

$$=\frac{p_4}{p_5^2}(x_4p_4+2x_5p_5)$$

$$=\frac{p}{q^2}\ (pu_p+2qu_q),$$

which does not vanish ; consequently

$$p_3-\frac{p_4^2}{p_5}=0,$$

and the equations now are

$$p_3p_5-p_4^2=0,\quad p_1=0,\quad p_2=0.$$

These are a complete Jacobian system: the complete integral of the system is

$$u=k+ax_3+bx_4+cx_5,$$

where k, a, b, c are constants such that

$$ac=b^2.$$

Thus

$$u=k+ax_3+bx_4+\frac{b^2}{a}x_5$$

$$=k+az\ +bp\ +\frac{b^2}{a}q,$$

so that the intermediate integral provided by $u=0$ can be taken in the form

$$z+\alpha p+\alpha^2q+\beta=0,$$

where α and β are a couple of arbitrary constants.

IV. We have to consider the possibilities of the relation

$$pu_p + 2qu_q = 0.$$

We then find that, concurrently with this relation, all the equations are satisfied by

$$u_x = -2qu_p, \quad u_y = -2\frac{q^2}{p}u_p,$$

so that the equations for u are

$$p\frac{\partial u}{\partial p} + 2q\frac{\partial u}{\partial q} = 0,$$

$$\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + 2q\frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + 2\frac{q^2}{p}\frac{\partial u}{\partial p} = 0.$$

The Jacobian condition of coexistence of the first and second requires that

$$\frac{\partial u}{\partial x} = 0;$$

and the condition for the first and third requires that

$$\frac{\partial u}{\partial z} = 0.$$

Hence $\dfrac{\partial u}{\partial p} = 0$, $\dfrac{\partial u}{\partial q} = 0$, $\dfrac{\partial u}{\partial y} = 0$: no intermediate integral is thus provided.

Hence the given equation has

$$p - aq = 0,$$

$$z + ap + a^2q + \beta = 0,$$

for intermediate integrals.

Ex. 2. Prove that all the surfaces, satisfying the equation

$$(sq - tp)^2 = (rt - s^2)(sp - rq),$$

and touching the cone $x^2 + y^2 = (z + a)^2$ along the circle $x^2 + y^2 = a^2$, $z = 0$, are given by equating to zero the c-discriminant of

$$(z + cp + c^2q)^2 - c^2(1 + c^2).$$

(Math. Trip., Part II, 1904.)

Ex. 3. Obtain an integral of the equation

$$z(rt - s^2)^2 - (tp^2 - 2spq + rq^2)(rt - s^2) + (tp - sq)(sp - rq) = 0,$$

by constructing an intermediate integral

$$z = ap + bq + ab.$$

Does any other intermediate integral exist?

Ex. 4. Obtain an intermediate integral of the equation

$$z(rt - s^2) = q^2r - 2pqs + p^2t,$$

in the form

$$z = ap + bq,$$

where a and b are arbitrary constants.

Ex. 5. Prove that the equation

$$\left(x - a\frac{pt - qr}{rt - s^2}\right)^2 + \left(y - a\frac{qr - ps}{rt - s^2}\right)^2 = 1$$

has an intermediate integral of the form

$$px + qy - (a+1)z = p\cos\alpha + q\sin\alpha + b,$$

where α and b are arbitrary constants, and a is a constant.

Explain the result when $a = 0$.

Application of the General Method to Special Equations.

239. We proceed to apply the process, just indicated, to the equations

$$rt - s^2 + Rr + 2Ss + Tt = V,$$
$$Rr + 2Ss + Tt = V,$$

which have been considered in the earlier sections of this chapter. The immediate object is the determination of an intermediate integral if such an integral exists, no assumption being made as to the character of such an integral or as to its implicit influence (if any) upon the quantities R, S, T, V, which are supposed to be functions of x, y, z, p, q.

Taking the first of the two forms and assuming that an intermediate integral, if it exists, is an equation

$$u(x, y, z, p, q) = 0,$$

we eliminate r and t from the equation

$$rt - s^2 + Rr + 2Ss + Tt = V$$

by means of the derivatives of the intermediate integral, which are

$$\frac{du}{dx} + \frac{\partial u}{\partial p}r + \frac{\partial u}{\partial q}s = 0,$$
$$\frac{du}{dy} + \frac{\partial u}{\partial p}s + \frac{\partial u}{\partial q}t = 0;$$

and then the eliminant, as an equation in s, is made evanescent. The conditions for evanescence are

$$\frac{du}{dx}\frac{\partial u}{\partial p} + \frac{du}{dy}\frac{\partial u}{\partial q} - R\left(\frac{\partial u}{\partial q}\right)^2 + 2S\frac{\partial u}{\partial q}\frac{\partial u}{\partial p} - T\left(\frac{\partial u}{\partial p}\right)^2 = 0,$$
$$\frac{du}{dx}\frac{du}{dy} - R\frac{du}{dx}\frac{\partial u}{\partial q} - T\frac{du}{dy}\frac{\partial u}{\partial p} - V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0;$$

and these are the differential equations to be satisfied by u. Conversely, if these two equations do possess a common integral u which involves p and q, then $u=0$ is an intermediate integral of the original equation: the proof of this converse is an immediate inference from the analysis, taken in reverse course.

Now these equations are exactly the same equations as occur in Boole's method, there deduced upon a more extended assumption: hence, using the algebraical resolution before obtained (§ 236) so as to have the equivalent equations linear in the derivatives of u, we see that any common integral of the two equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} - \sigma\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} - \rho\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\},$$

or any common integral of the two equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\},$$

where ρ and σ are the roots of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0,$$

is an intermediate integral of the original equation.

Further, if the first pair of equations have a common integral u_1, and if the second pair have a common integral u_2, then the intermediate integrals

$$u_1 = 0, \quad u_2 = 0,$$

coexist. For the Jacobian condition of their coexistence is

$$[u_1, u_2] = 0,$$

that is,

$$\frac{du_1}{dx}\frac{\partial u_2}{\partial p} - \frac{du_2}{dx}\frac{\partial u_1}{\partial p} + \frac{du_1}{dy}\frac{\partial u_2}{\partial q} - \frac{du_2}{dy}\frac{\partial u_1}{\partial q} = 0;$$

on substitution from the two sets of equations for $\frac{du_1}{dx}$ and $\frac{du_1}{dy}$, $\frac{du_2}{dx}$ and $\frac{du_2}{dy}$, respectively, this relation is satisfied identically.

Moreover, we can take the most general forms of u_1 and u_2 that are admissible: thus, if the first system should possess two independent integrals (no matter how particular) represented by v_1 and w_1, we take

$$u_1 = F(v_1, w_1) = 0,$$

or

$$v_1 = f(w_1),$$

where F and f are arbitrary functions: and similarly for u_2. For our immediate purpose, however, the form of the intermediate integral (if any) is less important than the property that it is an integral common to two homogeneous linear equations of the first order, belonging to one or other of the two systems.

When the roots of the quadratic are equal, so that the equation is of the form

$$rt - s^2 + Rr + 2Ss + Tt + RT - S^2 = 0,$$

any intermediate integral is an integral of the equations

$$\left.\begin{aligned} \frac{du}{dx} - T\frac{\partial u}{\partial p} + S\frac{\partial u}{\partial q} &= 0 \\ \frac{du}{dy} + S\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\};$$

and conversely.

240. Similarly, for the equation

$$Rr + 2Ss + Tt = V,$$

if

$$u(x, y, z, p, q) = 0$$

is an intermediate integral, a corresponding process shews that u satisfies the equations

$$\left.\begin{aligned} R\left(\frac{\partial u}{\partial q}\right)^2 - 2S\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} + T\left(\frac{\partial u}{\partial p}\right)^2 &= 0 \\ R\frac{du}{dx}\frac{\partial u}{\partial q} + T\frac{du}{dy}\frac{\partial u}{\partial p} + V\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} &= 0 \end{aligned}\right\}.$$

Let ρ_1 and σ_1 be the roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0;$$

then the equations for u, when resolved, lead either to the system

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \rho_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \sigma_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\},$$

or to the system

$$\left.\begin{aligned} \frac{\partial u}{\partial q} - \sigma_1 \frac{\partial u}{\partial p} = 0 \\ \frac{du}{dx} + \rho_1 \frac{du}{dy} + \frac{V}{R}\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

These equations are exactly the same as the equations which occur in Boole's method: but they are now obtained merely on the assumption of the existence of an intermediate integral and without any assumption as to its form.

When the roots of the quadratic are equal, there is only a single system: it is

$$\left.\begin{aligned} R\frac{\partial u}{\partial q} - S\frac{\partial u}{\partial p} = 0 \\ R\frac{du}{dx} + S\frac{du}{dy} + V\frac{\partial u}{\partial p} = 0 \end{aligned}\right\}.$$

As for the former case, so for the present case, if u_1 is an integral common to the equations in the first system, and if u_2 is an integral common to the equations in the second system, then the equations

$$u_1 = 0, \quad u_2 = 0,$$

coexist: for the Jacobian condition of coexistence is satisfied identically.

We proceed from the intermediate integral or intermediate integrals in order to obtain a primitive. If there are two intermediate integrals

$$u_1 = 0, \quad u_2 = 0,$$

we resolve these equations with respect to p and q; the values so obtained are substituted in

$$dz = p\,dx + q\,dy,$$

and quadrature then leads to a primitive. If there is only a single intermediate integral

$$u = 0,$$

it is regarded as a partial equation of the first order: its integral, obtained by any of the customary processes, is a primitive of the original equation of the second order.

Three Integrals common to the Subsidiary System.

241. It thus appears that the determination of an intermediate integral (if any) of the equation

$$rt - s^2 + Rr + 2Ss + Tt = V$$

is bound up with the determination of a common integral (if any) of the equations

$$\left.\begin{array}{l} \dfrac{du}{dx} - T\dfrac{\partial u}{\partial p} - \rho\dfrac{\partial u}{\partial q} = 0 \\ \dfrac{du}{dy} - \sigma\dfrac{\partial u}{\partial p} - R\dfrac{\partial u}{\partial q} = 0 \end{array}\right\},$$

or of a common integral (if any) of the equations

$$\left.\begin{array}{l} \dfrac{du}{dx} - T\dfrac{\partial u}{\partial p} - \sigma\dfrac{\partial u}{\partial q} = 0 \\ \dfrac{du}{dy} - \rho\dfrac{\partial u}{\partial p} - R\dfrac{\partial u}{\partial q} = 0 \end{array}\right\}.$$

Now these equations are homogeneous and linear: and there are perfectly definite processes for determining whether the equations in such a system do possess a common integral and, if so, what is the number of algebraically independent integrals which they do possess.

Consider the first system of equations. We take

$$\rho = -S + \theta, \quad \sigma = -S - \theta,$$

where

$$\theta^2 = S^2 - RT - V;$$

and we write the equations in the form

$$\left.\begin{array}{l} \Delta(u) = \dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z} - T\dfrac{\partial u}{\partial p} - \rho\dfrac{\partial u}{\partial q} = 0 \\ \Delta'(u) = \dfrac{\partial u}{\partial y} + q\dfrac{\partial u}{\partial z} - \sigma\dfrac{\partial u}{\partial p} - R\dfrac{\partial u}{\partial q} = 0 \end{array}\right\}.$$

As there are five variables x, y, z, p, q which can occur in u, and as there are two equations initially, four different cases can arise; there can be three independent common integrals, or two, or one, or none, according to the number of equations in the system when it is rendered complete.

In order that the two equations $\Delta(u)=0$, $\Delta'(u)=0$, may coexist, the Jacobian condition

$$(\Delta, \Delta')=0$$

must be satisfied, u now being regarded as the dependent variable and x, y, z, p, q as the independent variables. But

$$(\Delta, \Delta')=(\sigma-\rho)\frac{\partial u}{\partial z}+\{\Delta'(T)-\Delta(\sigma)\}\frac{\partial u}{\partial p}+\{\Delta'(\rho)-\Delta(R)\}\frac{\partial u}{\partial q};$$

and the right-hand side, which manifestly does not vanish in virtue of $\Delta(u)=0$ and $\Delta'(u)=0$, still must vanish. This requirement can be satisfied in one of two ways: the right-hand side may vanish identically: or, if not vanishing identically, it provides a new non-identical equation when equated to zero. We take the two cases separately.

When $(\Delta, \Delta')=0$ is satisfied identically, we have

$$\rho=\sigma,$$

as a first condition. The roots of the quadratic are equal, so that there is only a single subsidiary system: the common value of the equal roots is $-S$. The other conditions are

$$\Delta'(T)-\Delta(\sigma)=0, \quad \Delta'(\rho)-\Delta(R)=0;$$

and therefore the equations

$$\left.\begin{aligned}\Delta(u)=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-T\frac{\partial u}{\partial p}+S\frac{\partial u}{\partial q}=0\\ \Delta'(u)=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial p}-R\frac{\partial u}{\partial q}=0\end{aligned}\right\}$$

are such that

$$(\Delta, \Delta')=0,$$

provided

$$\Delta'(T)+\Delta(S)=0, \quad \Delta'(S)+\Delta(R)=0:$$

that is, *when these conditions are satisfied, the equations* $\Delta=0$, $\Delta'=0$ *are a complete Jacobian system,* and therefore they possess

three independent integrals in common. Let these three independent integrals be u_1, u_2, u_3, obtainable by any of the methods given in Chapter IV in the preceding volume, where u_1, u_2, u_3 are functions of x, y, z, p, q. Now

$$\begin{aligned}[u_1, u_2] &= \frac{du_1}{dx}\frac{\partial u_2}{\partial p} - \frac{du_2}{dx}\frac{\partial u_1}{\partial p} + \frac{du_1}{dy}\frac{\partial u_2}{\partial q} - \frac{du_2}{dy}\frac{\partial u_1}{\partial q} \\ &= 0,\end{aligned}$$

on substitution; and similarly

$$[u_1, u_3] = 0, \quad [u_2, u_3] = 0.$$

Consequently, the equations

$$u_1 = a, \quad u_2 = b, \quad u_3 = c,$$

where a, b, c are arbitrary constants, coexist: and the quantities u_1, u_2, u_3 are independent of one another, so that the three equations can be resolved with respect to any three of the variables. Let this resolution be effected with respect to z, p, q; then we have

$$\begin{aligned} z &= f(x, y, a, b, c), \\ p &= g(x, y, a, b, c), \\ q &= h(x, y, a, b, c), \end{aligned}$$

and these values of p and q are the derivatives of z.

Thus, with the conditions as satisfied, an integral involving three arbitrary constants has been obtained. This integral can be generalised so as to involve two arbitrary functions: and the generalised form is given by

$$\left.\begin{aligned} & b = \phi(a), \quad c = \psi(a) \\ & z = f(x, y, a, b, c) \\ & 0 = \frac{df}{da} \end{aligned}\right\},$$

where ϕ and ψ are arbitrary functions, and a, b, c are to be eliminated. The origin of this generalisation is to be found in Ampère's method, which will be expounded later: meanwhile, the statement can be verified as follows.

We first need the relations connected with the fact that

$$z = f, \quad p = g, \quad q = h,$$

constitute an integral of the original equation. These three equations are the equivalent of the three equations

$$u_1 = a, \quad u_2 = b, \quad u_3 = c,$$

so that the quantities z, p, q in these three equations are such that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q,$$

$$\frac{\partial p}{\partial x} = r, \quad \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = s, \quad \frac{\partial q}{\partial y} = t.$$

When we substitute in each of the three equations

$$u_\mu = \text{constant},$$

for $\mu = 1, 2, 3$, the values $z = f$, $p = g$, $q = h$, we have identities; and therefore

$$\frac{\partial u_\mu}{\partial x} + p\frac{\partial u_\mu}{\partial z} + r\frac{\partial u_\mu}{\partial p} + s\frac{\partial u_\mu}{\partial q} = 0,$$

or, taking account of one of the differential equations satisfied by u_μ, we have

$$(r + T)\frac{\partial u_\mu}{\partial p} + (s - S)\frac{\partial u_\mu}{\partial q} = 0.$$

As this holds for $\mu = 1, 2, 3$, and as u_1, u_2, u_3 are independent of one another, we have

$$r + T = 0, \quad s - S = 0.$$

Similarly, from

$$\frac{\partial u_\mu}{\partial y} + q\frac{\partial u_\mu}{\partial z} + s\frac{\partial u_\mu}{\partial p} + t\frac{\partial u_\mu}{\partial q} = 0,$$

we have

$$s - S = 0, \quad t + R = 0.$$

Consequently,

$$-T = \frac{\partial^2 f}{\partial x^2}, \quad S = \frac{\partial^2 f}{\partial x \partial y}, \quad -R = \frac{\partial^2 f}{\partial y^2};$$

and

$$V = S^2 - RT = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}.$$

With these values, the equation

$$rt - s^2 + Rr + 2Ss + Tt = V = S^2 - RT$$

is satisfied by

$$z = f(x, y, a, b, c).$$

Passing now to the equations of the generalised integral, we have

$$\frac{df}{da}=0,$$

and therefore

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)+\frac{d^2f}{da^2}\frac{\partial a}{\partial x}=0,\quad \frac{\partial}{\partial y}\left(\frac{df}{da}\right)+\frac{d^2f}{da^2}\frac{\partial a}{\partial y}=0\,;$$

thus

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)=\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)=-\frac{d^2f}{da^2}\frac{\partial a}{\partial x},$$

$$\frac{\partial}{\partial y}\left(\frac{df}{da}\right)=\frac{d}{da}\left(\frac{\partial f}{\partial y}\right)=-\frac{d^2f}{da^2}\frac{\partial a}{\partial y}.$$

Now from the equation

$$z=f,$$

we have

$$p=\frac{\partial f}{\partial x}+\frac{df}{da}\frac{\partial a}{\partial x}=\frac{\partial f}{\partial x},$$

$$q=\frac{\partial f}{\partial y}+\frac{df}{da}\frac{\partial a}{\partial y}=\frac{\partial f}{\partial y}.$$

Again,

$$r=\frac{\partial^2 f}{\partial x^2}+\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)\frac{\partial a}{\partial x}=\frac{\partial^2 f}{\partial x^2}-\frac{d^2f}{da^2}\left(\frac{\partial a^2}{\partial x}\right),$$

$$s=\frac{\partial^2 f}{\partial x\partial y}+\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)\frac{\partial a}{\partial y}=\frac{\partial^2 f}{\partial x\partial y}-\frac{d^2f}{da^2}\frac{\partial a}{\partial x}\frac{\partial a}{\partial y},$$

$$t=\frac{\partial^2 f}{\partial y^2}+\frac{d}{da}\left(\frac{\partial f}{\partial y}\right)\frac{\partial a}{\partial y}=\frac{\partial^2 f}{\partial y^2}-\frac{d^2f}{da^2}\left(\frac{\partial a}{\partial y}\right)^2.$$

On the right-hand sides, the partial derivatives of f with regard to x and y are taken on the hypothesis that a is constant: thus we may put

$$\frac{\partial^2 f}{\partial x^2}=-T,\quad \frac{\partial^2 f}{\partial x\partial y}=S,\quad \frac{\partial^2 f}{\partial y^2}=-R,$$

and therefore

$$r=-T-\frac{d^2f}{da^2}\left(\frac{\partial a}{\partial x}\right)^2,$$

$$s=\quad S-\frac{d^2f}{da^2}\frac{\partial a}{\partial x}\frac{\partial a}{\partial y},$$

$$t=-R-\frac{d^2f}{da^2}\left(\frac{\partial a}{\partial y}\right)^2.$$

Consequently,

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT,$$

on substituting these values of r, s, t: that is, the original differential equation is satisfied by

$$\left.\begin{aligned} z &= f\{x, y, a, \phi(a), \psi(a)\} \\ 0 &= \frac{df}{da} \end{aligned}\right\},$$

which thus is an integral involving two arbitrary functions. The conditions for the existence of the integral are that the quantities R, S, T, being functions of x, y, z, p, q, shall satisfy the relations

$$\Delta(R) + \Delta'(S) = 0, \quad \Delta(S) + \Delta'(T) = 0,$$

identically, where

$$\Delta = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - T\frac{\partial}{\partial p} + S\frac{\partial}{\partial q},$$

$$\Delta' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + S\frac{\partial}{\partial p} - R\frac{\partial}{\partial q}.$$

Ex. 1. Verify the converse, viz. that the equation given by the elimination of a between

$$z = f\{x, y, a, \phi(a), \psi(a)\}, \quad \frac{df}{da} = 0,$$

satisfies an equation of the second order, for which the two conditions are satisfied and for which

$$V = S^2 - RT.$$

Ex. 2. A surface is defined as the locus of the family of curves

$$f(x, y, z, a) = 0, \quad g(x, y, z, a) = 0$$

where a is a parameter; shew that z, regarded as a function of x and y along this surface, satisfies the partial equations

$$Ap + Bq - C = 0,$$

$$A^2r + 2ABs + B^2t = H,$$

where

$$A = J\left(\frac{f, g}{y, z}\right), \quad B = J\left(\frac{f, g}{z, x}\right), \quad C = J\left(\frac{f, g}{x, y}\right),$$

$$H = \delta A - p\delta B - q\delta C,$$

and

$$\delta = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial B} + C\frac{\partial}{\partial C}.$$

Give a geometrical interpretation of the two partial equations : and find the condition or conditions which must be satisfied by quantities A', B', H', in order that the equation

$$A'^2r+2A'B's+B'^2t=H'$$

may possess an integral of the foregoing type.

242. It is not difficult to construct equations of the type just discussed. The equation is

$$rt-s^2+Rr+2Ss+Tt=S^2-RT;$$

and the quantities R, S, T must satisfy the equations

$$\Delta(R)+\Delta'(S)=0, \quad \Delta(S)+\Delta'(T)=0.$$

Hence S may be assumed arbitrarily; and then R and T are given by the two simultaneous equations

$$\left.\begin{aligned}\Delta(R)&=-\Delta'(S)\\ \Delta'(T)&=-\Delta(S)\end{aligned}\right\}.$$

A. The simplest set of cases occurs when

$$S=a,$$

where a is a constant: then $\Delta'(S)=0$, $\Delta(S)=0$; so that R and T are given by the two equations

$$\left.\begin{aligned}\frac{\partial R}{\partial x}+p\frac{\partial R}{\partial z}-T\frac{\partial R}{\partial p}+a\frac{\partial R}{\partial q}=0\\ \frac{\partial T}{\partial y}+q\frac{\partial T}{\partial z}+a\frac{\partial T}{\partial p}-R\frac{\partial T}{\partial q}=0\end{aligned}\right\}.$$

Individual forms are easily obtainable.

I. Let

$$R=b,$$

where b is a constant; the first equation is satisfied identically, and T then is any integral of

$$\frac{\partial T}{\partial y}+q\frac{\partial T}{\partial z}+a\frac{\partial T}{\partial p}-b\frac{\partial T}{\partial q}=0,$$

so that we can take

$$T=F(x,\ p-ay,\ q+by,\ z-qy-\tfrac{1}{2}by^2),$$

where F is any function of its arguments. As R, S, T, are now known, the differential equation is known.

When we proceed to obtain the primitive by the method in the text, we have to obtain three integrals common to

$$\Delta(u)=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-T\frac{\partial u}{\partial p}+a\frac{\partial u}{\partial q}=0,$$

$$\Delta'(u)=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+a\frac{\partial u}{\partial p}-b\frac{\partial u}{\partial q}=0.$$

The latter is the equation defining T; hence it has the four independent integrals

$$v_1 = x,$$
$$v_2 = p - ay,$$
$$v_3 = q + by,$$
$$v_4 = z - qy - \tfrac{1}{2} by^2,$$

and any function of these will satisfy $\Delta'(u) = 0$. Let

$$u = g(v_1, v_2, v_3, v_4)$$

be a function of them satisfying $\Delta(u) = 0$: then

$$\frac{\partial g}{\partial v_1} - \frac{\partial g}{\partial v_2} F + \frac{\partial g}{\partial v_3} a + \frac{\partial g}{\partial v_4} v_2 = 0.$$

To obtain the form of g, we construct the subsidiary equations

$$\frac{dv_1}{1} = \frac{dv_2}{-F} = \frac{dv_3}{a} = \frac{dv_4}{v_2}.$$

Let

$$w_1 = \alpha, \quad w_2 = \beta, \quad w_3 = \gamma,$$

be three independent integrals of these equations, where α, β, γ are constants: then, if we eliminate p and q between the three equations

$$w_1 = \alpha, \quad w_2 = \beta, \quad w_3 = \gamma,$$

leading to an equation

$$H(x, y, z, \alpha, \beta, \gamma) = 0,$$

the integral of the differential equation

$$rt - s^2 + br + 2as - tF = a^2 + bF$$

is given by

$$\left.\begin{aligned} H\{x, y, z, \alpha, \phi(\alpha), \psi(\alpha)\} = 0 \\ \frac{dH}{d\alpha} = 0 \end{aligned}\right\}.$$

A particular case of this form, viz.

$$(p + q + r)(1 + t) = (1 - s)^2$$

is given by Imschenetsky*: it is obtained by taking

$$a = 1, \quad b = 1, \quad F = v_2 + v_3,$$

in what precedes. The quantities w_1, w_2, w_3 arise in the integrals of

$$\frac{dv_2}{dv_1} = -v_2 - v_3, \quad \frac{dv_3}{dv_1} = 1, \quad \frac{dv_4}{dv_1} = v_2;$$

we easily find

$$v_3 = v_1 + \alpha,$$
$$v_2 = \beta e^{-v_1} - v_1 - \alpha + 1,$$
$$v_4 = \gamma - \beta e^{-v_1} - \tfrac{1}{2} v_1^2 - (\alpha - 1) v_1.$$

* At p. 299 of his frequently quoted memoir.

Substituting for v_1, v_2, v_3, v_4, and eliminating p and q, we have

$$z=\gamma-\beta e^{-x}+x-\tfrac{1}{2}(x-y)^2-a(x-y).$$

The generalised integral is

$$\left.\begin{aligned} z&=\phi(a)-e^{-x}\psi(a)+x-\tfrac{1}{2}(x-y)^2-a(x-y) \\ 0&=\phi'(a)-e^{-x}\psi'(a)-x+y \end{aligned}\right\}.$$

Both results are given by Imschenetsky.

II. Still keeping $S=a$, let

$$R=f(y,\ q-ax),$$

where f denotes any function at our disposal. Then the condition

$$\Delta(R)=-\Delta'(S)=0$$

is satisfied identically; and the equation for T is

$$\frac{\partial T}{\partial y}+q\frac{\partial T}{\partial z}+a\frac{\partial T}{\partial p}-f(y,\ q-ax)\frac{\partial T}{\partial q}=0.$$

Then

$$T=G(x,\ p-ay,\ \theta,\ \zeta),$$

where G is any function at our disposal, and

$$\theta=\theta(y,\ t)=\text{constant},\quad \zeta=z-axy-\phi(y,\ t)=\text{constant},$$

are two independent integrals of the ordinary equations

$$\frac{dt}{dy}=-f(y,\ t),$$

$$\frac{dz}{dy}=t+ax,$$

in which x is parametric and t denotes $q-ax$. As S, R, T are known, the form of the differential equation is given explicitly.

In order to obtain the primitive by the method in the text, we have to obtain three integrals common to

$$\Delta(u)=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-G\frac{\partial u}{\partial p}+a\frac{\partial u}{\partial q}=0,$$

$$\Delta'(u)=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+a\frac{\partial u}{\partial p}-f\frac{\partial u}{\partial q}=0.$$

The latter is the equation defining T: we know four independent integrals in the form

$$v_1=x,$$

$$v_2=p-ay,$$

$$v_3=\theta,$$

$$v_4=\zeta;$$

and any functional combination of these will satisfy the second equation for u. Let

$$u=g(v_1,\ v_2,\ v_3,\ v_4)$$

be a functional combination satisfying the first equation ; then, as

$$\Delta(v_1)=1, \quad \Delta(v_2)=-T=-G(v_1,\ v_2,\ v_3,\ v_4),$$
$$\Delta(v_3)=0, \quad \Delta(v_4)=v_2,$$

the equation for g is

$$\frac{\partial g}{\partial v_1}-G\frac{\partial g}{\partial v_2}+v_2\frac{\partial g}{\partial v_4}=0.$$

One integral evidently is

$$v_3:$$

let

$$w_1=\beta, \quad w_2=\gamma,$$

be two independent integrals of the ordinary equations

$$dv_1=\frac{dv_2}{-G}=\frac{dv_4}{v_2},$$

in which v_3 is parametric. An integral of the original equation is given by eliminating p and q between

$$v_3=a, \quad w_1=\beta, \quad w_2=\gamma:$$

and it can be generalised in the usual way.

As an example, let

$$R=q-ax=t;$$

to determine T, we integrate the equations

$$\frac{dt}{dy}=-t, \quad \frac{dz}{dy}=t+ax,$$

so that

$$\theta=te^y, \quad \zeta=z-axy+t,$$

and then

$$T=G(x,\ p-ay,\ te^y,\ z-axy+t),$$

where G is at our disposal, being any function whatever of its arguments.

In particular, let

$$T=p-ay,$$

so that the differential equation is

$$rt-s^2+(q-ax)r+2as+(p-ay)t=a^2-(p-ay)(q-ax).$$

To obtain a primitive, we have

$$(q-ax)e^y=v_3=a;$$

and we have to integrate

$$\frac{dv_2}{dv_1}=-G=-v_2, \quad \frac{dv_4}{dv_1}=v_2.$$

Thus

$$w_1=v_2e^{v_1}=\beta, \quad w_3=v_4+\beta e^{-v_1}=\gamma;$$

the primitive is given by eliminating p and q between

$$(q-ax)e^y=a,$$
$$(p-ay)e^x=\beta,$$
$$z-axy+q-ax=\gamma-\beta e^{-x},$$

that is, it is

$$z = axy - ae^{-y} - \beta e^{-x} + \gamma.$$

The generalised integral comes by taking

$$\beta = \phi(\gamma), \quad a = \psi(\gamma).$$

III. Still keeping $S = a$, write

$$\delta = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + a\frac{\partial}{\partial q},$$

$$\delta' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + a\frac{\partial}{\partial p};$$

then the equations for R and T are

$$\delta R = T\frac{\partial R}{\partial p},$$

$$\delta' T = R\frac{\partial T}{\partial q}.$$

If $\dfrac{\partial R}{\partial p}$ is not zero, then

$$T = \frac{\delta R}{\dfrac{\partial R}{\partial p}},$$

and therefore

$$\delta'\left(\frac{\delta R}{\dfrac{\partial R}{\partial p}}\right) = R\frac{\partial}{\partial q}\left(\frac{\delta R}{\dfrac{\partial R}{\partial p}}\right),$$

which is a differential equation for R of the second order involving five independent variables. The general primitive of this equation does not appear to be obtainable: but special integrals can be obtained.

B. Another set of cases is given by

$$R = 0, \quad T = 0,$$

provided S can be determined so as to satisfy the equations

$$\Delta(S) = 0, \quad \Delta'(S) = 0,$$

which in the present instance are

$$\Delta = \frac{\partial S}{\partial x} + p\frac{\partial S}{\partial z} + S\frac{\partial S}{\partial q} = 0,$$

$$\Delta' = \frac{\partial S}{\partial y} + q\frac{\partial S}{\partial z} + S\frac{\partial S}{\partial p} = 0.$$

The condition of coexistence is that

$$[\Delta, \Delta'] = 0:$$

it is easy to verify that

$$[\Delta, \Delta'] = p\Delta - q\Delta',$$

and therefore the condition is satisfied.

Suppose that S is determined by an equation

$$\sigma(x, y, z, p, q, S)=0:$$

then

$$\frac{\partial \sigma}{\partial x}+\frac{\partial \sigma}{\partial S}\frac{\partial S}{\partial x}=0,$$

and so for other derivatives of S: thus the equations for σ become

$$\left.\begin{aligned}\frac{\partial \sigma}{\partial x}+p\frac{\partial \sigma}{\partial z}+S\frac{\partial \sigma}{\partial q}=0\\ \frac{\partial \sigma}{\partial y}+q\frac{\partial \sigma}{\partial z}+S\frac{\partial \sigma}{\partial p}=0\end{aligned}\right\}.$$

Proceeding in the usual manner, we find that there are four independent integrals of these two equations, viz.

$$S, \quad p-yS, \quad q-xS, \quad z-xp-yq+xyS:$$

and therefore the most general value of σ is

$$\sigma=\Phi(S, p-yS, q-xS, z-px-qy+xyS),$$

where Φ is a completely arbitrary function. But S is given by $\sigma=0$, that is, S is determined by the equation

$$\Phi(S, p-yS, q-xS, z-px-qy+xyS)=0.$$

The differential equation to be integrated is

$$rt=(s-S)^2;$$

and if u_1, u_2, u_3 be three independent integrals of the two equations

$$\left.\begin{aligned}\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial q}=0\\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial p}=0\end{aligned}\right\},$$

an integral of the differential equation is obtained by eliminating p and q between

$$u_1=a, \quad u_2=\beta, \quad u_3=\gamma.$$

As an example, take

$$S=p-yS,$$

so that

$$S=\frac{p}{1+y};$$

the quantities u_1, u_2, u_3 are found to be

$$u_1=\frac{p}{1+y}, \quad u_2=q-\frac{px}{1+y}, \quad u_3=z-qy-\frac{px}{1+y};$$

and the integral is

$$z=ax(1+y)+y\beta+\gamma.$$

Ex. Integrate the equations :—

(i) $rt - s^2 + br + 2as - a^2 = (b+t) f''(x)$;

(ii) $rt - s^2 + br + 2as - a^2 = \dfrac{b+t}{(p-ay)^n}$,

where n is constant;

(iii) $rt - s^2 + br + 2as - a^2 = \dfrac{b+t}{(q+by)^m}$,

where m is constant;

(iv) $rt - s^2 + br + 2as - a^2 = c(b+t)(z - qy - \frac{1}{2}y^2)$,

where c is constant;

(v) $rt - s^2 + br + 2as - a^2 = \kappa \dfrac{q+by}{p-ay}(b+t)$,

where κ is constant;

(vi) $rt - s^2 + br + 2as - a^2 = \lambda (p-ay)(z - qy - \frac{1}{2}y^2)(b+t)$,

where λ is constant;

(vii) $rt - s^2 + (q - ax) r + 2as + tf''(x) = a^2 - (q - ax) f''(x)$;

(viii) $rt - s^2 + (q - ax) r + 2as - a^2 = (cx + ay - p)(t + q - ax)$;

(ix) $rt - s^2 + rf(x) + 2as + tg(y) = a^2 - f(x) g(y)$;

(x) $rt = \left(s - \dfrac{q}{1+x}\right)^2$;

(xi) $rt = \left(s - \dfrac{xp + yq - z}{xy - 1}\right)^2$;

(xii) $rt = \left(s - \dfrac{pq}{z}\right)^2$.

Two Integrals common to the Subsidiary System.

243. It was seen that the two equations $\Delta(u) = 0$, $\Delta'(u) = 0$, coexist only if the equation

$$(\Delta, \Delta') = 0$$

is satisfied. From its form, it clearly cannot be satisfied in virtue of $\Delta(u) = 0$, $\Delta'(u) = 0$; and we have discussed the case in which it is satisfied identically. It remains to discuss the case in which it is a new equation and for which therefore not all the three quantities

$$\sigma - \rho, \quad \Delta'(T) - \Delta(\sigma), \quad \Delta'(\rho) - \Delta(R),$$

vanish.

We shall deal with only some of the possibilities in detail: for the present purpose, we shall assume that no one of the three quantities vanishes, so that, writing

$$P=\frac{\Delta'(T)-\Delta(\sigma)}{\rho-\sigma},\quad Q=\frac{\Delta'(\rho)-\Delta(R)}{\rho-\sigma},$$

the new equation has the form

$$\Delta''(u)=\frac{\partial u}{\partial z}-P\frac{\partial u}{\partial p}-Q\frac{\partial u}{\partial q}=0.$$

It is easy to see that, if the system of equations in u is to possess only two independent integrals (so as to justify the assumptions in Monge's method and in Boole's method as regards the origin of the equation), the quantity $\rho-\sigma$ must not vanish. Assume the contrary, so that $\rho=\sigma$: then, for our present purpose, we cannot have both the quantities $\Delta'(T)+\Delta(S)$, $\Delta'(S)+\Delta(R)$, equal to zero. Let the former be different from zero, and write

$$-\mu=\frac{\Delta'(S)+\Delta(R)}{\Delta'(T)+\Delta(S)};$$

the new equation is

$$\nabla(u)=\frac{\partial u}{\partial p}+\mu\frac{\partial u}{\partial q}=0.$$

The equations

$$\Delta(u)=0,\quad \Delta'(u)=0,\quad \nabla(u)=0,$$

are to be a complete system if they possess two independent integrals: consequently the equations

$$(\Delta,\ \Delta')=0,\quad (\Delta,\ \nabla)=0,\quad (\Delta',\ \nabla)=0,$$

must be satisfied in virtue of the equations of the system. Now

$$(\Delta,\ \Delta')=\{\Delta'(T)+\Delta(S)\}\,\nabla(u)=0,$$

$$(\Delta,\ \nabla)=\ \frac{\partial u}{\partial z}-\nabla(T)\frac{\partial u}{\partial p}+\{\nabla(S)-\Delta\ (\mu)\}\frac{\partial u}{\partial q},$$

$$(\Delta',\ \nabla)=\mu\frac{\partial u}{\partial z}+\nabla(S)\frac{\partial u}{\partial p}-\{\nabla(R)+\Delta'(\mu)\}\frac{\partial u}{\partial q}:$$

it is obvious that the quantities $(\Delta,\ \nabla)$, $(\Delta',\ \nabla)$, do not vanish in virtue of $\Delta=0$, $\Delta'=0$, $\nabla=0$. The three equations are not a complete system: they cannot possess two independent integrals.

Accordingly, we are justified (for our immediate purpose) in assuming that $\rho-\sigma$ does not vanish.

We thus have three equations

$$\Delta(u)=0,\quad \Delta'(u)=0,\quad \Delta''(u)=0,$$

and these equations are to coexist: consequently, the relations

$$(\Delta,\ \Delta')=0,\quad (\Delta,\ \Delta'')=0,\quad (\Delta',\ \Delta'')=0,$$

must be satisfied, either identically, or in virtue of the equations of the system, or as new equations. Now

$$(\Delta, \Delta') = (\sigma - \rho)\,\Delta''(u) = 0,$$

thus providing no new condition. Also

$$(\Delta, \Delta'') = P\frac{\partial u}{\partial z} - \{\Delta(P) - \Delta''(T)\}\frac{\partial u}{\partial p} - \{\Delta(Q) - \Delta''(\rho)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \Delta'') = Q\frac{\partial u}{\partial z} - \{\Delta'(P) - \Delta''(\sigma)\}\frac{\partial u}{\partial p} - \{\Delta'(Q) - \Delta''(R)\}\frac{\partial u}{\partial q}.$$

These equations manifestly do not vanish identically; if they vanish in virtue of the equations of the system, we evidently must have

$$(\Delta, \Delta'') = P\Delta''(u), \quad (\Delta', \Delta'') = Q\Delta''(u),$$

and therefore we must have

$$\left.\begin{aligned} \Omega_1 &= P^2 - \Delta(P) + \Delta''(T) = 0 \\ \Omega_2 &= PQ - \Delta(Q) + \Delta''(\rho) = 0 \\ \Omega_3 &= PQ - \Delta'(P) + \Delta''(\sigma) = 0 \\ \Omega_4 &= Q^2 - \Delta'(Q) + \Delta''(R) = 0 \end{aligned}\right\}.$$

Taking account of the relations between the operators $\Delta, \Delta', \Delta''$, viz.

$$\Delta\Delta' - \Delta'\Delta = (\sigma - \rho)\,\Delta'',$$
$$\Delta\Delta'' - \Delta''\Delta = P\Delta'',$$
$$\Delta'\Delta'' - \Delta''\Delta' = Q\Delta'',$$

we find

$$\left.\begin{aligned} \Delta'\Omega_1 - \Delta\Omega_3 &= Q\Omega_1 + P\Omega_2 - 2P\Omega_3 \\ \Delta'\Omega_2 - \Delta\Omega_4 &= P\Omega_4 + Q\Omega_3 - 2Q\Omega_2 \end{aligned}\right\}.$$

Accordingly, the four relations may be really equivalent to only two relations: and they are the conditions that the system of equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

shall be a complete system.

Suppose that the conditions are satisfied and that therefore the three equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

are a complete system: they involve five variables x, y, z, p, q, and they therefore possess two independent integrals in common. Let

v and w be these common integrals, taken as simply as possible: then the most general integral is of the form

$$u = F(v, w),$$

where F is an arbitrary function. But

$$u = 0$$

is an equation of the first order compatible with the original differential equation; hence an intermediate integral is

$$F(v, w) = 0,$$

that is,

$$v = f(w),$$

where f is an arbitrary function.

But this is precisely the result obtained by Monge's method and by Boole's method on the assumption, made for the analysis in each method, that the appropriate conditions (there left undetermined) are satisfied. Consequently we have the theorem:

The equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

where $S^2 - RT - V$ *is not zero, possesses an intermediate integral of the form*

$$v = f(w),$$

provided the relations

$$P^2 - \Delta(P) + \Delta''(T) = 0, \quad PQ - \Delta'(P) + \Delta''(\sigma) = 0,$$
$$PQ - \Delta(Q) + \Delta''(\rho) = 0, \quad Q^2 - \Delta'(Q) + \Delta''(R) = 0,$$

are satisfied; and these relations may be equivalent to only two conditions. The quantities ρ *and* σ *are the (unequal) roots of the quadratic*

$$\mu^2 + 2\mu S + RT + V = 0;$$

the operators Δ *and* Δ' *are given by*

$$\Delta = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - T\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial q};$$

$$\Delta' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} - \sigma\frac{\partial}{\partial p} - R\frac{\partial}{\partial q};$$

the quantities P *and* Q *are*

$$P = \frac{\Delta'(T) - \Delta(\sigma)}{\rho - \sigma}, \quad Q = \frac{\Delta'(\rho) - \Delta(R)}{\rho - \sigma},$$

and the operator Δ'' *is given by*

$$\Delta'' = \frac{\partial}{\partial z} - P\frac{\partial}{\partial p} - Q\frac{\partial}{\partial q}.$$

The preceding result is the definite establishment of the theorem as to the possession of an intermediate integral, involving in its expression an arbitrary function.

No discrimination has been made between the two unequal roots of the quadratic equation: an intermediate integral will be possessed, if the essential conditions are satisfied for either arrangement of the two roots.

If the essential conditions are satisfied for each of the arrangements of the roots, then each arrangement of the roots leads to an intermediate integral involving an arbitrary function. These intermediate integrals coexist, according to an earlier theorem (§ 233). The construction of the primitive is then a matter of mere quadrature of the relation

$$dz = p\,dx + q\,dy,$$

after substitution of the values of p and q derived from the simultaneous intermediate integrals. To such equations we shall recur later.

The more frequent case arises when the essential conditions are satisfied for one, but not for both, of the arrangements of the roots. We then have one intermediate integral: the construction of the primitive requires the integration of that equation of the first order, and it will appear (from a theorem of Ampère's which will presently be proved) that, in this integration, the equations connected with the unsatisfying arrangement of the roots of the quadratic occur.

Ex. 1. One very simple case arises when

$$P=0, \quad Q=0.$$

The operator Δ'' is merely $\frac{\partial}{\partial z}$: one of the equations for the intermediate integral is

$$\frac{\partial u}{\partial z}=0,$$

so that z does not occur explicitly in the integral: and the (four) conditions are

$$\frac{\partial T}{\partial z}=0, \quad \frac{\partial \rho}{\partial z}=0, \quad \frac{\partial \sigma}{\partial z}=0, \quad \frac{\partial R}{\partial z}=0.$$

As ρ and σ are unequal to one another, these conditions require that R, S, T, V do not explicitly involve z. We therefore take R, S, T, V free from z; and then, as both P and Q vanish, we have

$$\Delta'(T) = \Delta(\sigma), \quad \Delta'(\rho) = \Delta(R),$$

where now

$$\left.\begin{aligned} \Delta &= \frac{\partial}{\partial x} - T\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial q} \\ \Delta' &= \frac{\partial}{\partial y} - \sigma\frac{\partial}{\partial p} - R\frac{\partial}{\partial q} \end{aligned}\right\}.$$

Four quantities, being functions of x, y, p, q, are at our disposal, subject to the limitation that ρ and σ are not equal to one another.

Thus ρ and σ can be assumed arbitrarily: the quantities R and T are determined by means of the equations

$$\Delta'(T) = \Delta(\sigma), \quad \Delta'(\rho) = \Delta(R):$$

and S and V are given by the equations

$$2S = -\rho - \sigma,$$
$$V = \rho\sigma - RT.$$

A special case will suffice as an illustration. Let

$$\rho = ap, \quad \sigma = cq:$$

then

$$\frac{\partial T}{\partial y} - cq\frac{\partial T}{\partial p} - R\frac{\partial T}{\partial q} = -acp,$$
$$\frac{\partial R}{\partial x} - T\frac{\partial R}{\partial p} - ap\frac{\partial R}{\partial q} = -acq.$$

Without attempting to obtain the general values of R and T, we note that the values

$$R = \lambda p, \quad T = \mu q,$$

satisfy the equations, provided

$$\lambda\mu = ac.$$

Thus the equation

$$rt - s^2 + \lambda pr - (ap + cq)s + \mu qt = 0,$$

that is, the equation

$$(r + \mu q)(t + \lambda p) = (s + ap)(s + cq),$$

where $\lambda\mu = ac$, possesses an intermediate integral involving an arbitrary function.

The construction of the intermediate integral is left as an exercise.

Ex. 2. Prove that the equation

$$(xp + yq)(rt - s^2) + aq^2r + (a + c)pq + cp^2t = 0,$$

where

$$\frac{1}{a} + \frac{1}{c} = 2,$$

has an intermediate integral involving an arbitrary function: and obtain the integral.

AMPÈRE'S THEOREM ON AN INTERMEDIATE INTEGRAL.

244. Before proceeding to consider the properties of equations which possess two intermediate integrals each involving an arbitrary function, one characteristic property of equations possessing only a single intermediate integral may be noticed here. It was first obtained by Ampère, and it is as follows:—

When the differential equation possesses an intermediate integral involving an arbitrary function, so that the qualifying conditions are satisfied for one of the two systems of subsidiary equations, then the Charpit relations leading to the integration of the intermediate integral include the other system of subsidiary equations.

Denote by v and w the independent integrals of the subsidiary system

$$\Delta(u) = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0,$$

$$\Delta'(u) = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0,$$

$$\Delta''(u) = \frac{\partial u}{\partial z} - P\frac{\partial u}{\partial p} - Q\frac{\partial u}{\partial q} = 0,$$

the system being complete: then the intermediate integral can be taken in the form

$$F(v, w) = 0,$$

where F is any arbitrary function; and the three equations are satisfied when F is substituted for u. Now $F = 0$ is an equation of the first order; to integrate it, we form the Charpit subsidiary equations (§ 68)

$$\frac{dx}{\dfrac{\partial F}{\partial p}} = \frac{dy}{\dfrac{\partial F}{\partial q}} = \frac{dz}{p\dfrac{\partial F}{\partial p} + q\dfrac{\partial F}{\partial q}} = \frac{-dp}{\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z}} = \frac{-dq}{\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z}},$$

and we need to obtain some integral of these equations. When we substitute for the denominators of the last two fractions, we have the modified set

$$\frac{dx}{\dfrac{\partial F}{\partial p}} = \frac{dy}{\dfrac{\partial F}{\partial q}} = \frac{dz}{p\dfrac{\partial F}{\partial p} + q\dfrac{\partial F}{\partial q}} = \frac{-dp}{T\dfrac{\partial F}{\partial p} + \rho\dfrac{\partial F}{\partial q}} = \frac{-dq}{\sigma\dfrac{\partial F}{\partial p} + R\dfrac{\partial F}{\partial q}};$$

and therefore the system includes the equations

$$\left.\begin{aligned} dz &= p\,dx + q\,dy \\ -dp &= T\,dx + \rho\,dy \\ -dq &= \sigma\,dx + R\,dy \end{aligned}\right\}.$$

Let U, = constant, be an integral of this set; then U satisfies the equations

$$\frac{\partial U}{\partial x} + p\frac{\partial U}{\partial z} - T\frac{\partial U}{\partial p} - \sigma\frac{\partial U}{\partial q} = 0,$$

$$\frac{\partial U}{\partial y} + q\frac{\partial U}{\partial z} - \rho\frac{\partial U}{\partial p} - R\frac{\partial U}{\partial q} = 0.$$

These are the equations of the other subsidiary system: hence Ampère's proposition.

We have supposed (though the supposition does not affect the preceding analysis and is postulated solely as providing the least favourable circumstances) that the qualifying conditions are not satisfied for this alternative subsidiary system. On that hypothesis, two independent integrals of this subsidiary system do not exist, for otherwise they would lead to an additional intermediate integral: but the subsidiary system may possess one integral, and that integral is one of the integrals of the Charpit equations. Now this is precisely what is required for the integration of the intermediate integral: we need one integral of those equations, which shall be distinct from $F=0$ and shall involve p or q.

It therefore appears that, if only a single integral of the alternative subsidiary system can be obtained, say

$$U = c,$$

it can be combined with the intermediate integral

$$F(v, w) = 0$$

so as to give values of p and q which, when substituted in the relation

$$dz = p\,dx + q\,dy,$$

make that relation exact. Quadrature of this exact equation leads to the primitive.

If the alternative system should lead to an intermediate integral

$$G(U_1, U_2) = 0,$$

where G is an arbitrary function, it is clear that special primitives are derivable from any of the combinations

$$\left.\begin{array}{l} v=a \\ U_1=\alpha \end{array}\right\}, \quad \left.\begin{array}{l} w=b \\ U_1=\alpha \end{array}\right\}, \quad \left.\begin{array}{l} v=\alpha \\ U_2=\beta \end{array}\right\}, \quad \left.\begin{array}{l} w=b \\ U_2=\beta \end{array}\right\}.$$

Ex. 1. Consider the equation

$$rt-s^2+pr-(p+q)s+qt=0.$$

The equation for μ is

$$\mu^2-\mu(p+q)+pq=0,$$

so that

$$\rho,\ \sigma=p,\ q.$$

With the assignment

$$\rho=p, \quad \sigma=q,$$

the equations leading to the intermediate integral (if any) are

$$\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0,$$

$$\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0.$$

The condition of coexistence is found to be

$$\frac{\partial u}{\partial z}=0;$$

and this equation, together with

$$\frac{\partial u}{\partial x}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0,$$

$$\frac{\partial u}{\partial y}-q\frac{\partial u}{\partial p}-p\frac{\partial u}{\partial q}=0,$$

makes a complete Jacobian system. Two independent integrals are easily found to be

$$p^2-q^2, \quad (p+q)e^{x+y},$$

so that an intermediate integral is

$$(p+q)e^{x+y}=f(p^2-q^2),$$

where f is an arbitrary function.

With the assignment

$$\rho=q, \quad \sigma=p,$$

the integrals leading to an intermediate integral (if any) are

$$\frac{\partial U}{\partial x}+p\frac{\partial U}{\partial z}-q\frac{\partial U}{\partial p}-q\frac{\partial U}{\partial q}=0,$$

$$\frac{\partial U}{\partial y}+q\frac{\partial U}{\partial z}-p\frac{\partial U}{\partial p}-p\frac{\partial U}{\partial q}=0.$$

The condition of coexistence is found to be

$$\frac{\partial U}{\partial z}-\frac{\partial U}{\partial p}-\frac{\partial U}{\partial q}=0;$$

this equation, together with

$$\frac{\partial U}{\partial x}+(p-q)\frac{\partial U}{\partial z}=0,$$

$$\frac{\partial U}{\partial y}-(p-q)\frac{\partial U}{\partial z}=0,$$

makes a complete Jacobian system. Two independent integrals are easily found to be

$$z+p-(p-q)(x-y), \quad p-q,$$

so that an intermediate integral is

$$z+p-(p-q)(x-y)=g(p-q),$$

where g is an arbitrary function.

It is easy to verify that

$$p-q, \quad z+p-(p-q)(x-y),$$

are integrals of the Charpit system subsidiary to the integration of

$$(p+q)e^{x+y}=f(p^2-q^2),$$

and that

$$p^2-q^2, \quad (p+q)e^{x+y},$$

are integrals of the Charpit system subsidiary to the integration of

$$z+p-(p-q)(x-y)=g(p-q).$$

The construction of various primitives is left as an exercise.

Corresponding properties and limitations belong to the equation

$$Rr+2Ss+Tt=V;$$

the equations for the determination of an intermediate integral (if any) are

$$\Delta'(u)=\frac{\partial u}{\partial q}-\rho\frac{\partial u}{\partial p}=0,$$

$$\Delta(u)=\frac{du}{dx}+\sigma\frac{du}{dy}+\frac{V}{R}\frac{\partial u}{\partial p}=0,$$

where ρ and σ are the roots of the quadratic

$$R\mu^2-2S\mu+T=0:$$

and, when the roots of the quadratic are unequal, there are two sets of equations in u corresponding to the two assignments of the unequal roots.

The development of the properties and limitations follows exactly the development in the case of the equation

$$rt-s^2+Rr+2Ss+Tt=V,$$

and so need not be given in detail: we subjoin some of the results.

Ex. 2. Prove that, if the equation possesses three independent intermediate integrals of the form

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3,$$

then

$$RT = S^2.$$

Ex. 3. Shew that, if the equation

$$r + 2Ss + S^2 t = V$$

possesses three independent intermediate integrals, then S is determined by an equation

$$f(S,\ p + qS,\ x,\ y,\ z) = 0,$$

where f is any arbitrary function (which must, however, involve S): and that V is determined by an equation

$$\frac{\partial V}{\partial q} - S\frac{\partial V}{\partial p} + V\frac{\partial S}{\partial p} + \frac{\partial S}{\partial x} + p\frac{\partial S}{\partial z} + S\frac{\partial S}{\partial y} + qS\frac{\partial S}{\partial z} = 0.$$

Prove that, if

$$S = -\frac{a+p}{b+q},$$

then

$$V = (a+p)\, g\left(\frac{a+p}{b+q},\ x,\ y,\ z\right),$$

where g is any function of its arguments; and determine the form of V when

$$S = F(x,\ y,\ z).$$

Ex. 4. Integrate the equations :—

(i) $$r - 2\frac{a+p}{b+q}s + \left(\frac{a+p}{b+q}\right)^2 t = \lambda\,(b+q)\,z,$$

where λ is a constant;

(ii) $$r + 2zs + z^2 t = (z-q)\,(p+qz).$$

Ex. 5. Shew that, if the equation

$$r + 2Ss + Tt = V$$

possesses an intermediate integral involving an arbitrary function, and if $S^2 - T$ is not zero; also, if

$$\Delta = \frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} + (p + \sigma q)\frac{\partial}{\partial z} + V\frac{\partial}{\partial p},$$

$$\Delta' = \frac{\partial}{\partial q} - \rho\frac{\partial}{\partial p},$$

$$\Delta'' = (1 + q\lambda)\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial y} + P\frac{\partial}{\partial p},$$

where ρ and σ (in one or other of the two possible assignments) are the roots of the quadratic

$$\mu^2 - 2S\mu + T = 0,$$

and where the quantities λ and P are given by

$$\lambda = \frac{\Delta'(\sigma)}{\sigma - \rho}, \quad P = \frac{\Delta'(V) + \Delta(\rho)}{\sigma - \rho};$$

then the necessary and sufficient conditions for the possession of the specified integral are that the equations

$$\left.\begin{array}{r} \Delta''(\sigma) - \Delta(\lambda) = 0 \\ \Delta''(\rho) + \Delta'(P) = \lambda P \\ \Delta''(V) - \Delta(P) = 0 \\ \Delta'(\lambda) = \lambda^2 \end{array}\right\}$$

are satisfied.

Can these equations be equivalent to only two independent conditions?

Ex. 6. Shew that the equation

$$r - sp\left(q + \frac{1}{q}\right) + tp^2 = pf(pq),$$

where f is any function, possesses an intermediate integral: and integrate the equation

$$r - sp\left(q + \frac{1}{q}\right) + tp^2 = p(p^2q^2 - 1).$$

One Integral common to the Subsidiary System.

245. In § 243, it was seen that the equation

$$rt - s^2 + Rr + 2Ss + Tt = V$$

possesses an intermediate integral of the type contemplated by Monge and by Boole, if the system of equations denoted by

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

is a complete Jacobian system: and the necessary conditions were duly set out, together with the limitation that the quadratic

$$\mu^2 + 2\mu S + RT + V = 0$$

should have unequal roots. If, however, in any given case the quadratic has equal roots: or if only some, but not all, of the essential conditions are satisfied for each of the unequal roots: then the system of three equations is not a complete Jacobian system. In either case, the three equations do not possess two independent integrals: and consequently the original equation possesses no intermediate integral that involves an arbitrary function.

Suppose then, that, the three equations are not a complete Jacobian system; any further analytical developments have no significance in connection with the problem as initiated by Monge and by Boole. They, however, do possess significance for the problem as propounded in § 238: for there we are concerned with equations which possess an intermediate integral of any kind whatever, there being no limitation and no requirement as to its functional character. An example was given in which an equation, definitely not of the postulated form, possessed an intermediate integral involving two arbitrary constants: and it is easy to see that an equation

$$u(x, y, z, p, q, a, b) = 0,$$

where a and b are arbitrary constants, can be an intermediate integral of an appropriate equation

$$F(x, y, z, p, q, r, s, t) = 0.$$

The matter will be sufficiently illustrated by briefly continuing the development of the analysis, which is connected with the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

on the hypothesis that the equations

$$\Delta(u) = 0, \quad \Delta'(u) = 0, \quad \Delta''(u) = 0,$$

do not constitute a complete Jacobian system. Two cases have to be considered, according as the quadratic

$$\mu^2 + 2\mu S + RT + V = 0$$

does not, or does, possess equal roots.

Assuming that the roots of the quadratic are unequal, so that $\rho - \sigma$ does not vanish, we have the three equations in the form

$$\left.\begin{aligned} \Delta(u) &= \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0 \\ \Delta'(u) &= \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0 \\ \Delta''(u) &= \frac{\partial u}{\partial z} - P\frac{\partial u}{\partial p} - Q\frac{\partial u}{\partial q} = 0 \end{aligned}\right\},$$

where

$$P = \frac{\Delta'(T) - \Delta(\sigma)}{\rho - \sigma}, \quad Q = \frac{\Delta'(\rho) - \Delta(R)}{\rho - \sigma},$$

and

$$(\Delta, \Delta') = -(\rho - \sigma)\,\Delta''.$$

Also

$$(\Delta, \Delta'') = P\frac{\partial u}{\partial z} - \{\Delta(P) - \Delta''(T)\}\frac{\partial u}{\partial p} - \{\Delta(Q) - \Delta''(\rho)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \Delta'') = Q\frac{\partial u}{\partial z} - \{\Delta'(P) - \Delta''(\sigma)\}\frac{\partial u}{\partial p} - \{\Delta'(Q) - \Delta''(R)\}\frac{\partial u}{\partial q};$$

so that $(\Delta, \Delta') = 0$ in virtue of the equations retained, and we must have

$$(\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0,$$

also satisfied because the equations are retained.

Now the case, next in importance after those which already have been discussed, is that in which there is only a single integral common to the system. As there are five independent variables x, y, z, p, q in the system, the complete system will involve four linearly independent equations: hence the two new equations will effectively add one extra equation to the set $\Delta = 0$, $\Delta' = 0$, $\Delta'' = 0$. As derivatives with regard to x and y do not occur in Δ and Δ', it follows that a linear relation connects Δ'', (Δ, Δ''), (Δ', Δ''); hence

$$\Theta = \begin{vmatrix} P^2 - \Delta(P) + \Delta''(T), & PQ - \Delta(Q) + \Delta''(\rho) \\ PQ - \Delta'(P) + \Delta''(\sigma), & Q^2 - \Delta'(Q) + \Delta''(R) \end{vmatrix} = 0.$$

Any one of the four constituents in this determinantal form of Θ may vanish, though no one need vanish; if one does vanish, then $\Theta = 0$ will be satisfied by making one other vanish. But not all four constituents can vanish: for then we have

$$(\Delta, \Delta'') = P\Delta'',$$

$$(\Delta', \Delta'') = Q\Delta'',$$

and the complete system would consist of three equations, thus leading to the preceding case.

Moreover, as there is a linear relation connecting Δ'', (Δ, Δ''), (Δ', Δ''), there is effectively one new equation, so that the system has become

$$\Delta = 0, \quad \Delta' = 0, \quad \Delta'' = 0, \quad \Delta''' = 0.$$

The conditions

$$(\Delta, \Delta') = 0, \quad (\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0,$$

are satisfied; we shall assume that the further necessary conditio
$$(\Delta, \Delta''')=0, \quad (\Delta', \Delta''')=0, \quad (\Delta'', \Delta''')=0,$$
also are satisfied; the system is then complete, and it possess one integral. Consequently, *the subsidiary system has one integr common to all its equations if, when the roots of the quadratic a unequal, certain relations are satisfied: one of these is that t equation*
$$\Theta=0$$
should be satisfied, without all the constituents in the determinant form of Θ *vanishing.* Let u denote this common integral: then
$$u=a,$$
where a is an arbitrary constant, is an intermediate integral the differential equation.

Next, let the roots of the quadratic be equal, so that th equation is
$$rt-s^2+Rr+2Ss+Tt=S^2-RT,$$
or, what is the same thing,
$$(r+T)(t+R)=(s-S)^2:$$
then $\rho=\sigma=-S$, and
$$\Delta(u)=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}-T\frac{\partial u}{\partial p}+S\frac{\partial u}{\partial q}=0,$$
$$\Delta'(u)=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+S\frac{\partial u}{\partial p}-R\frac{\partial u}{\partial q}=0.$$
Now both the quantities A and B, where
$$A=\Delta'(T)+\Delta(S), \quad B=\Delta'(S)+\Delta(R),$$
do not vanish, for then $\Delta=0$ and $\Delta'=0$ would be a complet system; suppose that A does not vanish, and take
$$-\mu=\frac{B}{A}=\frac{\Delta'(S)+\Delta(R)}{\Delta'(T)+\Delta(S)}.$$
Then we have an equation
$$\nabla(u)=\frac{\partial u}{\partial p}+\mu\frac{\partial u}{\partial q}=0,$$
and
$$(\Delta, \Delta')=\{\Delta'(T)+\Delta(S)\}\nabla.$$

Also, as before, we have

$$(\Delta, \nabla) = \frac{\partial u}{\partial z} - \nabla(T)\frac{\partial u}{\partial p} + \{\nabla(S) - \Delta(\mu)\}\frac{\partial u}{\partial q},$$

$$(\Delta', \nabla) = \mu\frac{\partial u}{\partial z} + \nabla(S)\frac{\partial u}{\partial p} - \{\nabla(R) + \Delta'(\mu)\}\frac{\partial u}{\partial q};$$

so that $(\Delta, \Delta') = 0$ in virtue of $\nabla = 0$: and we must have

$$(\Delta, \nabla) = 0, \quad (\Delta', \nabla) = 0,$$

if $\nabla = 0$ is to coexist with $\Delta = 0$, $\Delta' = 0$.

For the present purpose, the complete Jacobian system is to contain four linearly independent equations; and therefore as derivatives with regard to x and y occur only in Δ and Δ', there must be one linear relation connecting ∇, (Δ, ∇), (Δ', ∇). The necessary and sufficient condition is easily found to be

$$\Phi = \mu^2\nabla(T) + 2\mu\nabla(S) - \mu\Delta(\mu) + \Delta'(\mu) + \nabla(R) = 0.$$

Moreover, we then have effectively one new equation, so that the system can be taken in a form

$$\Delta = 0, \quad \Delta' = 0, \quad \nabla = 0, \quad \Delta'' = 0.$$

The conditions

$$(\Delta, \Delta') = 0, \quad (\Delta, \nabla) = 0, \quad (\Delta', \nabla) = 0,$$

are satisfied, in virtue of the system; we shall assume that the further necessary conditions

$$(\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0, \quad (\nabla, \Delta'') = 0,$$

also are satisfied. The system then is complete, and so it possesses one integral. Hence *the subsidiary system of the equation*

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT$$

has one integral common to its equations if the relation

$$\Phi = 0$$

is satisfied, as well as certain other relations. Let v be this common integral: then

$$v = c,$$

where c is an arbitrary constant, is an intermediate integral of the differential equation.

Note. In the case of both these intermediate integrals

$$u = a, \quad v = c,$$

the *intermediate integral leads, not to a single equation, but to two equations of the second order*

$$\left.\begin{array}{l} u_x + u_p r + u_q s = 0 \\ u_y + u_p s + u_q t = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} v_x + v_p r + v_q s = 0 \\ v_y + v_p s + v_q t = 0 \end{array}\right\};$$

from each of these, the single equation can be compounded.

Ex. 1. The equation

$$r + 2Ss + S^2 t = V$$

possesses an intermediate integral, involving an arbitrary constant in the form $u(x, y, z, p, q) = a$, but not involving an arbitrary function: prove that, if

$$\Delta = \frac{\partial}{\partial x} + S\frac{\partial}{\partial y} + (p + qS)\frac{\partial}{\partial z} + V\frac{\partial}{\partial p},$$

$$\Delta' = \frac{\partial}{\partial q} - S\frac{\partial}{\partial p},$$

$$\Delta'' = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + L\frac{\partial}{\partial p},$$

where

$$L = \frac{\Delta(S) + \Delta'(V)}{\Delta'(S)},$$

sufficient conditions are:—

(i) that $\Delta'(S)$ shall not vanish,

(ii) that the relation

$$L\Delta''(S) + \Delta(L) = \Delta''(V)$$

must be satisfied. Are these conditions necessary?

Ex. 2. The equation

$$r + 2Ss + Tt = V$$

possesses an intermediate integral involving an arbitrary constant in the form $u(x, y, z, p, q) = a$, but not involving an arbitrary function; and the roots, ρ and σ, of the quadratic $\mu^2 - 2S\mu + T = 0$ are unequal. Also, let

$$\Delta = \frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} + (p + \sigma q)\frac{\partial}{\partial z} + V\frac{\partial}{\partial p},$$

$$\Delta' = \frac{\partial}{\partial q} - \rho\frac{\partial}{\partial p},$$

$$\Delta'' = (1 + q\lambda)\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial y} + P\frac{\partial}{\partial p},$$

where

$$\lambda = \frac{\Delta'(\sigma)}{\sigma - \rho}, \quad P = \frac{\Delta'(V) + \Delta(\rho)}{\sigma - \rho}.$$

Prove that the necessary and sufficient condition is that the equation

$$\Psi = \begin{vmatrix} \Delta(\lambda) - \Delta''(\sigma), & \Delta(P) - \Delta''(V) \\ \Delta'(\lambda) - \lambda^2 \quad , & \Delta'(P) + \Delta''(\rho) - \lambda P \end{vmatrix} = 0$$

shall be satisfied, without the vanishing of the four constituents in the determinant.

Ex. 3. Integrate (so far as to obtain an intermediate integral involving an arbitrary constant) the equations:—

(i) $rt - s^2 + (z - px - qy)(r - s) - pq(s - t) = 0$;

(ii) $r + 2Ss + S^2 t = z - qy$,

where S has the forms

(*a*) $S = qe^x - y$,

(*b*) $S = qe^{-x} + y$,

(*c*) $S = (q^2 + y^2)^{\frac{1}{2}}$.

NO INTEGRAL COMMON TO THE SUBSIDIARY SYSTEM.

246. Finally, it may happen that the condition

$$\Theta = 0$$

is not satisfied for the equation

$$rt - s^2 + Rr + 2Ss + Tt = V;$$

then the system can be replaced by

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0.$$

The possibility of the method has depended on the assumption that some intermediate integral exists, so that $\frac{\partial u}{\partial p}$ and $\frac{\partial u}{\partial q}$ do not vanish together. In the present circumstances, therefore, the equation does not possess an intermediate integral.

Similarly, when the condition

$$\Phi = 0$$

is not satisfied for the equation

$$(r + T)(t + R) = (s - S)^2,$$

no intermediate integral exists. Likewise, the equation

$$r + 2Ss + S^2t = V$$

possesses no intermediate integral when the condition

$$L\Delta''(S) + \Delta(L) = \Delta''(V)$$

is not satisfied, nor the equation

$$r + 2Ss + Tt = V$$

when the condition

$$\Psi = 0$$

is not satisfied.

In each of these cases, when the last condition for the existence of an intermediate integral involving only one arbitrary constant is not satisfied, we are led to the conclusion that the intermediate integral does not exist because of the equations

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0.$$

It is, however, not to be inferred that no integral of the form

$$u(x, y, z) = 0$$

exists; for our analysis has depended upon a non-zero value for $\frac{\partial u}{\partial p}$ or $\frac{\partial u}{\partial q}$ or both. All that can be inferred is that it is useless to proceed to the integration of the equation by attempting to determine an intermediate integral. Some other method for the integration of the equation must be devised, which does not depend upon the use of any supposed intermediate integral and which must avoid any assumption of this character. Such a method is the process constructed by Ampère, to which accordingly we shall now proceed.

Ex. 1. Integrate completely the equation

$$(1-xy)\{xr-(1+xy)s+yt\}+(1-x)p+(1-y)q=0. \qquad \text{(Boehm.)}$$

[The primitive is

$$z=\phi(xe^y)+\psi(ye^x).]$$

Ex. 2. Obtain an integral of the equation

$$2xqr+(yp-2xq)s+2ypt+2xy(rt-s^2)+3pq=0$$

in the form

$$z=ax^\lambda+by^\mu+c,$$

where a, b, c, λ, μ are arbitrary constants subject to the single relation

$$2\lambda\mu+1=0. \qquad \text{(Dixon.)}$$

Ex. 3. Shew that, if the equation

$$r-\lambda^2 t+\mu=0,$$

where λ and μ can be functions of x, y, z, p, q, possesses two intermediate integrals, then λ must be of the form

$$\frac{a+2bp+cp^2}{f+2gq+hq^2},$$

where the six functions a, b, c, f, g, h of the variables x, y, z are subject to the relation

$$b^2-ac=g^2-fh,$$

and that μ must be of the form

$$\mu=\lambda^2(F+2Gq+Hq^2)+A+2Bp+Cp^2,$$

where A, B, C, F, G, H are functions of x, y, z.

Obtain the complete expressions when a, b, c, f, g, h are constants: and construct the primitives of the equations so determined. (Kapteyn.)

Ex. 4. Integrate the equation

$$qr+(zq-p)s-pzt=0. \qquad \text{(Goursat.)}$$

Ex. 5. Shew that, if the equation

$$r=\lambda^2 t,$$

where λ is a function of x and y only, possesses an intermediate integral of the form $f(u, v)=0$, in which f denotes an arbitrary function, the quantity λ must satisfy an equation of the second order: and prove that the most general value of λ which satisfies this equation is given by the elimination of a between the equations

$$\left.\begin{aligned}\lambda\{x-\phi(a)\}&=y-\psi(a)\\ y\{x-\phi(a)\}&=-a+x\psi(a)\end{aligned}\right\},$$

where ϕ and ψ are arbitrary functions. (Goursat.)

SUPPLEMENTARY NOTE.

In § 180 a warning was given that the aggregate of the usual classes of integrals, which occur in the solution of various individual equations of the second order, and of the Cauchy integrals which are proved to exist for widely comprehensive classes of equations of the second order, does not necessarily exhaust all the integrals that are possessed. Other integrals may exist which, as in the case (§ 34) of corresponding integrals of equations of the first order, are not included among the kinds of integrals there specified.

An illustration of the warning can be given in the case of an equation

$$Rr+2Ss+Tt=V.$$

Even when it possesses two independent intermediate general integrals

$$f(u, v) = 0, \quad g(u', v') = 0,$$

and, *a fortiori*, when it possesses only a single intermediate general integral, it cannot be proved (and it is not in fact true) that the arbitrary function f or the arbitrary function g can be chosen so that any other intermediate integral is uniquely obtained.

For example, let

$$\vartheta(x, y, z, p, q) = 0$$

be any intermediate integral of the equation in question, supposed to possess a general intermediate integral

$$f(u, v) = 0.$$

Because the terms $rt - s^2$ are absent from the differential equation, we have (as in § 236)

$$J\left(\frac{u, v}{p, q}\right) = 0,$$

as may easily be verified by eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between the equations

$$\frac{\partial f}{\partial u}\left(\frac{du}{dx} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q}\right) + \frac{\partial f}{\partial v}\left(\frac{dv}{dx} + r\frac{\partial v}{\partial p} + s\frac{\partial v}{\partial q}\right) = 0,$$

$$\frac{\partial f}{\partial u}\left(\frac{du}{dy} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q}\right) + \frac{\partial f}{\partial v}\left(\frac{dv}{dy} + s\frac{\partial v}{\partial p} + t\frac{\partial v}{\partial q}\right) = 0.$$

Hence the two equations

$$u(x, y, z, p, q) = u, \quad v(x, y, z, p, q) = v,$$

cannot be resolved so as to express p and q in terms of x, y, z, u, v. But in the absence of any conditions upon R, S, T, V, other than those in Ex. 5, § 244, which secure the existence of u and v as integrals of the subsidiary equations, we can conceive the two equations resolved so as to express any other two of the arguments in terms of the remainder: say

$$x = g(z, p, q, u, v), \quad y = h(z, p, q, u, v).$$

Let these expressions be substituted in ϑ, provided $\vartheta = 0$ is not a singularity of u or of v: and let the modified form of the intermediate integral be

$$\vartheta(x, y, z, p, q) = \theta(z, p, q, u, v) = 0;$$

then, if the general intermediate integral is to become the given intermediate integral $\theta = 0$, the arguments z, p, q should not occur in θ, the necessary and sufficient conditions being that the relations

$$\frac{\partial \theta}{\partial z} = 0, \quad \frac{\partial \theta}{\partial p} = 0, \quad \frac{\partial \theta}{\partial q} = 0,$$

should be satisfied identically.

Now, from the intermediate integral in the form $\theta = 0$, we have

$$\frac{\partial \theta}{\partial z} p + \frac{\partial \theta}{\partial p} r + \frac{\partial \theta}{\partial q} s + \frac{\partial \theta}{\partial u}\left(\frac{du}{dx} + \frac{\partial u}{\partial p} r + \frac{\partial u}{\partial q} s\right) + \frac{\partial \theta}{\partial v}\left(\frac{dv}{dx} + \frac{\partial v}{\partial p} r + \frac{\partial v}{\partial q} s\right) = 0,$$

$$\frac{\partial \theta}{\partial z} q + \frac{\partial \theta}{\partial p} s + \frac{\partial \theta}{\partial q} t + \frac{\partial \theta}{\partial u}\left(\frac{du}{dy} + \frac{\partial u}{\partial p} s + \frac{\partial u}{\partial q} t\right) + \frac{\partial \theta}{\partial v}\left(\frac{dv}{dy} + \frac{\partial v}{\partial p} s + \frac{\partial v}{\partial q} t\right) = 0.$$

As in § 236, we denote the (unequal) roots of the quadratic

$$R\mu^2 - 2S\mu + T = 0$$

by ρ and σ. Multiplying the second of the derivatives of $\theta = 0$ by σ, and adding to the first, we have the combined relation

$$\begin{aligned} &\frac{\partial \theta}{\partial z}(p + \sigma q) + \frac{\partial \theta}{\partial p}(r + \sigma s) + \frac{\partial \theta}{\partial q}(s + \sigma t) \\ &\quad + \frac{\partial \theta}{\partial u}\left\{\frac{du}{dx} + \sigma \frac{du}{dy} + \frac{\partial u}{\partial p}(r + \sigma s) + \frac{\partial u}{\partial q}(s + \sigma t)\right\} \\ &\quad + \frac{\partial \theta}{\partial v}\left\{\frac{dv}{dx} + \sigma \frac{dv}{dy} + \frac{\partial v}{\partial p}(r + \sigma s) + \frac{\partial v}{\partial q}(s + \sigma t)\right\} = 0, \end{aligned}$$

which, on taking account of the subsidiary partial equations of the first order satisfied by u and by v, becomes

$$\begin{aligned} &\frac{\partial \theta}{\partial z}(p + \sigma q) + \frac{\partial \theta}{\partial p}(r + \sigma s) + \frac{\partial \theta}{\partial q}(s + \sigma t) \\ &\qquad + \frac{1}{R}\left(\frac{\partial \theta}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial \theta}{\partial v}\frac{\partial v}{\partial p}\right)\{-V + Rr + 2Ss + Tt\} = 0. \end{aligned}$$

The values of r, s, t consistent with $\theta = 0$ satisfy the original differential equation; hence the relation

$$\frac{\partial \theta}{\partial z}(p + \sigma q) + \frac{\partial \theta}{\partial p}(r + \sigma s) + \frac{\partial \theta}{\partial q}(s + \sigma t) = 0$$

must be satisfied, in association with the original differential equation and with the intermediate integral

$$\theta(z, p, q, u, v) = 0,$$

together with derivatives from the intermediate integral. It is clear that the conditions

$$\frac{\partial\theta}{\partial z} = 0, \quad \frac{\partial\theta}{\partial p} = 0, \quad \frac{\partial\theta}{\partial q} = 0,$$

cannot be assigned as identical equations merely as necessary consequences of the relation just obtained: and therefore we conclude that the general intermediate integral, when it is possessed, is not completely (though it may be largely) comprehensive of all the intermediate integrals that may be possessed.

A fortiori, a similar doubt extends as to the comprehensiveness of a general primitive.

Two examples will suffice.

Ex. 1. The equation

$$x^2r - y^2t = (px + qy - z)^2$$

has an intermediate general integral

$$f(ye^{\frac{-1}{px+qy-z}}, xy) = 0.$$

It also possesses a special intermediate integral

$$\vartheta = px + qy - z = 0.$$

Manifestly no form of the function f can be devised which will change $f=0$ into $\vartheta=0$: and even the preliminary transformation of ϑ into θ, as in the text, cannot be accomplished, for $\vartheta=0$ provides an essential singularity of one of the arguments of f.

Ex. 2. The equation

$$x(r+s) - y(s+t) = p + q - xy$$

has an intermediate general integral

$$f(u, v) = 0,$$

where

$$u = x(p+q-xy),$$

$$v = y(p+q-xy),$$

and the value of σ in the text is unity. It also possesses a special intermediate integral

$$\vartheta = p + q - xy = 0.$$

Effecting upon ϑ the transformation which changes it into θ, we find

$$\theta = p + q - \rho,$$

where ρ is given in terms of p, q, u, v by the equation

$$(p+q-\rho)^2 \rho - uv = 0.$$

Thus θ does not involve z: but it does involve p and q: and we therefore do not have

$$\frac{\partial \theta}{\partial p} = 0, \quad \frac{\partial \theta}{\partial q} = 0,$$

satisfied identically, that is, the intermediate general integral does not comprehend the intermediate special integral.

The equation of condition comes to be (with the value of σ)

$$\frac{\partial \theta}{\partial p}(r+s) + \frac{\partial \theta}{\partial q}(s+t) = 0:$$

it is easy to verify that the quantities $\frac{\partial \theta}{\partial p}$, $\frac{\partial \theta}{\partial q}$, while not vanishing identically, do vanish in virtue of $\theta = 0$: and so the equation of condition is satisfied.

CHAPTER XVII.

Ampère's Method applied to Equations of the Second Order in Two Independent Variables.

The method, which Ampère constructed for the integration of partial differential equations, is contained in the two important memoirs presented* to the Institute of France in 1814. The application of the method to equations of the first order is now relatively unimportant, owing to the subsequent discovery of other methods of treating such equations. The application to equations of the second order is still of fundamental importance. The memoirs seem more complicated than they are in fact, the principal cause being the cumbrous character of the notation.

Reference, in this connection, should also be made to the valuable memoir by Imschenetsky†, who was among the earliest writers to indicate the importance of Ampère's researches on the subject.

Limitations on the Integral.

247. The methods given by Monge and by Boole were applied by them to equations of a restricted form: and the assumption that an intermediate integral exists is essential to the practical success of their methods. As has been seen in the last chapter, equations of that restricted form occur which do not satisfy all the conditions needed to justify the assumption; and accordingly it follows that those methods are of limited application.

The method devised by Ampère, though explained only for equations of the second order involving two independent variables, and illustrated mostly by application to equations of the forms

* They are contained in the *Journal de l'École Polytechnique*; one of them, in *Cahier* XVII (1815), pp. 549—611, deals with Ampère's general theory; the other, in *Cahier* XVIII (1819), pp. 1—188, contains the application of the theory to particular equations.

† *Grunert's Archiv*, t. LIV (1872), pp. 209—360.

discussed by Monge and subsequently by Boole, can be extended to equations in any number of independent variables and of any order: moreover, it neither makes nor requires any initial assumption as to the character of the equation or to the existence of an intermediate integral. Further, when the method proves effective for the practical construction of the primitive of a given equation, the primitive is not necessarily provided by means of a single explicit equation between the variables in finite form: but it need hardly be remarked that generality of the equations to be treated by a method is more important than simplicity of form in the primitive.

In the present chapter we shall deal with equations of the second order involving two independent variables; and we shall begin with the general equation

$$f(x, y, z, p, q, r, s, t) = 0,$$

where a sufficiently wide class of equations will be provided by supposing that f is merely polynomial in r, s, t. The integral provided by Cauchy's theorem contains two arbitrary functions which may have definite arguments: we shall assume that the arguments are definite. The arguments may be different from one another, or they may be the same as one another; and derivatives of the arbitrary functions with respect to their arguments may occur. We shall assume that the highest derivative of an arbitrary function, which occurs in the integral system, is of finite order and that the integral system is free from partial quadratures which essentially cannot be performed. Lastly, we shall assume that the occurrence of the derivatives of the arbitrary functions is of such a character that (§ 181) the formation of the derivatives of z of successively increasing orders introduces derivatives of the arbitrary functions of successively increasing orders*: but there is no assumption as to the existence of an intermediate integral.

AMPÈRE'S METHOD.

248. Ampère's method is based upon a transformation of the independent variables as the stage of initial departure. Let a new independent variable α be introduced; it is not made determinate until the effect of the transformation is being considered. This

* This aggregate of conditions should be compared with the aggregate of conditions in § 221, where, however, it is specified that the integral shall be given by a single equation resoluble with regard to the dependent variable.

variable α may be a function of both variables x and y, though it may involve not more than one of the variables: let it be used to make x and α the independent variables so that, in the least restricted circumstances, y is a function of x and α. Denoting partial differentiations by $\frac{\delta}{\delta x}$ and $\frac{\delta}{\delta \alpha}$ when x and α are the independent variables, we have

$$\frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta \alpha} d\alpha = du$$
$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y}\left(\frac{\delta y}{\delta x} dx + \frac{\delta y}{\delta \alpha} d\alpha\right),$$

for any function u; and therefore

$$\frac{\delta u}{\delta x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\delta y}{\delta x}, \qquad \frac{\delta u}{\delta \alpha} = \frac{\partial u}{\partial y}\frac{\delta y}{\delta \alpha}.$$

Consequently,

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}, \qquad \frac{\delta z}{\delta \alpha} = q\frac{\delta y}{\delta \alpha},$$

$$\frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x}, \qquad \frac{\delta p}{\delta \alpha} = s\frac{\delta y}{\delta \alpha},$$

$$\frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x}, \qquad \frac{\delta q}{\delta \alpha} = t\frac{\delta y}{\delta \alpha}.$$

Then, keeping the value of t given by

$$t = \frac{\delta q}{\delta \alpha} \div \frac{\delta y}{\delta \alpha},$$

we have

$$s = \frac{\delta q}{\delta x} - t\frac{\delta y}{\delta x},$$

$$r = \frac{\delta p}{\delta x} - \frac{\delta q}{\delta x}\frac{\delta y}{\delta x} + t\left(\frac{\delta y}{\delta x}\right)^2.$$

Let these values of r and s be substituted in $f = 0$, when f will become a polynomial in t: let this polynomial be arranged in powers of t, so that the equation then is

$$P_0 + P_1 t + \ldots + P_n t^n = 0,$$

where the original degree of f, as a polynomial in r, s, t, is n at least.

Thus far, the quantity α is quite unrestricted, and so it is completely at our disposal: let it be chosen to be the (as yet unknown) argument of one of the arbitrary functions in the integral system. Now, by the hypothesis concerning the character of the integral system, the quantities p and q contain derivatives of that arbitrary function of one order higher than those which occur in the integral system. When we change the independent variables so that they become x and α, the partial derivatives of p and q with regard to x contain only the same derivatives of the arbitrary function as do p and q: while the partial derivative of q with regard to α (which does occur in the transformed equation, being introduced by t, while the partial derivative of p with regard to α does not occur there) contains a derivative of the arbitrary function of α, that is of order higher by one unit than the derivatives occurring in p, q, $\frac{\delta p}{\delta x}$, $\frac{\delta q}{\delta x}$. Hence, in the transformed equation, the quantity t contains a derivative of the arbitrary function of α of one order higher than the derivatives that occur in $P_0, P_1, \ldots, P_n$. Moreover, the differential equation is to be satisfied identically in connection with the integral system, and this must take place whether the independent variables be x and y or x and α. Taking account of the successive degrees of that highest derivative which occurs in t alone, we see that the equation can be satisfied only if

$$P_n = 0, \quad P_{n-1} = 0, \ldots, P_1 = 0, \quad P_0 = 0.$$

We have seen (§ 186) that $\frac{\delta y}{\delta x}$, which is the derivative of y on the supposition that α is constant, also satisfies the equation

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}\frac{\delta y}{\delta x} + \frac{\partial f}{\partial r}\left(\frac{\delta y}{\delta x}\right)^2 = 0.$$

249. Thus there is a number of simultaneous equations. If these equations are consistent with one another, and with the original equation $f = 0$, regard being paid to the relations between derivatives with reference to the old independent variables and the new, then the equation can possess an integral system of the specified type. The quantities $P_n, P_{n-1}, \ldots, P_1, P_0$ contain z, p, q, and the derivatives $\frac{\delta y}{\delta x}$, $\frac{\delta p}{\delta x}$, $\frac{\delta q}{\delta x}$: also

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}:$$

and therefore the system of equations contains no derivatives with regard to α, so that it can be regarded as a simultaneous system of ordinary equations.

If the arguments of the two arbitrary functions in the integral system are the same, the preceding discussion is complete as regards the inferences to be drawn from the occurrence of the highest argument of an arbitrary function: the inference is merely duplicated by taking each of the arbitrary functions in turn.

If the arguments of the two arbitrary functions in the integral system are different, let the other argument be denoted by β. A corresponding discussion of the equations, after making x and β the independent variables, leads to the same set of equations involving only derivatives with regard to x.

Hence the system, if it is self-consistent, applies to both arguments α and β when they are distinct from one another. As in § 186, they then are distinct integrals of the equation

$$\frac{\partial f}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial f}{\partial s}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial r}\left(\frac{\partial u}{\partial y}\right)^2 = 0.$$

We thus have a number of equations

$$P_n = 0, \quad P_{n-1} = 0, \;\ldots,\; P_1 = 0, \quad P_0 = 0,$$

which are to be consistent with one another and with the original equation in virtue of

$$\frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x}, \quad \frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x}.$$

These equations, other than the original equation, involve $x, y, z, p, q, \frac{\delta y}{\delta x}, \frac{\delta p}{\delta x}, \frac{\delta q}{\delta x}$; and we also have

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}.$$

Hence there cannot be more than three essentially independent equations: otherwise, it would be possible to eliminate $\frac{\delta y}{\delta x}, \frac{\delta p}{\delta x}, \frac{\delta q}{\delta x}$, and to obtain a relation between x, y, z, p, q alone, involving no arbitrary constant. There is no guarantee that such a relation is an intermediate integral of the original equation: even if it is an intermediate integral, it contains no arbitrary element, and therefore it is of the nature of a special integral.

We put aside such special integrals when they exist; and therefore we cannot have more than three independent equations in the subsidiary Ampère system.

The system contains partial derivatives with regard to x alone, these being taken on the supposition that an argument of an arbitrary function is constant in their formation.

The system may be irresoluble in the sense that no simpler system or systems can be formed which are equivalent to it: the inference is that there is only a single argument common to the two arbitrary functions in the general integral, the equation

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}\frac{\delta y}{\delta x} + \frac{\partial f}{\partial r}\left(\frac{\delta y}{\delta x}\right)^2 = 0$$

then having equal roots for $\dfrac{\delta y}{\delta x}$.

The system may be resoluble in the sense that it can be replaced by simpler systems, which are its equivalent. There cannot be more than two such systems, one to be associated with an argument α, the other to be associated with an argument β; in this case, the preceding quadratic has unequal roots.

Whether there be one system or whether there be two systems, the first object is to obtain some integral of a system. When that integral is obtained, the arbitrary element is made either the argument (or some arbitrary function of the argument) associated with the system.

If there is only a single system, further integrals of the system are required, the arbitrary elements being made arbitrary functions of the one argument: the simultaneous integrals provide an integral of the original equation.

If there are two systems, and an integral of each has been obtained, then the arguments α and β are made the independent variables for a new set of equations. Two of these equations are

$$\left.\begin{aligned} \frac{\partial z}{\partial \alpha} &= p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha} \\ \frac{\partial z}{\partial \beta} &= p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta} \end{aligned}\right\};$$

the others of the set are selected from the respective systems, to express relations among some of the partial derivatives of x, y,

p, q, when the respective arguments are constant. This new set of equations is to be treated as a set of simultaneous equations to express, ultimately, x, y, z in terms of α and β: the equations, which give these expressions, constitute an integral of the original differential equation.

The method is illustrated by the following examples, which correspond to various cases.

Ex. 1. Consider the equation*

$$(r-pt)^2=q^2rt.$$

The arguments of the arbitrary function are the same as one another only if the relation

$$4\frac{\partial f}{\partial t}\frac{\partial f}{\partial r}=\left(\frac{\partial f}{\partial s}\right)^2$$

is satisfied concurrently with the equation, that is, if

$$4p+q^2=0.$$

Putting this possibility on one side for the present, we assume that the arguments are different from one another: and we denote them by α and β.

Denoting partial derivatives with regard to x, when α and x are the independent variables, by z', y', p', q', and substituting

$$r=p'-q'y'+ty'^2$$

in the equation, it becomes

$$\{p'-q'y'+t(y'^2-p)\}^2=q^2t(p'-q'y'+ty'^2);$$

and therefore, if the integral system is of the specified finite type, we must have

$$(y'^2-p)^2=q^2y'^2,$$

$$2(p'-q'y')(y'^2-p)=q^2(p'-q'y'),$$

$$(p'-q'y')^2=0,$$

which are satisfied by

$$p'-q'y'=0,$$

$$(y'^2-p)^2=q^2y'^2.$$

These two equations are consistent with one another: and, in virtue of the equation

$$r=p'-q'y'+ty'^2,$$

they are consistent with the original equation. The integral system can therefore be of the specified type.

We require an integral combination of the system, taken with

$$z'=p+qy'.$$

* It is discussed by Ampère in his second memoir, p. 48.

Now we have

$$\left.\begin{aligned} p'-q'y'&=0\\ y'^2-p&=\pm qy'\end{aligned}\right\},$$

and therefore

$$2y'y''-q'y'=\pm(qy''+q'y').$$

Taking the lower sign, we have

$$(2y'+q)\,y''=0;$$

hence* we can take

$$y''=0;$$

and therefore, as the derivatives with regard to x are taken on the supposition that a is constant, we can write

$$y'=a,$$
$$y=ax+\phi(a),$$

where ϕ is an arbitrary function. Also, with this value, we have

$$a^2+qa-p=0;$$

and

$$p'-aq'=0,$$

that is,

$$\begin{aligned} p-aq&=\psi(a)\\ &=a^2,\end{aligned}$$

from the preceding equation.

Next, we take the equations when x and β are the independent variables. Denoting by z_1, y_1, p_1, q_1 the derivatives with regard to x when β is constant, we still have

$$z_1=p+qy_1,$$
$$p_1-q_1y_1=0,$$
$$y_1^2-p=\pm qy_1:$$

but as the lower sign was taken in the former case, we must take the upper sign now, or the two sets of equations will be identical. Hence our equations are

$$\left.\begin{aligned} z_1&=p+qy_1\\ p_1-q_1y_1&=0\\ y_1^2-p&=qy_1\end{aligned}\right\},$$

when β is constant.

Eliminating y_1 between the last two equations, we have

$$\left(\frac{p_1}{q_1}\right)^2-p=q\frac{p_1}{q_1},$$

that is,

$$\left(\frac{dp}{dq}\right)^2-p=q\frac{dp}{dq},$$

or, writing

$$p=Y,\quad q=X,\quad \frac{dp}{dq}=P,$$

* The value $y'=-\frac{1}{2}q$ leads to the temporarily excluded case.

we have an equation in the form

$$Y = P^2 - XP,$$

one of the recognised forms of equations of the first order that are simply integrable. Proceeding in the regular manner, we find

$$\begin{aligned}(3X - 2P)^2 P &= \text{constant}\\ &= \beta,\end{aligned}$$

say, for β has been assumed constant throughout the integrations. Thus

$$(3q - 2y_1)^2 y_1 = \beta$$

is an integrable combination of the system of equations. But

$$y_1^2 - qy_1 = p = a^2 + aq,$$

so that

$$(y_1 + a)(y_1 - a - q) = 0,$$

and therefore

$$y_1 = -a, \text{ or } y_1 = a + q;$$

also

$$a = \tfrac{1}{2}\{-q \pm (q^2 + 4p)^{\frac{1}{2}}\},$$

so that the first pair of values of y_1 are

$$\tfrac{1}{2}\{q \mp (q^2 + 4p)^{\frac{1}{2}}\},$$

and the second pair of values of y_1 are

$$\tfrac{1}{2}\{q \pm (q^2 + 4p)^{\frac{1}{2}}\},$$

that is, the same as the first pair. Hence, without loss of generality, we may take

$$y_1 = -a;$$

and the equations expressing p and q in terms of a and β are

$$(3q + 2a)^2 a = -\beta,$$

$$a^2 + aq - p = 0;$$

while we have

$$y = ax + \phi(a),$$

$$z' = p + qa,$$

$$z_1 = p - qa.$$

Now make x, y, z functions of a and β. Since

$$\frac{\partial y}{\partial x}, \text{ when } a \text{ is constant} = a,$$

$$\frac{\partial y}{\partial x}, \ldots\ldots \beta \ldots\ldots\ldots\ldots\ldots = -a,$$

we have, when a and β are made the variables

$$\frac{\partial y}{\partial a} = -a\frac{\partial x}{\partial a},$$

$$\frac{\partial y}{\partial \beta} = \quad a\frac{\partial x}{\partial \beta}.$$

For convenience, take a^2 in place of a, and $-\beta^2$ in place of β: our equations are

$$\frac{\partial y}{\partial a} = -a^2 \frac{\partial x}{\partial a},$$

$$\frac{\partial y}{\partial \beta} = a^2 \frac{\partial x}{\partial \beta},$$

$$\frac{\partial z}{\partial a} = p \frac{\partial x}{\partial a} + q \frac{\partial y}{\partial a},$$

$$\frac{\partial z}{\partial \beta} = p \frac{\partial x}{\partial \beta} + q \frac{\partial y}{\partial \beta}.$$

Also, the values of p and q are given by

$$(3q + 2a^2)\, a^2 = \beta^2,$$

$$a^4 + a^2 q = p,$$

so that

$$q = -\tfrac{2}{3} a^2 + \tfrac{1}{3} \frac{\beta}{a}, \quad p = \tfrac{1}{3} a^4 + \tfrac{1}{3} a\beta.$$

We have

$$\frac{\partial}{\partial a}\left(a^2 \frac{\partial x}{\partial \beta}\right) = \frac{\partial^2 y}{\partial a\, \partial \beta} = \frac{\partial}{\partial \beta}\left(-a^2 \frac{\partial x}{\partial a}\right),$$

so that

$$a \frac{\partial^2 x}{\partial a\, \partial \beta} + \frac{\partial x}{\partial \beta} = 0.$$

Hence

$$a \frac{\partial x}{\partial \beta} = \theta''(\beta),$$

and therefore

$$x = \frac{1}{a} \theta'(\beta) + \psi(a),$$

where θ and ψ are arbitrary so far as this equation is concerned. Also,

$$dy = -a^2 \frac{\partial x}{\partial a} da + a^2 \frac{\partial x}{\partial \beta} d\beta$$

$$= \{\theta'(\beta) - a^2 \psi'(a)\}\, da + a\theta''(\beta)\, d\beta,$$

and therefore

$$y = a\theta'(\beta) - \int a^2 \psi'(a)\, da.$$

But, with the changed value of a, we have (from the former relation)

$$y = a^2 x + \phi(a^2)$$

$$= a\theta'(\beta) + a^2 \psi(a) + \phi(a^2).$$

Taking a new arbitrary function χ, such that

$$\psi(a) = a\chi'(a) + 2\chi(a),$$

then

$$\int a^2 \psi'(a)\, da = \int \{a^3 \chi''(a) + 3a^2 \chi'(a)\}\, da$$

$$= a^3 \chi'(a);$$

and then

$$\left.\begin{aligned} y &= a\theta'(\beta) - a^3\chi'(a) \\ x &= \frac{1}{a}\theta'(\beta) + a\chi'(a) + 2\chi(a) \end{aligned}\right\}.$$

Lastly,

$$\begin{aligned} dz &= \left(p\frac{\partial x}{\partial a} + q\frac{\partial y}{\partial a}\right)da + \left(p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}\right)d\beta \\ &= (p - qa^2)\frac{\partial x}{\partial a}da + (p + qa^2)\frac{\partial x}{\partial \beta}d\beta \\ &= (-a^2\theta' + a^5\chi'' + 3a^4\chi')\,da + (-\tfrac{1}{3}a^3 + \tfrac{2}{3}\beta)\,\theta''\,d\beta\,; \end{aligned}$$

and therefore

$$z = \tfrac{1}{3}(2\beta - a^3)\,\theta'(\beta) - \tfrac{2}{3}\theta(\beta) + \int(a^5\chi'' + 3a^4\chi')\,da.$$

Taking

$$a\chi'(a) + 2\chi(a) = \mu(a),$$

then

$$a^3\chi'(a) = \int a^2\mu'(a)\,da,$$

and we have

$$\left.\begin{aligned} x &= \frac{1}{a}\theta'(\beta) + \mu(a) \\ y &= a\theta'(\beta) - \int a^2\mu'(a)\,da \\ z &= \tfrac{1}{3}(2\beta - a^3)\,\theta'(\beta) - \tfrac{2}{3}\theta(\beta) + \int a^4\mu'(a)\,da \end{aligned}\right\}.$$

We can effect the apparent partial quadratures after taking a new arbitrary function $\sigma(a)$, such that

$$\mu(a) = \frac{d^4\sigma(a)}{da^4}.$$

The integral system then is of the desired form.

Note 1. We have put the relation

$$q^2 + 4p = 0$$

on one side in the preceding investigation. Now suppose

$$q^2 + 4p = 0,$$

so that, differentiating, we have

$$r = -\tfrac{1}{2}qs, \quad s = -\tfrac{1}{2}qt,$$

and therefore

$$r = \tfrac{1}{4}q^2t = -pt,$$

which are consistent with the original differential equation. Thus the relation

$$q^2 + 4p = 0$$

is of the nature of a special integral: we have

$$z = -\tfrac{1}{4}c^2x + cy + a,$$

where c and a are arbitrary constants.

Note 2. The original equation

$$(r-pt)^2=q^2rt$$

can be resolved into the two equations

$$r-\tfrac{1}{2}t\{2p+q^2+q(4p+q^2)^{\frac{1}{2}}\}=0,$$

and

$$r-\tfrac{1}{2}t\{2p+q^2-q(4p+q^2)^{\frac{1}{2}}\}=0.$$

Now (with the earlier significance of a) we had

$$a^2+aq-p=0,$$

$$y=ax+\phi(a),$$

and therefore

$$2y+qx-x(4p+q^2)^{\frac{1}{2}}=\phi\{(4p+q^2)^{\frac{1}{2}}-q\},$$

which is of the nature of an intermediate integral. When it is differentiated, it leads to

$$2r=t\{2p+q^2-q(4p+q^2)^{\frac{1}{2}}\},$$

the sign of the radical being the same as in the intermediate integral. Thus the first branch of the differential equation has

$$2y+qx+x(4p+q^2)^{\frac{1}{2}}=\phi\{-(4p+q^2)^{\frac{1}{2}}-q\},$$

for an intermediate integral; and the second branch has

$$2y+qx-x(4p+q^2)^{\frac{1}{2}}=\phi\{(4p+q^2)^{\frac{1}{2}}-q\},$$

for an intermediate integral.

It will be noticed that the method of integration adopted nowhere uses these integrals.

Ex. 2. As another illustration, consider Ampère's* process of integration applied to

$$(1+q^2)r-2pqs+(1+p^2)t=0,$$

which is the equation of minimal surfaces.

The arguments of the arbitrary functions are the same only if

$$1+p^2+q^2=0.$$

We may put this relation† on one side, as before: it is not inconsistent with the differential equation, but it leads only to a trivial integral: and we shall therefore assume that $1+p^2+q^2$ does not vanish.

* Second memoir, p. 82.

† It arises also, (see § 340), as a subsidiary equation in the integration

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$$

being connected with the characteristics.

Keeping the same notation as in the last example by writing θ' for $\frac{\delta\theta}{\delta x}$, when α is constant, and θ_1 for $\frac{\delta\theta}{\delta x}$, when β is constant, we substitute

$$s=q'-ty', \quad r=p'-q'y'+ty'^2,$$

in the equation; and then we make it evanescent, quà relation in t. Thus

$$\left.\begin{array}{l}(1+q^2)(p'-q'y')-2pqq' \quad =0\\ (1+q^2)y'^2+2pqy'+1+p^2=0\end{array}\right\},$$

together with

$$z'=p+qy',$$

are our equations: and, as in the general case, these equations are satisfied also by p_1, q_1, y_1. When we write

$$w=(1+p^2+q^2)^{\frac{1}{2}},$$

the resolved equivalents can be taken

$$\left.\begin{array}{l} y'+\dfrac{pq-iw}{1+q^2} \quad =0\\[2ex] p'-\dfrac{pq+iw}{1+q^2}\,q'=0\\[2ex] z'-\dfrac{p+iqw}{1+q^2} \quad =0\end{array}\right\},$$

being the system when α is constant; and

$$\left.\begin{array}{l} y_1+\dfrac{pq+iw}{1+q^2} \quad =0\\[2ex] p_1-\dfrac{pq-iw}{1+q^2}\,q_1=0\\[2ex] z_1-\dfrac{p-iqw}{1+q^2} \quad =0\end{array}\right\},$$

being the system when β is constant. We need an integral equation for each system.

Now the second equation

$$p'-\frac{pq+iw}{1+q^2}\,q'=0$$

in the first system, that is,

$$\frac{dp}{dq}-\frac{pq+iw}{1+q^2}=0,$$

can be expressed as a Clairaut equation: and its integral is

$$\frac{pq+iw}{1+q^2}=\text{constant},$$

so that we take

$$\frac{pq+iw}{1+q^2}=a.$$

Similarly, the second equation in the other system has an integral

$$\frac{pq - iw}{1 + q^2} = \text{constant},$$

and we take

$$\frac{pq - iw}{1 + q^2} = \beta.$$

We now proceed to make a and β the independent variables, and we note that

$$y' = -\beta, \quad y_1 = -a:$$

that is, as y' is the value of $\frac{\delta y}{\delta x}$, when a is constant, and y_1 is the value of $\frac{\delta y}{\delta x}$, when β is constant, we have

$$\frac{\partial y}{\partial \beta} = -\beta \frac{\partial x}{\partial \beta}, \quad \frac{\partial y}{\partial a} = -a \frac{\partial x}{\partial a}.$$

Hence

$$\frac{\partial}{\partial a}\left(\beta \frac{\partial x}{\partial \beta}\right) = -\frac{\partial^2 y}{\partial a\, \partial \beta} = \frac{\partial}{\partial \beta}\left(a \frac{\partial x}{\partial a}\right),$$

and therefore

$$\frac{\partial^2 x}{\partial a\, \partial \beta} = 0.$$

Consequently, we can take

$$x = \phi'(a) + \psi'(\beta),$$

where ϕ and ψ are arbitrary functions; and then

$$\frac{\partial y}{\partial a} = -a\phi''(a), \quad \frac{\partial y}{\partial \beta} = -\beta\psi''(\beta),$$

so that

$$y = \phi(a) - a\phi'(a) + \psi(\beta) - \beta\psi'(\beta).$$

Now

$$\begin{aligned} dz &= \left(p \frac{\partial x}{\partial a} + q \frac{\partial y}{\partial a}\right) da + \left(p \frac{\partial x}{\partial \beta} + q \frac{\partial y}{\partial \beta}\right) d\beta \\ &= (p - aq) \frac{\partial x}{\partial a}\, da + (p - \beta q) \frac{\partial x}{\partial}\, d\beta \\ &= (p - aq)\, \phi''(a)\, da + (p - \beta q)\, \psi''(\beta)\, d\beta. \end{aligned}$$

Also,

$$a = \frac{pq + iw}{1 + q^2},$$

so that

$$\{(1 + q^2)\, a - pq\}^2 = -1 - p^2 - q^2,$$

that is,

$$(1 + q^2)\, a^2 - 2apq + 1 + p^2 = 0,$$

and therefore

$$p - aq = i\,(1 + a^2)^{\frac{1}{2}}.$$

Similarly

$$p - \beta q = i\,(1 + \beta^2)^{\frac{1}{2}};$$

and so

$$z = i\int (1+a^2)^{\frac{1}{2}} \phi''(a)\,da + i\int (1+\beta^2)^{\frac{1}{2}} \psi''(\beta)\,d\beta.$$

This equation, with

$$x = \phi'(a) + \psi'(\beta),$$
$$y = \phi(a) - a\phi'(a) + \psi(\beta) - \beta\psi'(\beta),$$

constitutes the integral of the equation.

This is the form given by Legendre and by Ampère; the apparent partial quadrature can be removed by taking

$$\phi(a) = (1+a^2)^{\frac{3}{2}} \Phi'(a),$$
$$\psi(\beta) = (1+\beta^2)^{\frac{3}{2}} \Psi'(\beta).$$

Ex. 3. Obtain the integral of the preceding equation in Monge's form, viz.

$$x = a + \beta,$$
$$y = f(a) + g(\beta),$$
$$z = i\int \{1 + f'^2(a)\}^{\frac{1}{2}}\,da + i\int \{1 + g'^2(\beta)\}^{\frac{1}{2}}\,d\beta,$$

where f and g are arbitrary.

Ex. 4. Verify that, if

$$p = \frac{uv-1}{u+v}, \quad iq = \frac{uv+1}{u+v},$$

then

$$a = i\frac{v^2+1}{v^2-1}, \quad \beta = -i\frac{u^2+1}{u^2-1}.$$

Apply these values to obtain the integral of the preceding equation in Weierstrass's form*, viz.

$$\left.\begin{aligned} x &= (1-v^2)V'' + 2vV' - 2V + (1-u^2)U'' + 2uU' - 2U \\ y &= i\{(1+v^2)V'' - 2vV' + 2V - (1+u^2)U'' + 2uU' - 2U\} \\ z &= 2vV'' - 2V' + 2uU'' - 2U' \end{aligned}\right\},$$

where U is any arbitrary function of u, and V is any arbitrary function of v.

Discuss the limitations for an integral that is entirely real.

Ex. 5. Consider the equation†

$$st + x(rt - s^2)^2 = 0.$$

When we substitute

$$s = q' - ty', \quad r = p' - q'y' + ty'^2,$$

and make the resulting equation evanescent in t, we find

$$xq'^2 = 0,$$
$$q' - 2xq'^2(p' + qy') = 0,$$
$$x(p' + q'y')^2 - y' = 0.$$

* This is the integrated form which lends itself most readily to the discussion of minimal surfaces.

† Discussed by Ampère in his first memoir, p. 608: see *ante*, § 183, Ex. 2.

These give

$$q'=0, \quad xp'^2-y'=0,$$

which are consistent with one another and with the original equation.

We take

$$q=\phi'(a),$$

where ϕ is an arbitrary function: then

$$\begin{aligned}\frac{\delta p}{\delta a}+y'\frac{\delta q}{\delta a}&=(s+ty')\frac{\delta y}{\delta a}\\&=q'\frac{\delta y}{\delta a}\\&=0,\end{aligned}$$

and therefore

$$\frac{\delta p}{\delta a}+x\left(\frac{\delta p}{\delta x}\right)^2\phi''(a)=0.$$

Hence

$$p=-k^2\phi'(a)+2kx^{\frac{1}{2}},$$

where k is a constant: and so

$$y'=xp'^2=k^2,$$

and

$$y=k^2x+\theta(a),$$

where θ is an arbitrary function. Now

$$\begin{aligned}\frac{\delta z}{\delta a}&=q\frac{\delta y}{\delta a}\\&=\phi'(a)\,\theta'(a),\end{aligned}$$

and

$$\begin{aligned}\frac{\delta z}{\delta x}&=p+qy'\\&=2kx^{\frac{1}{2}};\end{aligned}$$

hence

$$\begin{aligned}z&=\tfrac{4}{3}kx^{\frac{3}{2}}+\chi(a)\\&=\tfrac{4}{3}kx^{\frac{3}{2}}+F(y-k^2x),\end{aligned}$$

which is the most general integral that can thus be obtained, F denoting an arbitrary function, and k an arbitrary constant.

Ampère's Method applied to Special Equations.

250. The preceding examples give some indication of Ampère's method: it is of interest to apply it to the particular equations

$$rt-s^2+Rr+2Ss+Tt=V,$$
$$Rr+2Ss+Tt=V,$$

the initial assumption now being that the integral system is finite in form and free from partial quadratures. The subsidiary equations are substantially the same as in the methods of Monge and of Boole: but they now bear a new significance.

We write

$$y',\ z',\ p',\ q',\ \text{for}\ \frac{\delta y}{\delta x},\ \frac{\delta z}{\delta x},\ \frac{\delta p}{\delta x},\ \frac{\delta q}{\delta x},$$

the derivatives being taken on the supposition that α is constant: and, if there be two different arguments α and β of arbitrary functions, those derivatives (taken on the supposition that β is constant) are denoted by y_1, z_1, p_1, q_1, respectively. According to the method, we substitute

$$s = q' - ty', \quad r = p' - q'y' + ty'^2,$$

in the equation; and we make the modified equation evanescent as a relation involving t. When applied to the equation

$$rt - s^2 + Rr + 2Ss + Tt = V,$$

the process leads to the two equations

$$\left.\begin{aligned} p' + q'y' + Ry'^2 - 2Sy' + T = 0 \\ R(p' - q'y') + 2Sq' - q'^2 - V = 0 \end{aligned}\right\}.$$

Multiplying the first of these by R and subtracting the second from the product, we have

$$q'^2 + 2Rq'y' + R^2y'^2 - 2Sq' - 2RSy' + RT + V = 0,$$

that is,

$$(q' + Ry')^2 - 2S(q' + Ry') + RT + V = 0.$$

Denoting by μ, as before, a root of the quadratic

$$\mu^2 + 2\mu S + RT + V = 0,$$

we have

$$q' + Ry' + \mu = 0.$$

The first equation then gives

$$\begin{aligned} p' + T &= 2Sy' - q'y' - Ry'^2 \\ &= (2S + \mu)\, y' \\ &= -\nu y', \end{aligned}$$

if ν denote the other root of the quadratic.

Let the roots of the quadratic be unequal, being then denoted by ρ and σ. We have, as usual, a couple of linear systems: one of them can be associated with the argument α, the other with the argument β; thus

$$\left.\begin{aligned} p' + \rho y' + T &= 0 \\ q' + Ry' + \sigma &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} p_1 + \sigma y_1 + T &= 0 \\ q_1 + Ry_1 + \rho &= 0 \\ z_1 - qy_1 - p &= 0 \end{aligned}\right\},$$

are the linear systems equivalent to the two equations.

When the roots of the quadratic are equal, then

$$\rho = \sigma = -S,$$
$$S^2 = RT + V;$$

there is only a single system

$$\left.\begin{aligned} p' - Sy' + T &= 0 \\ q' + Ry' - S &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\},$$

equivalent to the two equations*.

The equations in form are substantially the same as those which occur in Monge's method: and, as was proved, the latter are equivalent to those which occur in Boole's method. Here, however, they have been obtained without the assumptions upon which

* It has been assumed that there are two arguments or only one, according as the roots of the quadratic are unequal or are equal. As a matter of fact, there are two arguments or only one, according (§ 186) as the equation

$$4\frac{\partial f}{\partial r}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial s}\right)^2,$$

that is, as the equation

$$rt - s^2 + Rr + 2Ss + Tt = S^2 - RT,$$

is not satisfied or is satisfied, that is, according as the relation

$$S^2 - RT = V$$

does not hold or does hold. These are the conditions which justify the assumption.

those methods are based; and they now are definitely connected with the arguments of the arbitrary functions which are to be found in the integral system.

Construction of the Primitive.

251. The actual process of proceeding to the integral system has already (§ 249) been indicated in general terms.

In the first place, let there be two linear systems, connected with α and β respectively. Suppose that an integral of the system

$$\left.\begin{aligned} p' + \rho y' + T &= 0 \\ q' + Ry' + \sigma &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\}$$

has been obtained: we make the arbitrary element in that integral equal to α. (If another distinct integral of the same system can be obtained, we make the arbitrary element equal to $\phi(\alpha)$, where ϕ is an arbitrary function: the elimination of α will lead to an intermediate integral of the original equation. This, however, is not the most general case: and it is not necessary for the success of the method.) Similarly, suppose that an integral of the system

$$\left.\begin{aligned} p_1 + \sigma y_1 + T &= 0 \\ q_1 + Ry_1 + \rho &= 0 \\ z_1 - qy_1 - p &= 0 \end{aligned}\right\}$$

has been obtained: we make the arbitrary element in that integral equal to β. (If a second integral of this system can be obtained, it leads to another intermediate integral of the original equation; but, again, this is not the general case, and it is not necessary for the success of the method.)

We now make α and β the independent variables. As p', q', y', z' are the derivatives of p, q, y, z with regard to x, when α is constant, we have

$$\frac{\partial p}{\partial \beta} = p' \frac{\partial x}{\partial \beta}, \quad \frac{\partial q}{\partial \beta} = q' \frac{\partial x}{\partial \beta}, \quad \frac{\partial y}{\partial \beta} = y' \frac{\partial x}{\partial \beta}, \quad \frac{\partial z}{\partial \beta} = z' \frac{\partial x}{\partial \beta};$$

and similarly for p_1, q_1, y_1, z_1. Hence we have

$$\frac{\partial p}{\partial \beta} + \rho \frac{\partial y}{\partial \beta} + T \frac{\partial x}{\partial \beta} = 0,$$

$$\frac{\partial q}{\partial \beta} + R \frac{\partial y}{\partial \beta} + \sigma \frac{\partial x}{\partial \beta} = 0,$$

$$\frac{\partial z}{\partial \beta} - q \frac{\partial y}{\partial \beta} - p \frac{\partial x}{\partial \beta} = 0,$$

$$\frac{\partial p}{\partial \alpha} + \sigma \frac{\partial y}{\partial \alpha} + T \frac{\partial x}{\partial \alpha} = 0,$$

$$\frac{\partial q}{\partial \alpha} + R \frac{\partial y}{\partial \alpha} + \rho \frac{\partial x}{\partial \alpha} = 0,$$

$$\frac{\partial z}{\partial \alpha} - q \frac{\partial y}{\partial \alpha} - p \frac{\partial x}{\partial \alpha} = 0,$$

which, however, are not six independent equations because of the integral combinations of the former sets that have been used. To obtain the integral system of the original equation of the second order, we have to integrate this set of simultaneous equations of the first order: it need hardly be said that this later integration is facilitated by a knowledge of further integrals (if any) of the original subsidiary systems, though such knowledge is not necessary for the purpose.

We have already (§ 244) seen that, if one system of subsidiary equations leads to an intermediate integral, and if we proceed to the integration of that integral regarded as an equation of the first order, the Charpit equations which are subsidiary for the latter purpose involve the other system of Ampère equations. Later (§ 254) we shall obtain a general property (established by Lie and by Darboux, independently of one another) characteristic of the equations, which possess an intermediate integral arising from each of the Ampère systems.

In the next place, let there be only a single linear system of subsidiary equations: it is

$$\left.\begin{aligned} p' - Sy' + T &= 0 \\ q' + Ry' - S &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\}.$$

When integrals connected with this system are obtained, the arbitrary elements in the integrals are made equal to α, $\phi(\alpha)$,

$\psi(\alpha)$, where ϕ and ψ are arbitrary functions; and it has already been proved (§ 241) that, if p and q be eliminated from these integrals so as to leave a relation

$$z = F\{x, y, \alpha, \phi(\alpha), \psi(\alpha)\},$$

then the general integral of the original equation is given by the elimination of α between the two equations

$$\left.\begin{aligned} z &= F \\ 0 &= \frac{dF}{d\alpha} \end{aligned}\right\}.$$

252. When the equation to be integrated is

$$Rr + 2Ss + Tt = V,$$

it can similarly be shewn that, if the quadratic equation

$$R\xi^2 - 2S\xi + T = 0$$

has unequal roots λ and μ, there are two sets of subsidiary equations

$$\left.\begin{aligned} y' &= \mu \\ Rp' + \lambda q' &= V \\ p + q\mu &= z' \end{aligned}\right\}, \qquad \left.\begin{aligned} y_1 &= \lambda \\ Rp_1 + \mu q_1 &= V \\ p + q\lambda &= z_1 \end{aligned}\right\},$$

leading to equations

$$\left.\begin{aligned} \frac{\partial y}{\partial \beta} &= \mu \frac{\partial x}{\partial \beta} \\ R\frac{\partial p}{\partial \beta} + \lambda \frac{\partial q}{\partial \beta} &= V\frac{\partial x}{\partial \beta} \\ p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta} &= \frac{\partial z}{\partial \beta} \end{aligned}\right\}, \quad \left.\begin{aligned} \frac{\partial y}{\partial \alpha} &= \lambda \frac{\partial x}{\partial \alpha} \\ R\frac{\partial p}{\partial \alpha} + \mu \frac{\partial y}{\partial \alpha} &= V\frac{\partial x}{\partial \alpha} \\ p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha} &= \frac{\partial z}{\partial \alpha} \end{aligned}\right\}.$$

If, however, the quadratic has equal roots, so that the equation can be taken in the form

$$r + 2Ss + S^2 t = V,$$

there is only a single system of subsidiary equations, viz.

$$\left.\begin{aligned} y' &= S \\ p' + Sq' &= V \\ p + Sq &= z' \end{aligned}\right\}.$$

The method of proceeding in the respective cases is the same as in the corresponding cases for the equation

$$rt - s^2 + Rr + 2Ss + Tt = V.$$

Ex. 1. Let Ampère's method be applied to

$$r-t=2\frac{p}{x},$$

which does not possess an intermediate integral.

Pursuing the usual process, the first form of the subsidiary equations is

$$y'^2-1=0,$$

$$p'-q'y'=2\frac{p}{x}.$$

Clearly, there are two systems: we have

$$\left.\begin{aligned} y'&=1\\ p'-q'&=2\frac{p}{x}\\ p+q&=z'\end{aligned}\right\} \qquad \left.\begin{aligned} y_1&=-1\\ p_1+q_1&=2\frac{p}{x}\\ p-q&=z_1\end{aligned}\right\}.$$

An integral of the first system is given by

$$y-x=\text{constant}=a,$$

and one of the second system is given by

$$y+x=\text{constant}=\beta,$$

so that

$$2x=\beta-a.$$

The equations of the first system are

$$\frac{\partial p}{\partial \beta}-\frac{\partial q}{\partial \beta}=2\frac{p}{x}\frac{\partial x}{\partial \beta}=\frac{p}{x},$$

$$\frac{\partial z}{\partial \beta}=(p+q)\frac{\partial x}{\partial \beta}=\tfrac{1}{2}(p+q);$$

and those of the second system are

$$\frac{\partial p}{\partial a}+\frac{\partial q}{\partial a}=2\frac{p}{x}\frac{\partial x}{\partial a}=-\frac{p}{x},$$

$$\frac{\partial z}{\partial a}=(p-q)\frac{\partial x}{\partial a}=-\tfrac{1}{2}(p-q).$$

Eliminating q between the equations

$$\frac{\partial p}{\partial \beta}-\frac{\partial q}{\partial \beta}=\frac{p}{x},\quad \frac{\partial p}{\partial a}+\frac{\partial q}{\partial a}=-\frac{p}{x},$$

we have

$$2\frac{\partial^2 p}{\partial a\,\partial \beta}=\frac{\partial}{\partial a}\left(\frac{p}{x}\right)-\frac{\partial}{\partial \beta}\left(\frac{p}{x}\right)$$

$$=\frac{1}{x}\frac{\partial p}{\partial a}-\frac{1}{x}\frac{\partial p}{\partial \beta}+\frac{p}{x^2},$$

and therefore

$$\frac{\partial^2}{\partial a\,\partial \beta}\left(\frac{p}{x}\right)=\frac{1}{x}\frac{\partial^2 p}{\partial a\,\partial \beta}-\frac{1}{2x^2}\frac{\partial p}{\partial a}+\frac{1}{2x^2}\frac{\partial p}{\partial \beta}-\frac{p}{2x^3}=0,$$

so that

$$\frac{p}{x} = 2\phi''(\alpha) + 2\psi''(\beta),$$

or

$$p = (\beta - \alpha)\{\phi''(\alpha) + \psi''(\beta)\},$$

where ϕ and ψ are arbitrary functions. Also, with this value of p, we have

$$\frac{\partial q}{\partial \alpha} = -\frac{p}{x} - \frac{\partial p}{\partial \alpha} = -\phi''(\alpha) - \psi''(\beta) - (\beta - \alpha)\phi'''(\alpha),$$

$$\frac{\partial q}{\partial \beta} = -\frac{p}{x} + \frac{\partial p}{\partial \beta} = -\phi''(\alpha) - \psi''(\beta) + (\beta - \alpha)\psi'''(\beta);$$

and therefore

$$q = -2\phi'(\alpha) - 2\psi'(\beta) - (\beta - \alpha)\{\phi''(\alpha) - \psi''(\beta)\}.$$

Lastly,

$$\frac{\partial z}{\partial \alpha} = -\tfrac{1}{2}(p - q) = -\phi'(\alpha) - \psi'(\beta) - (\beta - \alpha)\phi''(\alpha),$$

$$\frac{\partial z}{\partial \beta} = \tfrac{1}{2}(p + q) = -\phi'(\alpha) - \psi'(\beta) + (\beta - \alpha)\psi''(\beta);$$

and therefore

$$\begin{aligned} z &= -2\phi(\alpha) - 2\psi(\beta) - (\beta - \alpha)\{\phi'(\alpha) - \psi'(\beta)\} \\ &= f(y - x) + g(y + x) + xf'(y - x) - xg'(y + x), \end{aligned}$$

where f and g are arbitrary functions of their arguments. This is the primitive of the original equation.

Ex. 2. As an example in which there is only one argument for the arbitrary functions, so that there is only a single system, consider the equation already (Ex. 2, § 186) discussed, viz.

$$(b+q)^2 r - 2(a+p)(b+q)s + (a+p)^2 t = 0.$$

When we substitute

$$s = q' - ty', \quad r = p' - q'y' + ty'^2,$$

and make the resulting equation evanescent as regards t, we have

$$(b+q)^2(p' - q'y') - 2(a+p)(b+q)q' = 0,$$

$$\{(b+q)y' + a + p\}^2 = 0:$$

and therefore, neglecting the trivial forms

$$b + q = 0, \quad a + p = 0,$$

which lead to a trivial primitive, we take

$$(b+q)(p' - q'y') - 2(a+p)q' = 0,$$

$$(b+q)y' + a + p = 0.$$

Substituting for y' from the latter into the former, we have

$$(b+q)p' - (a+p)q' = 0,$$

that is,

$$\begin{aligned} \frac{a+p}{b+q} &= \text{constant} \\ &= -\alpha, \end{aligned}$$

as the quantity a is not yet specified. Then

$$y' = -\frac{a+p}{b+q}$$
$$= a,$$

so that

$$y = ax + \phi(a),$$

where ϕ is an arbitrary function. Also

$$z' = p + qy'$$
$$= p + qa$$
$$= -(a + ba),$$

and therefore

$$z = -x(a + ba) + \psi(a),$$

where ψ is another arbitrary function. Thus the integral system is

$$\left.\begin{aligned} y &= ax + \phi(a) \\ z &= -x(a+ba) + \psi(a) \end{aligned}\right\};$$

and it is easy to see that it can be exhibited by means of a single equation in any of the three forms

$$x = y\theta(ax + by + z) + \phi(ax + by + z),$$
$$y = xf(ax + by + z) + g(ax + by + z),$$
$$z = xF(ax + by + z) + G(ax + by + z),$$

where all the functional symbols imply arbitrary functions.

SIGNIFICANCE OF THE SUBSIDIARY EQUATIONS.

253. We have already indicated that the significance of the subsidiary equations, which formally are the same for all the methods, is wider in Ampère's method than in the methods of Monge and of Boole. The relation between the distinct uses of the subsidiary equations can be exhibited in a different manner as follows, it being assumed for this purpose that the original equation of the second order has an intermediate integral. Let this integral be supposed to occur in connection with the subsidiary system

$$\left.\begin{aligned} p' + \rho y' + T &= 0 \\ q' + Ry' + \sigma &= 0 \\ z' - qy' - p &= 0 \end{aligned}\right\},$$

expressed in Ampère's form, the derivatives with regard to x being taken on the assumption that α is constant. Let

$$u(x, y, z, p, q) = \text{constant}, \quad v(x, y, z, p, q) = \text{constant},$$

be two integrable combinations of the foregoing equations.

In Monge's method, we at once construct the intermediate integral

$$u = \phi(v);$$

in order to obtain the primitive, we proceed to integrate this equation, regarded as a partial equation of the first order.

In Ampère's method, remembering that differentiations with regard to x are effected on the hypothesis that α is kept constant in the subsidiary equations considered, we take

$$u = \alpha, \quad v = \phi(\alpha),$$

where ϕ is any arbitrary function. Instead of considering these equations as partial equations of the first order, we regard them merely as two equations connecting x, y, z, p, q, α; the last five quantities are functions of x and β, and their derivatives with regard to x on the supposition that β is constant satisfy the alternative subsidiary system of equations in the form

$$\left.\begin{aligned} p_1 + \sigma y_1 + T &= 0 \\ q_1 + R y_1 + \rho &= 0 \\ z_1 - q y_1 - p &= 0 \end{aligned}\right\}.$$

Moreover,

$$\alpha_1 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y_1 + \frac{\partial u}{\partial z} z_1 + \frac{\partial u}{\partial p} p_1 + \frac{\partial u}{\partial q} q_1,$$

$$\alpha_1 \phi'(\alpha) = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} y_1 + \frac{\partial v}{\partial z} z_1 + \frac{\partial v}{\partial p} p_1 + \frac{\partial v}{\partial q} q_1.$$

We thus have seven equations in all. Among these seven equations, let p, q, p_1, q_1 be eliminated: when the elimination has been effected, there remain three equations involving

$$x;\ y, y_1;\ z, z_1;\ \alpha, \alpha_1.$$

These three equations, in three dependent variables y, z, α, are integrated as an ordinary system of equations; in their integral, there will occur three quantities A, B, C, which are arbitrary constants so far as the integration is concerned.

Now the derivatives y_1, z_1, α_1 have been formed on the supposition that β is constant: hence we take

$$A = \beta, \quad B = \psi(\beta), \quad C = \chi(\beta),$$

where ψ and χ are arbitrary functions. Apparently, there now are three arbitrary functions in the three integral equations con-

necting x, y, z, α, β; and therefore, when α and β are eliminated among the three equations so as to give a primitive between x, y, z, the primitive would appear to contain three arbitrary functions, instead of merely two in accordance with Cauchy's theorem. The explanation is that we still have to take account of derivation with regard to β: thus

$$z_1 = \frac{dz}{dx} = q\frac{dy}{dx} + p, \qquad \frac{dz}{d\beta} = q\frac{dy}{d\beta},$$

and therefore

$$\frac{d}{dx}\left(q\frac{dy}{d\beta}\right) = \frac{d}{d\beta}\left(q\frac{dy}{dx} + p\right),$$

that is,

$$\frac{dp}{d\beta} = \frac{dq}{dx}\frac{dy}{d\beta} - \frac{dq}{d\beta}\frac{dy}{dx}$$

$$= q_1\frac{dy}{d\beta} - y_1\frac{dq}{d\beta}.$$

From the equations

$$u = \alpha, \quad v = \phi(\alpha),$$

we have p and q as functions of x, y, z, α, which become functions of $x, \beta, \psi(\beta), \chi(\beta)$, on using the integrals of the second set of subsidiary equations. When these values are substituted in the foregoing relation, which must be satisfied identically, one of the three functional forms in the integral system is expressible in terms of the other two: that is, the primitive equation contains only the necessary two arbitrary functional forms.

Note. The mode, in which derivation with regard to β is taken into account, need not necessarily be that which precedes: thus it might be more convenient in practice to substitute directly in the relation

$$\frac{dz}{d\beta} = q\frac{dy}{d\beta}.$$

Ex. 1. Consider the equation

$$(q + yt)(r + 1) = s(ys - p - x).$$

Arranged so that the coefficient of $rt - s^2$ is unity, the equation is

$$rt - s^2 + \frac{q}{y}r + \frac{p+x}{y}s + t = -\frac{q}{y}.$$

The critical quadratic is

$$\mu^2 + \frac{p+x}{y}\mu = 0;$$

the roots are unequal, and therefore there are two distinct systems of subsidiary equations; and neither of the systems can possess three integrable combinations. The two systems are respectively

$$\left.\begin{array}{l} p' - \dfrac{p+x}{y}\, y' + 1 = 0 \\ q' + \dfrac{q}{y}\, y' \qquad = 0 \\ z' - qy' \quad - p = 0 \end{array}\right\}, \quad \left.\begin{array}{l} p_1 \qquad\qquad + 1 = 0 \\ q_1 + \dfrac{q}{y}\, y_1 - \dfrac{p+x}{y} = 0 \\ z_1 - qy_1 - p \qquad = 0 \end{array}\right\}.$$

The former system possesses the two integrals

$$\frac{p+x}{y} = \text{constant}, \quad qy = \text{constant};$$

and the latter system possesses the two integrals

$$p + x = \text{constant}, \quad qy - x\,(p+x) = \text{constant}.$$

When the Monge method of using these integrals is adopted, the equation

$$\frac{p+x}{y} = \phi\,(qy),$$

where ϕ is an arbitrary function, is an intermediate integral: and the equation

$$qy - x\,(p+x) = \psi\,(p+x),$$

where ψ is an arbitrary function, is another intermediate integral. The two integrals coexist.

Though it is not possible to resolve the two equations for p and q, yet quadrature of the relation

$$dz = p\,dx + q\,dy$$

becomes simple, on the introduction of new variables

$$qy = a, \quad p + x = \beta.$$

The intermediate integrals give

$$y = \frac{\beta}{\phi\,(a)},$$

$$x = \frac{a - \psi\,(\beta)}{\beta},$$

and therefore

$$\begin{aligned} d\,(z + \tfrac{1}{2}x^2) &= \beta\,dx + \frac{a}{y}\,dy \\ &= \left\{1 - a\,\frac{\phi'(a)}{\phi\,(a)}\right\} da + \left\{\frac{\psi\,(\beta)}{\beta} - \psi'(\beta)\right\} d\beta. \end{aligned}$$

When quadrature is effected, the explicit form being obtained most easily by the introduction of new functions f and g such that

$$\phi\,(a) = e^{f'(a)}, \quad \psi\,(\beta) = \beta g'(\beta),$$

we have the primitive given by the equations

$$\left.\begin{aligned} x &= \frac{a}{\beta} - g'(\beta) \\ y &= \beta e^{-f'(a)} \\ \tfrac{1}{2}x^2 + z &= a + f(a) - af'(a) + g(\beta) - \beta g'(\beta) \end{aligned}\right\}.$$

If, still adopting Monge's use of the integrals of a subsidiary system, we proceed to integrate

$$\frac{p+x}{y} = \phi(qy),$$

regarded as a partial equation of the first order, we find

$$p + x = a$$

as an integral of the Charpit subsidiary equations: and then

$$q = \frac{1}{y}\phi^{-1}\left(\frac{a}{y}\right).$$

Hence

$$\begin{aligned} z &= \int (p\,dx + q\,dy) \\ &= ax - \tfrac{1}{2}x^2 + \int \frac{1}{y}\phi^{-1}\left(\frac{a}{y}\right)dy + b \\ &= ax - \tfrac{1}{2}x^2 + \chi\left(\frac{a}{y}\right) + b, \end{aligned}$$

where χ is a new arbitrary function and b is a constant. To obtain the general primitive, we take $b = \omega(a)$: and we have that primitive in the form

$$\left.\begin{aligned} z &= ax - \tfrac{1}{2}x^2 + \chi\left(\frac{a}{y}\right) + \omega(a) \\ 0 &= x + \frac{1}{y}\chi'\left(\frac{a}{y}\right) + \omega'(a) \end{aligned}\right\}.$$

Now consider the use of the integrals of the subsidiary systems in Ampère's method. We take

$$\frac{p+x}{y} = a, \quad qy = \phi(a).$$

The equations of the subsidiary system associated with β are

$$\left.\begin{aligned} p_1 + 1 &= 0 \\ q_1 y + q y_1 - (p+x) &= 0 \\ z_1 - q y_1 - p &= 0 \end{aligned}\right\},$$

the derivatives with regard to x being taken on the hypothesis that β is constant. We have

$$\frac{d}{dx}(ya) = \frac{d}{dx}(p+x) = p_1 + 1 = 0,$$

so that

$$\begin{aligned} ya &= \text{constant, when } \beta \text{ is constant,} \\ &= \beta, \end{aligned}$$

say. Again,

$$\phi'(a)\,a_1 = qy_1 + yq_1$$
$$= p + x$$
$$= ya$$
$$= \beta,$$

so that

$$\phi(a) = \beta x + \psi(\beta),$$

where ψ is an arbitrary function of β. Again,

$$y = \frac{\beta}{a},$$

so that

$$y_1 = -\frac{\beta}{a^2}a_1,$$

and therefore

$$z_1 = qy_1 + p$$
$$= -\frac{\phi(a)}{a}a_1 + \beta - x;$$

consequently,

$$z = \beta x - \tfrac{1}{2}x^2 - \Phi(a) + \theta(\beta),$$

where θ is arbitrary, and

$$\phi(a) = a\Phi'(a).$$

We thus have three arbitrary functions θ, ψ, Φ : one of them is dependent upon the other two.

To determine this dependence, we substitute from the equations

$$z = \beta x - \tfrac{1}{2}x^2 - \Phi(a) + \theta(\beta),$$
$$y = \frac{\beta}{a},$$
$$q = \frac{a}{\beta}\phi(a),$$

in the relation

$$\frac{dz}{d\beta} = q\frac{dy}{d\beta}.$$

It gives

$$x - \Phi'(a)\frac{da}{d\beta} + \theta'(\beta) = \frac{a}{\beta}\phi(a)\left\{\frac{1}{a} - \frac{\beta}{a^2}\frac{da}{d\beta}\right\},$$

so that, taking account of the relation between ϕ and Φ', we have

$$x + \theta'(\beta) = \frac{1}{\beta}\phi(a)$$
$$= \frac{a}{\beta}\Phi'(a),$$

which gives the required relation. Hence the primitive is

$$\left.\begin{aligned} z &= \beta x - \tfrac{1}{2}x^2 - \Phi\left(\frac{\beta}{y}\right) + \theta(\beta) \\ 0 &= x \qquad\quad - \frac{1}{y}\Phi'\left(\frac{\beta}{y}\right) + \theta'(\beta) \end{aligned}\right\},$$

involving two arbitrary functions.

Ex. 2. Compare the Monge method and the Ampère method, by detailed reference to the equation

$$2pqyr + (p^2y + qx)\,s + xpt - p^2q\,(rt - s^2) = xy.$$

Lie's Theorem on Equations of the Second Order, possessing Two Intermediate Integrals.

254. In the case of the equations which have just been considered, the process of integration is materially simplified when either of the subsidiary systems leads to an intermediate integral: in particular, we know that, when each of these systems leads to an intermediate integral, the two integrals can be treated as coexistent; and quadrature then will suffice to lead to the primitive. *When an equation of the second order has this property of possessing two independent intermediate integrals, it is reducible to the form*

$$s = 0,$$

by contact transformations. This theorem, which is due to Lie*, can be proved as follows. Let

$$F(u_1, v_1) = 0, \quad G(u_2, v_2) = 0,$$

be the intermediate integrals: as we know (§ 239), these coexist for all functional forms of F and G, and therefore

$$[F, G] = 0.$$

Consequently, taking $F = u_1$ and v_1, $G = u_2$ and v_2, all in turn, we have

$$[u_1, u_2] = 0, \quad [v_1, u_2] = 0, \quad [u_1, v_2] = 0, \quad [u_2, v_2] = 0.$$

(It may be remarked incidentally that these equations verify Ampère's theorem that u_2 and v_2 are integrals of the equations subsidiary to the integration of $F = 0$, regarded as an equation of the first order.) To compare them with the equations of contact transformation, we write

$$v_1 = X, \quad v_2 = Y;$$

and they then become

$$[u_1, u_2] = 0, \quad [X, u_2] = 0, \quad [Y, u_1] = 0, \quad [X, Y] = 0.$$

* *Arch. f. Math. og Nat.*, t. II (1877), pp. 1—9. It was afterwards discovered independently by Darboux: see §§ 38, 39, of his memoir quoted on p. 302 hereafter.

Because the last equation is satisfied, it follows from the theory of contact transformations that a function Z exists, which satisfies

$$[Z, X]=0, \quad [Z, Y]=0,$$

and which is functionally distinct from u_1 and u_2, because of the number of independent variables involved. Further, we know, from the same theory, that it is possible to determine functions P and Q such that the relation

$$dZ - PdX - QdY = \rho\,(dz - pdx - qdy)$$

is satisfied identically, where ρ is a non-vanishing quantity not dependent upon the differential elements. Suppose that these quantities Z, P, Q are known: and effect, alike upon the differential equation and its intermediate integrals, the contact transformation which replaces the variables x, y, z, p, q by X, Y, Z, P, Q; and suppose that, in consequence of this transformation, u_1 and u_2 become U_1 and U_2 respectively, so that the intermediate integrals become

$$F(U_1, X)=0, \quad G(U_2, Y)=0,$$

these being intermediate integrals of the transformed equation. The equation

$$[Y, u_1]=0$$

thus becomes

$$[Y, U_1]=0,$$

that is,

$$\frac{\partial U_1}{\partial Q}=0,$$

so that U_1 is independent of Q. Similarly, the equation

$$[X, u_2]=0$$

becomes

$$[X, U_2]=0,$$

that is,

$$\frac{\partial U_2}{\partial P}=0,$$

so that U_2 is independent of P. Also, the relation

$$[u_1, u_2]=0$$

is replaced by

$$[U_1, U_2]=0:$$

when this is expressed in full, account being taken of the established facts that U_1 and U_2 are independent of Q and of P respectively, we have

$$-\left(\frac{\partial U_2}{\partial X} + P\frac{\partial U_2}{\partial Z}\right)\frac{\partial U_1}{\partial P} + \left(\frac{\partial U_1}{\partial Y} + Q\frac{\partial U_1}{\partial Z}\right)\frac{\partial U_2}{\partial Q} = 0,$$

so that

$$\frac{\dfrac{\partial U_2}{\partial X} + P\dfrac{\partial U_2}{\partial Z}}{\dfrac{\partial U_2}{\partial Q}} = \frac{\dfrac{\partial U_1}{\partial Y} + Q\dfrac{\partial U_1}{\partial Z}}{\dfrac{\partial U_1}{\partial P}}.$$

Because U_2 is independent of P, the first of the fractions is a linear function of P; and because U_1 is independent of Q, the second of the fractions is a linear function of Q. Moreover, the relation is satisfied identically, and therefore each fraction is of the form

$$APQ + BP + CQ + D,$$

where A, B, C, D do not involve P or Q: thus

$$\left.\begin{aligned}\frac{\partial U_1}{\partial Y} &= (BP+D)\frac{\partial U_1}{\partial P}\\ \frac{\partial U_1}{\partial Z} &= (AP+C)\frac{\partial U_1}{\partial P}\\ \frac{\partial U_1}{\partial Q} &= 0\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{\partial U_2}{\partial X} &= (CQ+D)\frac{\partial U_2}{\partial Q}\\ \frac{\partial U_2}{\partial Z} &= (AQ+B)\frac{\partial U_2}{\partial Q}\\ \frac{\partial U_2}{\partial P} &= 0\end{aligned}\right\}.$$

Now U_1 is to be functionally distinct from X, because u_1 and v_1 are functionally distinct from one another; and U_2 is to be functionally distinct from Y, because u_2 and v_2 are functionally distinct from one another. Hence the equations for U_1 must be a complete Jacobian system as they stand; and likewise those for U_2 must be a complete Jacobian system as they stand.

The Jacobi-Poisson condition that the equations for U_1 should be complete is

$$\frac{\partial U_1}{\partial P}\left\{P\left(\frac{\partial B}{\partial Z} - \frac{\partial A}{\partial Y}\right) + \frac{\partial D}{\partial Z} - \frac{\partial C}{\partial Y} + AD - BC\right\} = 0;$$

and therefore, as A, B, C, D do not involve P, we must have

$$\frac{\partial B}{\partial Z} - \frac{\partial A}{\partial Y} = 0,$$

$$\frac{\partial D}{\partial Z} - \frac{\partial C}{\partial Y} = BC - AD$$

Assuming these conditions satisfied, we have

$$U_1 = KP + L,$$

where K and L are independent of P and Q, and

$$A = \frac{1}{K}\frac{\partial K}{\partial Z}, \quad B = \frac{1}{K}\frac{\partial K}{\partial Y}, \quad \frac{\partial L}{\partial Y} = KD, \quad \frac{\partial L}{\partial Z} = KC,$$

the conditions for the coexistence of these equations being satisfied. Similarly, we have

$$U_2 = MQ + N,$$

where M and N are independent of P and Q. Consequently, the two intermediate integrals of the transformed equation are

$$F(KP + L, X) = 0, \quad G(MQ + N, Y) = 0;$$

and they may be taken in the forms

$$KP + L - \xi = 0, \quad MQ + N - \eta = 0,$$

where ξ is an arbitrary function of X, and η of Y.

The condition for coexistence of these two equations, viz.

$$[KP + L - \xi, \; MQ + N - \eta] = 0,$$

now becomes

$$-K\left\{Q\frac{\partial M}{\partial X} + \frac{\partial N}{\partial X} + P\left(Q\frac{\partial M}{\partial Z} + \frac{\partial N}{\partial Z}\right)\right\}$$
$$+ M\left\{P\frac{\partial K}{\partial Y} + \frac{\partial L}{\partial Y} + Q\left(P\frac{\partial K}{\partial Z} + \frac{\partial L}{\partial Z}\right)\right\} = 0,$$

and it is satisfied identically; hence

$$M\frac{\partial K}{\partial Z} - K\frac{\partial M}{\partial Z} = 0,$$

$$M\frac{\partial L}{\partial Z} - K\frac{\partial M}{\partial X} = 0,$$

$$M\frac{\partial K}{\partial Y} - K\frac{\partial N}{\partial Z} = 0,$$

$$M\frac{\partial L}{\partial Y} - K\frac{\partial N}{\partial X} = 0.$$

The first of these relations gives

$$M = \lambda K,$$

where λ is a function of X and Y only, not involving Z. From the second, using this result, we have

$$\lambda \frac{\partial L}{\partial Z} = \frac{\partial M}{\partial X} = \frac{\partial (\lambda K)}{\partial X},$$

and therefore

$$\frac{\partial L}{\partial Z} = \frac{\partial K}{\partial X} + K \frac{\partial (\log \lambda)}{\partial X}.$$

From the third, we have

$$\lambda \frac{\partial K}{\partial Y} = \frac{\partial N}{\partial Z},$$

and from the fourth

$$\lambda \frac{\partial L}{\partial Y} = \frac{\partial N}{\partial X}:$$

hence

$$\begin{aligned} \frac{\partial}{\partial X}\left(\lambda \frac{\partial K}{\partial Y}\right) &= \frac{\partial}{\partial Z}\left(\lambda \frac{\partial L}{\partial Y}\right) \\ &= \lambda \frac{\partial^2 L}{\partial Z \partial Y} \\ &= \lambda \left\{\frac{\partial^2 K}{\partial X \partial Y} + \frac{\partial K}{\partial y} \frac{\partial (\log \lambda)}{\partial X} + K \frac{\partial^2 (\log \lambda)}{\partial X \partial Y}\right\}, \end{aligned}$$

and therefore

$$\frac{\partial^2 (\log \lambda)}{\partial X \partial Y} = 0,$$

so that

$$\lambda = \frac{\xi_1}{\eta_1},$$

where ξ_1 is any function of X only, and η_1 is any function of Y only. Taking a new function Θ of X, Y, Z, such that

$$K\xi_1 = \frac{\partial \Theta}{\partial Z},$$

we have

$$L\xi_1 = \frac{\partial \Theta}{\partial X},$$

$$N\eta_1 = \frac{\partial \Theta}{\partial Y};$$

and so

$$KP + L - \xi = \frac{1}{\xi_1}\left(\frac{\partial \Theta}{\partial Z}\frac{\partial Z}{\partial X} + \frac{\partial \Theta}{\partial X}\right) - \xi = 0,$$

$$\lambda KQ + N - \eta = \frac{1}{\eta_1}\left(\frac{\partial \Theta}{\partial Z}\frac{\partial Z}{\partial Y} + \frac{\partial \Theta}{\partial Y}\right) - \eta = 0.$$

As a last transformation, we take a new dependent variable ζ (as may be done arbitrarily in contact transformations) such that

$$\zeta = \Theta;$$

and then the intermediate integrals are

$$\frac{\partial \zeta}{\partial X} = \xi \xi_1 = \text{a function of } X \text{ only},$$

$$\frac{\partial \zeta}{\partial Y} = \eta \eta_1 = \text{a function of } Y \text{ only},$$

so that

$$\frac{\partial^2 \zeta}{\partial X \partial Y} = 0,$$

thus establishing Lie's theorem that, if an equation of the second order possesses two independent intermediate integrals, it can be reduced to a form

$$s = 0,$$

by means of contact transformations.

Ex. 1. In Ex. 1, § 253, it was proved that the equation

$$(q+yt)(r+1) = s(ys-p-x)$$

possesses two intermediate integrals; and that the primitive, obtained by the Monge method, could be represented by the equations

$$\left.\begin{aligned} x &= \frac{a}{\beta} - g'(\beta) \\ y &= \beta e^{-f'(a)} \\ z + \tfrac{1}{2}x^2 &= a + f(a) - af'(a) + g(\beta) - \beta g'(\beta) \end{aligned}\right\}.$$

The quantity a is the X of the preceding investigation, and the quantity β is the Y: and, in particular,

$$qy = a, \quad p + x = \beta.$$

The contact transformation in question is

$$\left.\begin{aligned} z + \tfrac{1}{2}x^2 &= Z \\ p + x = \beta &= Y \\ qy = a &= X \\ x &= \frac{a}{\beta} - g'(\beta) \\ y &= \beta e^{-f'(a)} \end{aligned}\right\},$$

where the forms of f and g are arbitrary: the values of P and Q are

$$P = 1 - af''(a) \qquad Q = -\beta g''(\beta):$$

the relation

$$dz - p\,dx - q\,dy = dZ - P\,dX - Q\,dY$$

is satisfied, the value of Z being

$$Z = \alpha + f(\alpha) - \alpha f'(\alpha) + g(\beta) - \beta g'(\beta):$$

and

$$\frac{\partial^2 Z}{\partial X \partial Y} = 0.$$

Ex. 2. Obtain two intermediate integrals of the equation

$$Ar + Bs + Ct = 0,$$

where

$$A = pqx^2 + (1+q^2)\,xy, \quad C = -pqy^2 - (1+p^2)\,xy,$$
$$B = (1+q^2)\,y^2 - (1+p^2)\,x^2;$$

construct the contact transformations which change it into the equation

$$s' = 0;$$

and hence derive a primitive.

By means of the intermediate integrals, devise a geometrical interpretation of the equation and its primitive. (Goursat.)

Ex. 3. Surfaces (due to Monge) have one system of their lines of curvature situated upon concentric spheres: construct the partial differential equation of the second order satisfied by such surfaces. Prove that this equation possesses two intermediate integrals: and obtain the contact transformation which changes it into the equation $s' = 0$.

CHAPTER XVIII.

Darboux's Method, and other Methods, for Equations of the Second Order in Two Independent Variables.

The first considerable extension (as distinct from improvements) in the methods of solving partial equations of the second order, effected after the publication of Ampère's memoirs, was obtained by Darboux in a memoir published* in 1870. Previous methods, such as those of Monge and Ampère, had mainly or entirely sought for the primitive through the construction of a compatible equation of lower order; and, as has appeared from the preceding discussions, such a process is not always applicable. Using Ampère's ideas, and combining with them the spirit of Jacobi's method of solving an equation of the first order by associating with it a new equation of its own order, Darboux devised a method which, for wide classes of equations, can lead to a primitive. The central feature is the construction of an equation or equations of the second order, or even of order higher than the second, which are compatible with a given equation of the second order: in order to derive these equations, it is necessary to integrate one or more subsidiary systems. These systems have two forms, which are equivalent to one another: in one form, the equations are homogeneous and linear in the differential elements; in the other, the equations are homogeneous and linear in the first derivatives of the unknown function. To such systems, the Jacobian method of integration can be applied: it has the advantage of indicating the conditions which must be satisfied, if the process is to be effective.

Moreover, Darboux's method is progressive: that is to say, when the tests shew that no equations of a particular order can be associated with a given equation, then it can be applied equally to obtain (if that be possible) equations of the next higher order which are compatible with the given equation. Accordingly, it is effective for all equations of the second order when their primitive can be expressed in finite terms, whether by means of a single integral equation or by means of a number of simultaneous integral equations.

* *Ann. de l'Éc. Norm. Sup.*, t. VII (1870), pp. 163—173.

Further, it can be applied to equations of any order in two independent variables.

Since the publication of Darboux's investigations, many other memoirs upon the subject have appeared. Among them, special mention should be made of those by Hamburger*, Winckler†, König‡, and Sersawy§: references to other writers will be found in these memoirs. A historical summary of the methods devised by various writers for obtaining equations, which are the same as, or are equivalent to, Darboux's equations, is given by Speckman‖: Goursat's discussion¶ of the matter may be consulted with advantage; and a memoir by Sonin** should be consulted.

255. The substantial difference between the main aim of general methods, devised for the integration of partial equations of the first order, and the main aim of such methods, as are expounded in the immediately preceding chapters for the integration of partial equations of the second order, is of significance and importance. In the case of equations of the first order, the aim of general methods such as those devised by Charpit and by Jacobi is to construct equations that can be associated with the original equation: and all the admissible equations thus constructed are themselves of the first order, as is the original equation. In the case of equations of the second order, the aim of the methods that have been expounded is the construction of equations that are compatible with the original equation: in the methods of Monge and of Boole, the admissible equations (when they exist) are of the first order, being thus of order lower than the original equation: in the method of Ampère, the admissible equations may be of the first order and may be of no order at all: in no case, has the order of the associated equation or equations been the same as, or higher than, that of the original equation.

As regards the details of the methods applied to the construction of equations which are associated, or are compatible, with equations already propounded, there is the superficial resemblance that all of them, in so far as they involve inverse

* *Crelle*, t. LXXXI (1876), pp. 271—280; *ib.*, t. XCIII (1882), pp. 201—214.

† *Wien. Ber.*, t. LXXXVIII (1883), pp. 7—74; *ib.*, t. LXXXIX (1884), pp. 614—624.

‡ *Math. Ann.*, t. XXIV (1884), pp. 465—536.

§ *Wien. Denkschr.*, t. XLIX (1884), pp. 1—104.

‖ *Arch. Néerl.*, t. XXVII (1894), pp. 303—354.

¶ *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, ch. VI, VII.

** *Math. Ann.*, t. XLIX (1897), pp. 417—447: it was first published, in Russian, in 1874.

integrational operations, demand such operations in only the first degree. The reason of the resemblance, such as it is, lies in the facts, that even moderately general inverse operations can be effected only if they are of the first degree, and that the feasible operations of the second degree are exceedingly limited in scope. The resemblance, therefore, has nothing to do with the orders of the equations concerned and is due solely to exiguity of facility with inverse operations: it needs no further comment.

There is one outstanding difference between a system of subsidiary equations used in connection with an equation of the first order and such a system used in connection with an equation of the second order. All the equations in both kinds of systems are of the type called ordinary: they seek a provisional expression of all the variables in terms of a single variable. When the original equation is of the first order, the number of equations in the subsidiary system is equal to the number of variables provisionally regarded as dependent; when the original equation is of the second order, the number of equations in the subsidiary system (when it is effective for its purpose) is one less than the number of variables provisionally regarded as dependent. As has been seen in the discussion of the subsidiary systems, retaining the significance given them most widely by Ampère's method, this excess by one unit cannot be used to give an arbitrary provisional value to one of the dependent variables; it arises from the latency of the argument of the arbitrary function or functions, which occur in the primitive of the equations of the second order.

The outlook beyond these considerations, applied to the simplest case when an equation of the second order in two independent variables is propounded for solution, suggests two questions. On the one hand, is it possible to further the construction of a primitive by associating, with the original equation, an equation involving partial derivatives of order higher than the first? On the other hand, is it possible, by proceeding to derivatives of higher order, to construct a subsidiary system (which, presumably, shall be ordinary) that is complete?

Moreover, when the discussion is not restricted to equations in only two independent variables but extends to those involving any number, a further question will arise as to whether a subsidiary system (if the method of subsidiary equations is then of

any effective use) will be, not merely complete or incomplete, but ordinary or partial. Putting this question on one side for the present, as well as cognate questions that are easy enough to propound, we proceed to consider the two earlier questions. The first discussion, which these questions received, is contained in a memoir by Darboux*; since the publication of that memoir, they have received much attention, especially in regard to particular equations.

CAUCHY'S METHOD, RESTATED AND DISCUSSED AFTER DARBOUX.

256. It has already been seen, in Chapter VI of the preceding volume during the exposition of Cauchy's method for an original equation of the first order, that the subsidiary equations deduced by the process of changing the independent variables are a complete ordinary system. Let the same process, which was first suggested by Ampère, be applied to an equation of the second order

$$f(x, y, z, p, q, r, s, t) = 0.$$

Denoting $\dfrac{\partial f}{\partial x}$ by X, and so for the other derivatives of f, we have the total differential equation equivalent to $f=0$ in the form

$$X dx + Y dy + Z dz + P dp + Q dq + R dr + S ds + T dt = 0.$$

The independent variables x and y are changed to x and u, where u is left to be determined and is not a function of x alone. As usual, we have

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x}, \quad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u},$$

$$\frac{\partial p}{\partial x} = r + s\frac{\partial y}{\partial x}, \quad \frac{\partial p}{\partial u} = s\frac{\partial y}{\partial u},$$

$$\frac{\partial q}{\partial x} = s + t\frac{\partial y}{\partial x}, \quad \frac{\partial q}{\partial u} = t\frac{\partial y}{\partial u};$$

three of these are equations which, in the new derivatives, involve derivation with regard to x alone. Moreover,

$$\frac{\partial}{\partial u}\left(p + q\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(q\frac{\partial y}{\partial u}\right),$$

* *Ann. de l'Éc. Norm. Sup.*, 1re Sér., t. VII (1870), pp. 163—173; it is reproduced as Note X at the end of volume IV of his *Théorie générale des surfaces.*

and therefore

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial q}{\partial u}\frac{\partial y}{\partial x};$$

and, similarly,

$$\frac{\partial r}{\partial u} = \frac{\partial s}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial s}{\partial u}\frac{\partial y}{\partial x},$$

$$\frac{\partial s}{\partial u} = \frac{\partial t}{\partial x}\frac{\partial y}{\partial u} - \frac{\partial t}{\partial u}\frac{\partial y}{\partial x}.$$

Substituting the values of the differential elements, which occur in the total differential form of the original equation, and remembering that dx and du are independent, we have

$$Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + \ldots + T\frac{\partial t}{\partial u} = 0,$$

as one of the equations: and when in this equation we insert the values of all the derivatives with respect to u, expressed in terms of $\frac{\partial y}{\partial u}$ and $\frac{\partial t}{\partial u}$, we find

$$\left\{Y + Zq + Ps + Qt + R\left(\frac{\partial s}{\partial x} - \frac{\partial t}{\partial x}\frac{\partial y}{\partial x}\right) + S\frac{\partial t}{\partial x}\right\}\frac{\partial y}{\partial u} + \left\{R\left(\frac{\partial y}{\partial x}\right)^2 - S\frac{\partial y}{\partial x} + T\right\}\frac{\partial t}{\partial u} = 0.$$

The variable u is at our disposal; let it be so chosen that, when y is expressed as a function of x and u, the equation

$$R\left(\frac{\partial y}{\partial x}\right)^2 - S\frac{\partial y}{\partial x} + T = 0$$

is satisfied; then, as $\frac{\partial y}{\partial u}$ is not zero, we also have

$$Y + Zq + Ps + Qt + R\left(\frac{\partial s}{\partial x} - \frac{\partial t}{\partial x}\frac{\partial y}{\partial x}\right) + S\frac{\partial t}{\partial x} = 0.$$

When the coefficient of $\frac{\partial t}{\partial x}$ is modified by means of the immediately preceding equation, we have

$$Y + Zq + Ps + Qt + R\frac{\partial s}{\partial x} + T\frac{\frac{\partial t}{\partial x}}{\frac{\partial y}{\partial x}} = 0;$$

and thus there are two other equations involving derivatives with regard to x alone.

The remaining equation, derived from the total differential form of the original equation, is

$$X + Y\frac{\partial y}{\partial x} + Z\frac{\partial z}{\partial x} + P\frac{\partial p}{\partial x} + Q\frac{\partial q}{\partial x} + R\frac{\partial r}{\partial x} + S\frac{\partial s}{\partial x} + T\frac{\partial t}{\partial x} = 0,$$

another equation involving derivatives with regard to x alone. From the earlier equation, we have

$$Y\frac{\partial y}{\partial x} + Zq\frac{\partial y}{\partial x} + Ps\frac{\partial y}{\partial x} + Qt\frac{\partial y}{\partial x} + R\frac{\partial s}{\partial x}\frac{\partial y}{\partial x} + T\frac{\partial t}{\partial x} = 0;$$

subtracting this from the equation just obtained, and using the other equations already constructed, we have

$$X + Zp + Pr + Qs + R\frac{\partial r}{\partial x} + T\frac{\frac{\partial s}{\partial x}}{\frac{\partial y}{\partial x}} = 0,$$

which will be used to replace the immediately preceding equation.

We thus have six equations which involve no derivatives with regard to u but only derivatives with regard to x alone; and these are all the equations of this character which can be constructed among these quantities. The dependent variables, being unknown functions of x and u, are y, z, p, q, r, s, t, seven in number; and so the subsidiary system of six ordinary equations (leaving arbitrary constants to be made arbitrary functions of u) cannot completely determine the seven dependent variables. In the case of an original equation of the first order, the subsidiary system thus constructed was sufficient to determine the dependent variables involved: so that, for equations of the second order, there is a relative diminution in the efficiency of the subsidiary system.

257. In the investigations of Monge and Boole, reasons (which have been explained) led to the consideration of the equations

$$Ar + 2Bs + Ct + rt - s^2 = D,$$

$$Ar + 2Bs + Ct = D,$$

and of no others, the quantities A, B, C, D not involving r, s, t; it so happens that, when these equations are submitted to the preceding process, the subsidiary system is simplified very considerably in form. The equation

$$R\left(\frac{\partial y}{\partial x}\right)^2 - S\frac{\partial y}{\partial x} + T = 0,$$

for the former equation, becomes

$$(A + t)\left(\frac{\partial y}{\partial x}\right)^2 - 2(B - s)\frac{\partial y}{\partial x} + C + r = 0,$$

that is,

$$A\left(\frac{\partial y}{\partial x}\right)^2 - 2B\frac{\partial y}{\partial x} + C + t\left(\frac{\partial y}{\partial x}\right)^2 + 2s\frac{\partial y}{\partial x} + r = 0,$$

or, using the initial equations connected with the change of independent variables, we have

$$A\left(\frac{\partial y}{\partial x}\right)^2 - 2B\frac{\partial y}{\partial x} + C + \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x}\frac{\partial y}{\partial x} = 0.$$

Again, using the equations

$$\frac{\partial p}{\partial x} = r + s\frac{\partial y}{\partial x}, \quad \frac{\partial q}{\partial x} = s + t\frac{\partial y}{\partial x},$$

to remove r and s from the equation

$$Ar + 2Bs + Ct + rt - s^2 = D,$$

we find that, in consequence of the combination $rt - s^2$, there is no term in t^2 and, in consequence of the subsidiary equation just obtained, there is no term in t; the result is

$$A\left(\frac{\partial p}{\partial x} - \frac{\partial y}{\partial x}\frac{\partial q}{\partial x}\right) + 2B\frac{\partial q}{\partial x} - \left(\frac{\partial q}{\partial x}\right)^2 = D,$$

another equation of the subsidiary system. And we always have

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x}.$$

There thus are three equations in the subsidiary system*; it involves four dependent variables, viz. y, z, p, q.

Again, for the equation $Ar + 2Bs + Ct = D$, the subsidiary system is similarly obtainable in the form

$$A\left(\frac{\partial y}{\partial x}\right)^2 - 2B\frac{\partial y}{\partial x} + C = 0,$$

$$A\left(\frac{\partial p}{\partial x} - \frac{\partial y}{\partial x}\frac{\partial q}{\partial x}\right) + 2B\frac{\partial q}{\partial x} = D,$$

with

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x},$$

* It can be resolved so as to acquire the form given in earlier chapters: the resolution is irrelevant to the present discussion.

again a set of three equations in four dependent variables, viz. y, z, p, q.

Now though, in the case of both of these equations, the subsidiary system is ineffective for the complete determination of y, z, p, q, it may happen that integrable combinations can be constructed for special instances of those equations: examples have occurred freely in preceding chapters. In that event, the arbitrary constants that occur in the integrated combinations are to be regarded as functions of the other variable u, arbitrary so far as concerns the subsidiary system of derivatives with regard to x: but the arbitrary functions are subject to the equations

$$\frac{\partial p}{\partial u}=\frac{\partial q}{\partial x}\frac{\partial y}{\partial u}-\frac{\partial y}{\partial x}\frac{\partial q}{\partial u}, \quad \frac{\partial z}{\partial u}=q\frac{\partial y}{\partial u}.$$

The quantities r, s, t have disappeared from the subsidiary system belonging to the less special type and, in their disappearance, they have removed three equations.

Similar remarks apply when the subsidiary system belonging to the equation $f=0$ admits of integrable combinations; the arbitrary functions of u, into which the arbitrary constants in the integrated combinations are changed, are subject to the two preceding equations, as well as to the equations

$$\frac{\partial r}{\partial u}=\frac{\partial s}{\partial x}\frac{\partial y}{\partial u}-\frac{\partial y}{\partial x}\frac{\partial s}{\partial u}, \quad \frac{\partial p}{\partial u}=s\frac{\partial y}{\partial u},$$

$$\frac{\partial s}{\partial u}=\frac{\partial t}{\partial x}\frac{\partial y}{\partial u}-\frac{\partial y}{\partial x}\frac{\partial t}{\partial u}, \quad \frac{\partial q}{\partial u}=t\frac{\partial y}{\partial u}.$$

258. It thus follows that, when we restrict ourselves to derivatives of the second order during the construction of the particular kind of subsidiary system, the number of equations in the system is one less than the number of dependent variables which it contains. It is natural to inquire whether, by proceeding to derivatives of higher orders, it is possible at any stage to construct a complete subsidiary system involving derivatives with regard to x alone. The answer has been given by Darboux: it is to the negative effect, for *the number of equations involving derivatives of x alone is always less by unity than the number of dependent variables which are to be determined.* This theorem of Darboux's can be verified for the next succeeding order as follows:

and the course of the verification will shew how the unit deficiency is maintained in succeeding orders.

Denoting by α, β, γ, δ, the derivatives of z, which are of the third order with regard to the original variables x and y, and taking account of the change of independent variables effected in the method, we have

$$\frac{\partial r}{\partial x} = \alpha + \beta \frac{\partial y}{\partial x}, \quad \frac{\partial r}{\partial u} = \beta \frac{\partial y}{\partial u},$$

$$\frac{\partial s}{\partial x} = \beta + \gamma \frac{\partial y}{\partial x}, \quad \frac{\partial s}{\partial u} = \gamma \frac{\partial y}{\partial u},$$

$$\frac{\partial t}{\partial x} = \gamma + \delta \frac{\partial y}{\partial x}, \quad \frac{\partial t}{\partial u} = \delta \frac{\partial y}{\partial u};$$

these provide three new equations, involving derivatives of x alone, towards the amplified subsidiary system. Also, as before, we have

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial x} \frac{\partial y}{\partial u} - \frac{\partial \beta}{\partial u} \frac{\partial y}{\partial x},$$

$$\frac{\partial \beta}{\partial u} = \frac{\partial \gamma}{\partial x} \frac{\partial y}{\partial u} - \frac{\partial \gamma}{\partial u} \frac{\partial y}{\partial x},$$

$$\frac{\partial \gamma}{\partial u} = \frac{\partial \delta}{\partial x} \frac{\partial y}{\partial u} - \frac{\partial \delta}{\partial u} \frac{\partial y}{\partial x}.$$

Now from the equation $f = 0$, we have

$$f_1 = R\alpha + S\beta + T\gamma + U = 0,$$

$$f_2 = R\beta + S\gamma + T\delta + V = 0,$$

where U and V do not involve α, β, γ, δ: these equations are the complete derivatives of $f = 0$ with regard to the old independent variables. Forming the total derivative of

$$f_1 = R\alpha + S\beta + T\gamma + U = 0,$$

and introducing the new independent variables, we have

$$R \frac{\partial \alpha}{\partial u} + S \frac{\partial \beta}{\partial u} + T \frac{\partial \gamma}{\partial u} + \Theta = 0,$$

$$R \frac{\partial \alpha}{\partial x} + S \frac{\partial \beta}{\partial x} + T \frac{\partial \gamma}{\partial x} + \Phi = 0,$$

where Θ involves linearly the derivatives of x, y, z, p, q, r, s, t with regard to u. The latter equation belongs to the amplified system.

Substituting in the former so that no derivatives with regard to u survive except $\frac{\partial y}{\partial u}$ and $\frac{\partial \delta}{\partial u}$, we find that it takes a form

$$\left\{R\left(\frac{\partial y}{\partial x}\right)^3 - S\left(\frac{\partial y}{\partial x}\right)^2 + T\frac{\partial y}{\partial x}\right\}\frac{\partial \delta}{\partial u} + \Theta_1\frac{\partial y}{\partial u} = 0,$$

where Θ_1 involves derivatives with regard to x only. The coefficient of $\frac{\partial \delta}{\partial u}$ vanishes on account of an earlier equation; and $\frac{\partial y}{\partial u}$ is not zero, so that the relation is

$$\Theta_1 = 0,$$

thus providing another equation for the amplified system. Consequently, the equation

$$f_1 = R\alpha + S\beta + T\gamma + U = 0$$

provides two new equations for the amplified system: they can be denoted by

$$\frac{\partial f_1}{\partial x} = 0, \qquad \Theta_1 = 0.$$

Similarly, the equation

$$f_2 = R\beta + S\gamma + T\delta + V = 0$$

provides two new equations for the amplified system: they can be denoted by

$$\frac{\partial f_2}{\partial x} = 0, \qquad \Theta_2 = 0.$$

It therefore appears as if the completely amplified system contains seven more members than the system for the second order, while it involves only four new dependent variables $\alpha, \beta, \gamma, \delta$, in addition to the old dependent variables; and it might therefore be imagined that the unit deficiency, which marked the old system, is more than supplied by the new system. But this is not, in fact, the case: for the amplified system of thirteen equations, involving the eleven dependent variables $y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta$, contains four dependent equations when $f = 0$ is retained. Thus, in the old system, a linear combination of the equations leads to

$$df = 0;$$

and therefore one of the six can be rejected, because the equation

$$f = 0$$

is retained. Again, a linear combination of

$$\frac{\partial f_1}{\partial x} = 0, \qquad \Theta_1 = 0,$$

together with equations from the old system, leads to the equation

$$df_1 = 0;$$

and therefore one of these two new equations can be rejected because the equation $f_1 = 0$, with other equations of the system, is a necessary consequence of the retained equation $f = 0$. Similarly, a linear combination of

$$\frac{\partial f_2}{\partial x} = 0, \qquad \Theta_2 = 0,$$

together with equations from the old system, leads to the equation

$$df_2 = 0;$$

and therefore one of these two new equations can be rejected because the equation $f_2 = 0$, with other equations of the system, is a necessary consequence of the retained equation $f = 0$. Lastly, a bilinear combination of

$$\frac{\partial f_1}{\partial x} = 0, \quad \Theta_1 = 0, \quad \frac{\partial f_2}{\partial x} = 0, \quad \Theta_2 = 0,$$

together with equations from the old system, leads to the equation

$$d^2 f = 0,$$

which is a necessary consequence of the retained equation $f = 0$; and therefore one more of the new equations can be rejected. Hence the amplified system of ordinary equations subsidiary to $f = 0$, when derivatives of the third order are introduced, contains ten independent members (including $f = 0$) and involves eleven dependent variables: the unit deficiency, characteristic of the subsidiary system for derivatives of the second order, is characteristic of the subsidiary system for derivatives of the third order.

And so for the orders, in increasing succession: the result, as stated in Darboux's theorem, applies to all orders.

Ex. Verify Darboux's theorem for the equations

$$Ar + 2Bs + Ct + rt - s^2 = D,$$

$$Ar + 2Bs + Ct = D,$$

in the case of derivatives of the third order, the quantities A, B, C, D not involving r, s, t.

259. It thus appears that, at no stage in the succession of increasing orders, can we construct a subsidiary system which shall be complete if the original equation is of the second order. But it may happen that, just as the incomplete system in Monge's method and that same system (in a wider significance) in Ampère's method offer integrable combinations, so in the process indicated the incomplete subsidiary system may at some stage offer integrable combinations. If the subsidiary system for derivatives of the second order should offer no integrable combinations, it may happen that the subsidiary system for derivatives of the third order will do so; if not, then the subsidiary system for derivatives of the fourth order may do so: and so on.

Suppose that the incomplete subsidiary system for derivatives of order n offers integrable combinations

$$F = \text{constant}, \quad G = \text{constant}.$$

The constant quantities have their quality on the hypothesis that u is constant and, subject to this hypothesis, they are unrestricted so far as the subsidiary system is concerned; hence we may take

$$F = \phi(u), \quad G = \psi(u).$$

The forms of ϕ and ψ may be limited by the other subsidiary equations which involve derivatives with regard to u: whatever their forms may be, the preceding equations are consistent with the subsidiary system which itself is consistent with the original equation. When we eliminate u, we have an equation

$$V = 0,$$

which is compatible with the given equation: it is an equation of order n; and if either ϕ or ψ is arbitrary, while ϕ and ψ are independent of one another, the new equation $V = 0$ involves an arbitrary function.

We thus have an indication of a method of obtaining equations compatible with a given equation and so, as will be seen almost at once, of proceeding to the integration of the given equation: the method is associated with the name of Darboux. It is not of compelling effect, for its success depends upon contingencies that cannot be controlled: but its operation manifestly is wider than the operation of the methods previously expounded.

DARBOUX'S METHOD FOR CONSTRUCTING COMPATIBLE EQUATIONS.

260. Having thus been led to the inference that, in favouring circumstances, an equation

$$V=0$$

of order higher than the first, say of order n, may be compatible with, and not independent of, a given equation

$$f=0$$

of the second order, we naturally desire to have the means of constructing V: one method, due to Darboux, is as follows. Take all the derivatives of $f=0$, with regard to both independent variables, of all orders up to and including those of order $n-1$; among these will be n equations, which involve the $n+2$ derivatives of z which are of order $n+1$. Take the two first derivatives of $V=0$, supposed to be of order n: these give two equations also involving the $n+2$ derivatives of z which are of order $n+1$: so that, in all, there are $n+2$ equations in these $n+2$ derivatives. The equation $V=0$ is not merely compatible with $f=0$ and therefore with derivatives of $f=0$, but also it is not independent of $f=0$ and therefore of derivatives of $f=0$; hence the $n+2$ equations are not independent of one another. Consequently, when they are resolved so as to express the values of the $n+2$ highest derivatives of z, the values so obtained must be indeterminate; and thus there will be at least two conditions* which, as they involve the first derivatives of V, are a set of simultaneous partial equations of the first order for the determination of V. If they possess a common integral (and the tests, as to the possession of a common integral by a set of simultaneous equations of the first order in a single dependent variable, are known), then an equation compatible with the original equation can be constructed.

* Darboux points out that, if the $n+2$ equations are independent of one another so that the $n+2$ derivatives of z of order $n+1$ can be obtained from them, then all derivatives of z of order higher than n can be expressed in terms of derivatives of order not higher than n. Having obtained these, we should then (by the process of successive quadratures) obtain a value of z which contains only a limited number of arbitrary constants at most and therefore could not imply the existence of an arbitrary function: an integral of the type which he requires would not then be given. Accordingly, the $n+2$ equations must not be independent of one another.

It may be noted that the method in § 238 is the special case of the above method when $n=1$.

The new equations which may thus be obtainable are of two kinds. If it should happen that the equations for V possess only a single integral, say V_1, then the new equation is of the form

$$V_1 = a,$$

where a is an arbitrary constant. If it should happen that the equations for V possess simultaneous integrals, say V_1 and V_2, then the new equation is of the form

$$\phi(V_1, V_2) = 0,$$

where ϕ is an arbitrary function. If it should happen that the equations for V possess two sets of simultaneous integrals, say V_1 and V_2, V_3 and V_4, there are two new equations of the form

$$\phi(V_1, V_2) = 0, \quad \psi(V_3, V_4) = 0,$$

where ϕ and ψ are arbitrary functions; this is the most effective case. Moreover, the equations for V cannot possess more than two sets of integrals, because they are quadratic in form. And it may happen that the equations for V possess no integrals: we should then proceed to the next higher order.

As regards the use to be made of the new equations thus obtained, consider the most effective case, when there are two new equations

$$\phi(V_1, V_2) = 0, \quad \psi(V_3, V_4) = 0,$$

which, for the present, we shall assume* to be compatible with one another. These equations are of order n, so that they are two equations among the $n+1$ derivatives of z of that order. When we take all derivatives of $f=0$ of order $n-2$, there result $n-1$ equations involving the derivatives of z of order n: so that there are, in all, $n+1$ equations in the same number of those derivatives, and they therefore suffice to express those derivatives in terms of the derivatives which are of lower order. When substitution takes place in the last of the $n-2$ sets of differential relations of the type

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$
$$dr = \alpha\,dx + \beta\,dy, \quad ds = \beta\,dx + \gamma\,dy, \quad dt = \gamma\,dx + \delta\,dy,$$
$$\vdots$$

we have quadratures to effect: when these are effected, the final primitive will, in some form or other, involve the two arbitrary functions ϕ and ψ introduced by the new equations.

* The assumption is justified in § 265.

261. The simplest case of course occurs when the equation

$$f(x, y, z, p, q, r, s, t) = 0$$

possesses an intermediate integral, being an equation of the first order: this possibility has been fully discussed under the methods of Monge, Boole, and Ampère. We proceed to the alternative case when the equation $f = 0$ possesses no intermediate integral. In order to consider whether an equation

$$V = V(x, y, z, p, q, r, s, t) = 0$$

coexists with $f = 0$, though it is not resoluble into $f = 0$, we must, after the preceding explanations, form the equations

$$\left.\begin{aligned}
0 &= \frac{df}{dx} + \alpha\frac{\partial f}{\partial r} + \beta\frac{\partial f}{\partial s} + \gamma\frac{\partial f}{\partial t} \\
0 &= \frac{df}{dy} \qquad\qquad + \beta\frac{\partial f}{\partial r} + \gamma\frac{\partial f}{\partial s} + \delta\frac{\partial f}{\partial t} \\
0 &= \frac{dV}{dx} + \alpha\frac{\partial V}{\partial r} + \beta\frac{\partial V}{\partial s} + \gamma\frac{\partial V}{\partial t} \\
0 &= \frac{dV}{dy} \qquad\qquad + \beta\frac{\partial V}{\partial r} + \gamma\frac{\partial V}{\partial s} + \delta\frac{\partial V}{\partial t}
\end{aligned}\right\},$$

where

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q},$$

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q},$$

and so for $\dfrac{dV}{dx}$ and $\dfrac{dV}{dy}$; and then we express the conditions that the values of α, β, γ, δ, furnished by these four equations, are indeterminate. The necessary conditions are

$$\left\| \begin{matrix}
\dfrac{dV}{dx}, & \dfrac{\partial V}{\partial r}, & \dfrac{\partial V}{\partial s}, & \dfrac{\partial V}{\partial t}, & 0 \\[2ex]
\dfrac{dV}{dy}, & 0, & \dfrac{\partial V}{\partial r}, & \dfrac{\partial V}{\partial s}, & \dfrac{\partial V}{\partial t} \\[2ex]
\dfrac{df}{dx}, & \dfrac{\partial f}{\partial r}, & \dfrac{\partial f}{\partial s}, & \dfrac{\partial f}{\partial t}, & 0 \\[2ex]
\dfrac{df}{dy}, & 0, & \dfrac{\partial f}{\partial r}, & \dfrac{\partial f}{\partial s}, & \dfrac{\partial f}{\partial t}
\end{matrix} \right\| = 0,$$

which in effect are equivalent to two conditions formally independent of one another: and each of these conditions is of the second degree in the derivatives of V.

The condition

$$\begin{vmatrix} \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t}, & 0 \\ 0\ , & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t} \\ \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t}, & 0 \\ 0\ , & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t} \end{vmatrix} = 0$$

is easily seen to be

$$\left(\frac{\partial f}{\partial r}\right)^2 \left(\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r}\right) \left(\frac{\partial V}{\partial t} - \mu \frac{\partial V}{\partial s} + \mu^2 \frac{\partial V}{\partial r}\right) = 0,$$

where λ and μ are the roots of the quadratic

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0\,;$$

so that, assuming (as will be assumed) that $\frac{\partial f}{\partial r}$ is not zero, we have either

$$\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r} = 0,$$

or

$$\frac{\partial V}{\partial t} - \mu \frac{\partial V}{\partial s} + \mu^2 \frac{\partial V}{\partial r} = 0.$$

Next, the other independent condition can be taken in the form

$$\begin{vmatrix} \frac{dV}{dx}, & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s}, & \frac{\partial V}{\partial t} \\ \frac{dV}{dy}, & 0\ , & \frac{\partial V}{\partial r}, & \frac{\partial V}{\partial s} \\ \frac{df}{dx}, & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s}, & \frac{\partial f}{\partial t} \\ \frac{df}{dy}, & 0\ , & \frac{\partial f}{\partial r}, & \frac{\partial f}{\partial s} \end{vmatrix} = 0.$$

With this condition, we associate either of the two equations that replace the earlier condition, say

$$\frac{\partial V}{\partial t} - \lambda \frac{\partial V}{\partial s} + \lambda^2 \frac{\partial V}{\partial r} = 0;$$

and we notice that

$$\frac{\partial f}{\partial t} = \lambda\mu \frac{\partial f}{\partial r}, \quad \frac{\partial f}{\partial s} = (\lambda + \mu) \frac{\partial f}{\partial r}.$$

Expanding the condition, using these relations, and removing a factor

$$\frac{\partial f}{\partial s}\frac{\partial V}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial V}{\partial s},$$

(which must not vanish because, taken in conjunction with

$$\frac{\partial V}{\partial t} - \theta\frac{\partial V}{\partial s} + \theta^2 \frac{\partial V}{\partial r} = 0,$$

$$\frac{\partial f}{\partial t} - \theta\frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0,$$

its vanishing would imply that f and V are not functionally independent of one another, quà functions of r, s, t), the equation reduces to

$$\left(\frac{dV}{dx} + \mu\frac{dV}{dy}\right)\frac{\partial f}{\partial r} - \frac{\partial V}{\partial r}\frac{df}{dx} - \frac{1}{\lambda}\frac{\partial V}{\partial t}\frac{df}{dy} = 0.$$

Accordingly, there are two linear systems for the determination of V; they are

$$\left.\begin{aligned} &\frac{\partial V}{\partial t} - \lambda\frac{\partial V}{\partial s} + \lambda^2\frac{\partial V}{\partial r} = 0 \\ &\frac{dV}{dx} + \mu\frac{dV}{dy} - \frac{\frac{\partial V}{\partial r}}{\frac{\partial f}{\partial r}}\frac{df}{dx} - \mu\frac{\frac{\partial V}{\partial t}}{\frac{\partial f}{\partial t}}\frac{df}{dy} = 0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} &\frac{\partial V}{\partial t} - \mu\frac{\partial V}{\partial s} + \mu^2\frac{\partial V}{\partial r} = 0 \\ &\frac{dV}{dx} + \lambda\frac{dV}{dy} - \frac{\frac{\partial V}{\partial r}}{\frac{\partial f}{\partial r}}\frac{df}{dx} - \lambda\frac{\frac{\partial V}{\partial t}}{\frac{\partial f}{\partial t}}\frac{df}{dy} = 0 \end{aligned}\right\},$$

respectively, where λ and μ are the roots of the critical quadratic

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0.$$

The two linear systems become one and the same when this quadratic has equal roots.

These results can be obtained quite simply by assigning the conditions that one of the four equations involving α, β, γ, δ is a linear combination of the other three. In order that this condition may be satisfied, quantities ρ, μ, τ must exist such that

$$\left.\begin{aligned} \frac{dV}{dx} + \mu \frac{dV}{dy} + \tau \frac{df}{dx} + \rho \frac{df}{dy} &= 0 \\ \frac{\partial V}{\partial r} + \tau \frac{\partial f}{\partial r} &= 0 \\ \frac{\partial V}{\partial s} + \mu \frac{\partial V}{\partial r} + \tau \frac{\partial f}{\partial s} + \rho \frac{\partial f}{\partial r} &= 0 \\ \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial s} + \tau \frac{\partial f}{\partial t} + \rho \frac{\partial f}{\partial s} &= 0 \\ \mu \frac{\partial V}{\partial t} + \rho \frac{\partial f}{\partial t} &= 0 \end{aligned}\right\} ;$$

when we substitute in the third equation the value of $\frac{\partial V}{\partial r}$ as given by the second equation, and in the fourth equation the value of $\frac{\partial V}{\partial t}$ as given by the fifth equation, and then eliminate $\frac{\partial V}{\partial s}$ between the two equations thus modified, we have (on removing a non-zero factor $\mu\tau - \rho$) the equation

$$\mu^2 \frac{\partial f}{\partial r} - \mu \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = 0.$$

Let μ and λ be the roots of this equation : then

$$-\tau \frac{\partial f}{\partial r} = \frac{\partial V}{\partial r},$$

$$-\mu \frac{\partial V}{\partial t} = \rho \frac{\partial f}{\partial t} = \rho\mu\lambda \frac{\partial f}{\partial r},$$

giving values of ρ and τ which change the first equation to the form

$$\left(\frac{dV}{dx} + \mu \frac{dV}{dy}\right) \frac{\partial f}{\partial r} = \frac{\partial V}{\partial r} \frac{df}{dx} + \frac{1}{\lambda} \frac{\partial V}{\partial t} \frac{df}{dy},$$

agreeing with the foregoing result. Similarly for the other equation.

Each of the two systems, so obviously similar to the subsidiary systems in the earlier methods, is homogeneous and linear in the derivatives of V; and therefore it is only necessary to apply the

tests, already established for such sets of simultaneous equations as are homogeneous and linear of the first degree, in order to determine whether either set or both sets can possess a common integral or common integrals.

One or two results are immediately obtained. In the first place, we know that, if u be an argument of an arbitrary function in the primitive of $f=0$, and if y be expressed as a function of x and u, then

$$\frac{\partial f}{\partial r}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{\partial f}{\partial s}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial t} = 0,$$

where $\frac{\partial y}{\partial x}$ is formed on the supposition that u is constant; thus $\frac{\partial y}{\partial x}$ is the μ of the preceding investigation, and we have

$$\frac{\partial V}{\partial r}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{\partial V}{\partial s}\frac{\partial y}{\partial x} + \frac{\partial V}{\partial t} = 0.$$

In other words, *the characteristic equation for the argument of an arbitrary function in the primitive of $f=0$ has the same form for any equation of the second order that is compatible with $f=0$.*

In the next place, if

$$u = \text{constant}, \quad v = \text{constant},$$

are two distinct integrals belonging to either of the two systems, then

$$\phi(u, v) = 0,$$

where ϕ is arbitrary, is also an integral of the system. This property obviously follows from the fact that each of the equations in the system is homogeneous and linear.

In the third place, it is clear that all the equations for V are satisfied by taking

$$V = f,$$

which accordingly will be an integrable combination for each of the systems. If, therefore, either system is to furnish an integral of the form

$$\phi(u, v) = 0,$$

which is functionally distinct from $f=0$, the system must provide three integrable combinations, viz.

$$f=0, \quad u = \text{constant}, \quad v = \text{constant}:$$

in other words, when the system for V is made complete in the Jacobian sense, it must possess three independent integrals. The variables which can enter into the expression for V are x, y, z, p, q, r, s, t, being eight in number; and therefore, when the system is made complete, it cannot contain more than five independent equations, all of these being linear and homogeneous in the derivatives of V.

Ex. 1. Consider the equation

$$f=r-t-n\frac{p}{x}=0,$$

where n is a constant. The equation is not integrable by Monge's method: it is integrable in finite terms after change of the variables, by Laplace's method, when n is an even integer.

We have

$$\frac{\partial f}{\partial r}=1, \quad \frac{\partial f}{\partial s}=0, \quad \frac{\partial f}{\partial t}=-1,$$

so that the characteristic equation for the argument of an arbitrary function in the primitive is

$$\theta^2-1=0:$$

hence there are two systems, which belong to the arrangements $\lambda=1$ and $\mu=-1$, $\lambda=-1$ and $\mu=1$, respectively.

Take the arrangement $\lambda=1$, $\mu=-1$. Since

$$\frac{df}{dx}=-n\frac{r}{x}+n\frac{p}{x^2}, \quad \frac{df}{dy}=-n\frac{s}{x},$$

the equations for V (if it exists) are

$$\Delta_1(V)=\frac{\partial V}{\partial r}-\frac{\partial V}{\partial s}+\frac{\partial V}{\partial t}=0,$$

$$\Delta_2(V)=\frac{dV}{dx}-\frac{dV}{dy}+n\left(\frac{r}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial r}+n\frac{s}{x}\frac{\partial V}{\partial t}$$

$$=\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}+(p-q)\frac{\partial V}{\partial z}+(r-s)\frac{\partial V}{\partial p}+(s-t)\frac{\partial V}{\partial q}+n\left(\frac{r}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial r}+n\frac{s}{x}\frac{\partial V}{\partial t}=0.$$

In order that these two equations may have a common integral, they must satisfy the Jacobian condition

$$(\Delta_1, \Delta_2)=0,$$

which, not being satisfied in virtue solely of $\Delta_1=0$ and $\Delta_2=0$, gives a new equation

$$\frac{n}{x}\left(\frac{\partial V}{\partial r}-\frac{\partial V}{\partial t}\right)+2\left(\frac{\partial V}{\partial p}-\frac{\partial V}{\partial q}\right)=0.$$

We combine this with $\Delta_1=0$, and we write

$$\nabla_1=2\frac{\partial V}{\partial r}-\frac{\partial V}{\partial s}+2\frac{x}{n}\frac{\partial V}{\partial p}-2\frac{x}{n}\frac{\partial V}{\partial q}=0,$$

$$\nabla_2=2\frac{\partial V}{\partial t}-\frac{\partial V}{\partial s}-2\frac{x}{n}\frac{\partial V}{\partial p}+2\frac{x}{n}\frac{\partial V}{\partial q}=0;$$

and then, substituting in $\Delta_2=0$ the value of $\dfrac{\partial V}{\partial r}$ from $\nabla_1=0$ and the value of $\dfrac{\partial V}{\partial t}$ from $\nabla_2=0$, we have

$$\nabla_3=\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}+(p-q)\frac{\partial V}{\partial z}+\frac{p}{x}\frac{\partial V}{\partial p}+\left(r-t-\frac{p}{x}\right)\frac{\partial V}{\partial q}+\tfrac{1}{2}n\left(\frac{r+s}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial s}=0.$$

Then

$$(\nabla_1,\ \nabla_2)=0,$$

$$(\nabla_1,\ \nabla_3)=\nabla_4=0,$$

$$(\nabla_2,\ \nabla_3)=-\nabla_4=0,$$

where

$$\nabla_4=4\frac{x}{n}\frac{\partial V}{\partial z}+2\frac{\partial V}{\partial q}+\frac{n-2}{2x}\frac{\partial V}{\partial s}=0.$$

Further,

$$(\nabla_1,\ \nabla_4)=0,$$

$$(\nabla_2,\ \nabla_4)=0,$$

$$(\nabla_3,\ \nabla_4)=\frac{4}{n}\frac{\partial V}{\partial z}-\frac{n-2}{2x^2}\frac{\partial V}{\partial s};$$

and therefore we take

$$\Delta_4=\frac{4}{n}\frac{\partial V}{\partial z}-\frac{n-2}{2x^2}\frac{\partial V}{\partial s}=0,$$

$$\Delta_5=\frac{\partial V}{\partial q}+\frac{n-2}{2x}\frac{\partial V}{\partial s}=0,$$

so that

$$\nabla_4=x\Delta_4+2\Delta_5,$$

and consequently the equation $\nabla_4=0$ can now be omitted.

At the present stage, our equations are

$$\nabla_1=0,\quad \nabla_2=0,\quad \nabla_3=0,\quad \Delta_4=0,\quad \Delta_5=0;$$

they are such that

$$(\nabla_1,\ \nabla_2)=0,\quad (\nabla_1,\ \nabla_3)=0,\quad (\nabla_1,\ \Delta_4)=0,\quad (\nabla_1,\ \Delta_5)=0;$$

$$(\nabla_2,\ \nabla_3)=0,\quad (\nabla_2,\ \Delta_4)=0,\quad (\nabla_2,\ \Delta_5)=0,\quad (\Delta_4,\ \Delta_5)=0;$$

and

$$(\nabla_3,\ \Delta_4)=\tfrac{1}{4}(n-2)(n+4)\frac{1}{x^3}\frac{\partial V}{\partial s},$$

$$(\nabla_3,\ \Delta_5)=\tfrac{1}{4}n\Delta_4-\tfrac{1}{8}(n-2)(n+4)\frac{1}{x^2}\frac{\partial V}{\partial s}.$$

We already have five equations in the system; and this is the greatest number which it can contain, if it is to provide the proper number of integrals:

hence both (∇_3, Δ_4) and (∇_3, Δ_6) must vanish, either identically or in virtue of the equations already retained. This condition can be satisfied only if

$$(n-2)(n+4)=0.$$

We take $n=2$, so that the original equation is

$$r-t-2\frac{p}{x}=0\,;$$

and then a complete system of equations for V is constituted by

$$\nabla_1=0, \quad \nabla_2=0, \quad \nabla_3=0, \quad \Delta_4=0, \quad \Delta_5=0.$$

These can be replaced by an equivalent linear combination of the five, in the form

$$\frac{\partial V}{\partial z}=0, \quad \frac{\partial V}{\partial q}=0,$$

$$2\frac{\partial V}{\partial r}-\frac{\partial V}{\partial s}+x\frac{\partial V}{\partial p}=0, \quad 2\frac{\partial V}{\partial t}-\frac{\partial V}{\partial s}-x\frac{\partial V}{\partial p}=0,$$

$$\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}+\frac{p}{x}\frac{\partial V}{\partial p}+\left(\frac{r+s}{x}-\frac{p}{x^2}\right)\frac{\partial V}{\partial s}=0\,;$$

and we obtain three independent integrals of this complete set in the form

$$r-t-2\frac{p}{x}, \quad x+y,$$

$$\frac{r+2s+t}{x}+\frac{1}{x}\left(r-t-2\frac{p}{x}\right).$$

The original differential equation is

$$r-t-2\frac{p}{x}=0\,;$$

hence the integral, provided by the system, is

$$\frac{r+2s+t}{x}=\text{arbitrary function of } x+y$$

$$=-4f'''(x+y),$$

where f is arbitrary.

Similarly dealing with the system for the arrangement $\lambda=-1$ and $\mu=1$, we find an integral

$$\frac{r-2s+t}{x}=4g'''(y-x).$$

Hence, treating the original equation and the two deduced integrals as simultaneous equations to determine r, s, t, we find

$$\left.\begin{aligned} r &= \frac{p}{x}-xf'''+xg''' \\ s &= \quad -xf'''-xg''' \\ t &= -\frac{p}{x}-xf'''+xg''' \end{aligned}\right\}.$$

They satisfy the conditions

$$\frac{\partial r}{\partial y}=\frac{\partial s}{\partial x}, \quad \frac{\partial s}{\partial y}=\frac{\partial t}{\partial x},$$

so that they are appropriate values for quadratures of

$$dp=r\,dx+s\,dy,$$
$$dq=s\,dx+t\,dy.$$

Effecting the quadratures, we find

$$\frac{p}{x}=-f''-g'',$$
$$q=-x(f''-g'')+f'+g';$$

when these values of p and q are substituted in

$$dz=p\,dx+q\,dy,$$

and quadrature is effected, we have

$$\begin{aligned} z&=f+g-x(f'-g') \\ &=f(x+y)+g(y-x)-x\{f'(x+y)-g'(y-x)\}, \end{aligned}$$

which is the primitive of the equation

$$r-t=2\frac{p}{x};$$

it involves two arbitrary functions f and g.

Note. If n is neither 2 nor -4 in value, then the system has no integral of the type required: and it is found that then the other system also has no such integral. In that event, we cannot obtain equations of the second order which can be associated with $f=0$: if the method is to be effective, it could then be so, only when we proceed to construct equations of higher orders.

Ex. 2. Integrate similarly the equation

$$r-t+4\frac{p}{x}=0.$$

Ex. 3. Shew that the equation

$$s=f(z)$$

cannot be integrated by Darboux's method, when the quantity

$$f(z)f''(z)-f'^2(z)$$

is different from zero. Discuss the cases when this quantity vanishes.

(Lie.)

Ex. 4. Can the equation

$$rt-s^2=c(1+p^2+q^2)^2,$$

where c is a non-vanishing constant, be integrated by Darboux's method?

(Lie.)

Integration of Simultaneous Equations of the Second Order.

262. In the preceding method of dealing with an equation of the second order in two independent variables, when the equation is known not to possess an intermediate integral, it appears that the process leads (for some classes of equations) to the construction of an integral in the form of an equation of the second order, which can be associated with the original equation: and that, in particular, when two integrals of such a form can be constructed, the derivation of the primitive is then merely a matter of quadratures. If, however, only one such integral is obtained, then quadratures will not be sufficient; and the use of the integral for the derivation of the primitive has still to be developed. The position thus created raises a similar question as to the determination of the integral (or integrals) common to two compatible equations of the second order in two independent variables.

Taking the later question indicated as the more general, we denote the two equations by

$$f=0, \quad f'=0,$$

where f and f' are given functions of x, y, z, p, q, r, s, t. Let

$$u=a,$$

where u is another (unknown) function of the same variables, and where a is a constant, be an equation which is compatible with $f=0$ and $f'=0$ and is algebraically independent of them. The three equations usually suffice to give values of r, s, t in terms of x, y, z, p, q: the conditions, that the values so obtained are the second derivatives of z, are that the relations

$$\frac{dr}{dy}=\frac{ds}{dx}, \qquad \frac{ds}{dy}=\frac{dt}{dx},$$

should be satisfied, where

$$\left.\begin{aligned}\frac{d}{dx}&=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q}\\ \frac{d}{dy}&=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}\end{aligned}\right\}.$$

Now, from the two given equations and the assumed equation, we have

$$\frac{\partial f}{\partial r}\frac{dr}{dx}+\frac{\partial f}{\partial s}\frac{ds}{dx}+\frac{\partial f}{\partial t}\frac{dt}{dx}=-\frac{df}{dx},$$

$$\frac{\partial f'}{\partial r}\frac{dr}{dx}+\frac{\partial f'}{\partial s}\frac{ds}{dx}+\frac{\partial f'}{\partial t}\frac{dt}{dx}=-\frac{df'}{dx},$$

$$\frac{\partial u}{\partial r}\frac{dr}{dx}+\frac{\partial u}{\partial s}\frac{ds}{dx}+\frac{\partial u}{\partial t}\frac{dt}{dx}=-\frac{du}{dx};$$

and therefore

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{ds}{dx}=-J\left(\frac{f, f', u}{r, x, t}\right),$$

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{dt}{dx}=-J\left(\frac{f, f', u}{r, s, x}\right).$$

Similarly, we have

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{dr}{dy}=-J\left(\frac{f, f', u}{y, s, t}\right),$$

$$J\left(\frac{f, f', u}{r, s, t}\right)\frac{ds}{dy}=-J\left(\frac{f, f', u}{r, y, t}\right).$$

When we use the relations of condition, we have the equations

$$J\left(\frac{f, f', u}{r, x, t}\right)=J\left(\frac{f, f', u}{y, s, t}\right),$$

$$J\left(\frac{f, f', u}{r, s, x}\right)=J\left(\frac{f, f', u}{r, y, t}\right).$$

Conversely, these two equations are sufficient to secure the two relations of condition.

The two equations thus obtained for u are homogeneous and linear of the first order. When we write

$$\frac{df}{dx}=X,\quad \frac{df}{dy}=Y,\quad \frac{\partial f}{\partial r}=R,\quad \frac{\partial f}{\partial s}=S,\quad \frac{\partial f}{\partial t}=T,$$

$$\frac{df'}{dx}=X',\quad \frac{df'}{dy}=Y',\quad \frac{\partial f'}{\partial r}=R',\quad \frac{\partial f'}{\partial s}=S',\quad \frac{\partial f'}{\partial t}=T',$$

the equations for u take the form

$$\begin{aligned}\Delta_1(u)=\frac{\partial u}{\partial r}(XT'-TX')&+\frac{\partial u}{\partial s}(YT'-TY')\\ &+\frac{\partial u}{\partial t}(RX'-XR'+SY'-YS')\\ &+\frac{du}{dx}(TR'-RT')+\frac{du}{dy}(TS'-ST')=0,\end{aligned}$$

$$\Delta_2(u) = \frac{\partial u}{\partial t}(RY' - YR') + \frac{\partial u}{\partial s}(RX' - XR')$$
$$+ \frac{\partial u}{\partial r}(XS' - SX' + YT' - TY')$$
$$+ \frac{du}{dx}(SR' - RS') + \frac{du}{dy}(TR' - RT') = 0.$$

Here u is a function of the eight variables x, y, z, p, q, r, s, t. If the simultaneous equations $\Delta_1 = 0$, $\Delta_2 = 0$, require other n equations in order to make them a complete Jacobian system, which then will consist of $2 + n$ equations, the equations possess $6 - n$ common integrals. Two such integrals are provided by $u = f$, $u = f'$; and so the equations possess $4 - n$ common integrals, which are algebraically distinct from f and f'.

The method manifestly is effective, if n is less than four; and the conditions for the coexistence of $\Delta_1 = 0$, $\Delta_2 = 0$, are conditions that involve the derivatives of f and f'. Consequently, the conditions that $n < 4$, so that the completed Jacobian system contains not more than five equations, are effectively the conditions that the two equations $f = 0$, $f' = 0$, are compatible with one another. If $f' = 0$ has been constructed by Darboux's method so as to be associated with $f = 0$, the conditions are satisfied; but if $f' = 0$ be given as a new equation, independent of any indication as to the mode of its construction, it is clear that $f' = 0$ cannot be taken arbitrarily.

The simplest case occurs when $n = 0$, so that $\Delta_1 = 0$, $\Delta_2 = 0$, are then a complete Jacobian system of themselves. In that case, there are four common integrals, say, u_1, u_2, u_3, u_4; hence we have six equations in all, viz.

$$f = 0, \quad f' = 0,$$
$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4.$$

When p, q, r, s, t are eliminated among these equations, we have a relation between z, x, y, which involves four arbitrary constants.

The least simple case, when the method is effective, occurs for $n = 1$. We then have three equations

$$f = 0, \quad f' = 0, \quad u = a,$$

which suffice to determine r, s, t as functions of the other five variables; when these are substituted in

$$\begin{aligned} dz &= p\,dx + q\,dy, \\ dp &= r\,dx + s\,dy, \\ dq &= s\,dx + t\,dy, \end{aligned}$$

and the quadratures are effected, we again obtain a primitive involving four arbitrary constants*.

Note. It should, however, be remarked that the two equations $\Delta_1 = 0$, $\Delta_2 = 0$, represent only a single equation, if we retain the equations which can be deduced by Darboux's method applied to the equations $f = 0$, $f' = 0$. From the results of § 261, it follows that, when a quantity θ is chosen so as to satisfy the equation

$$T - \theta S + \theta^2 R = 0,$$

(with the preceding notation), then, because $f' = 0$ is compatible with $f = 0$, the equations

$$\begin{aligned} T' - \theta S' + \theta^2 R' &= 0, \\ T'Y - TY' &= \theta (RX' - R'X), \end{aligned}$$

must be satisfied for one or other of the two values of θ. Consider the expression $\Delta_1 - \theta\Delta_2$. In this expression, the coefficient of $\frac{\partial u}{\partial r}$ is

$$\begin{aligned} & XT' - TX' - \theta (XS' - SX' + YT' - TY') \\ = {} & X (T' - \theta S' + \theta^2 R') - X' (T - \theta S + \theta^2 R) \\ = {} & 0; \end{aligned}$$

the coefficient of $\frac{\partial u}{\partial s}$

$$= YT' - Y'T - \theta (RX' - R'X) = 0;$$

the coefficient of $\frac{\partial u}{\partial t}$

$$\begin{aligned} & = RX' - XR' + SY' - YS' - \theta (RY' - YR') \\ & = \frac{1}{\theta} Y (T' - \theta S' + \theta^2 R') - \frac{1}{\theta} Y' (T - \theta S + \theta^2 R) \\ & = 0; \end{aligned}$$

* The investigation is due to Vályi, *Crelle*, t. xcv (1883), pp. 99—101. Some notes by Bianchi, *Atti d. Reale Acc. d. Lincei*, Ser. 4ta, t. II, 2° Sem., (1886), pp. 218—223, 237—241, 307—310, and a memoir by von Weber, *Münch. Sitzungsb.*, t. xxv (1895—6), pp. 101—113; may also be consulted. See also Goursat's treatise, quoted on p. 303, chapter VI.

the coefficients of $\frac{du}{dx}$ and of $\frac{du}{dy}$ are easily seen to vanish. Hence

$$\Delta_1 = \theta\Delta_2,$$

in virtue of the equations connecting $f = 0$ and $f' = 0$ in Darboux's method.

It thus appears that the Vályi process does not add to the theory: and indeed, analysis similar to that which precedes will, when applied to the equations in Darboux's method expressing the conditions of coexistence of

$$f = 0, \quad u = 0,$$

lead to Vályi's equations. The importance of the results rather lies in the fact that, when two simultaneous equations of the second order

$$f = 0, \quad f' = 0,$$

are given, the common primitive (if any) can be obtained by integrating simultaneous equations of the first order, followed (if need be) by quadratures.

263. In the preceding investigation, the conditions of compatibility have been assumed to be satisfied, although no attempt has been made to construct these conditions. Some, at least, of them can be obtained by constructing the successive derivatives of the equations. Let the two equations $f = 0$ and $f' = 0$ be supposed resolved, so as to express r and t explicitly in a form

$$r + \theta(x, y, z, p, q, s) = 0, \quad t + \phi(x, y, z, p, q, s) = 0.$$

Denoting the four third derivatives of z by α, β, γ, δ, and writing

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \theta\frac{\partial}{\partial p} + s\frac{\partial}{\partial q},$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} - \phi\frac{\partial}{\partial q},$$

we have

$$\alpha + \beta\frac{\partial\theta}{\partial s} + \frac{d\theta}{dx} = 0,$$

$$\beta + \gamma\frac{\partial\theta}{\partial s} + \frac{d\theta}{dy} = 0,$$

$$\gamma + \beta\frac{\partial\phi}{\partial s} + \frac{d\phi}{dx} = 0,$$

$$\delta + \gamma\frac{\partial\phi}{\partial s} + \frac{d\phi}{dy} = 0.$$

These four equations determine values of α, β, γ, δ, unless the quantity

$$1 - \frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s}$$

vanishes.

Assuming, in the first place, that this quantity does not vanish, the values of α, β, γ, δ are definitely expressible in terms of the derivatives of θ and ϕ; and they must obey the relations

$$\frac{d\alpha}{dy} + \frac{\partial\alpha}{\partial s}\gamma = \frac{d\beta}{dx} + \frac{\partial\beta}{\partial s}\beta,$$

$$\frac{d\beta}{dy} + \frac{\partial\beta}{\partial s}\gamma = \frac{d\gamma}{dx} + \frac{\partial\gamma}{\partial s}\beta,$$

$$\frac{d\gamma}{dy} + \frac{\partial\gamma}{\partial s}\gamma = \frac{d\delta}{dx} + \frac{\partial\delta}{\partial s}\beta.$$

When the values of α, β, γ, δ are substituted, we shall have relations affecting the quantities θ and ϕ alone: these must be satisfied. Assuming that the necessary conditions are satisfied, so that definite proper values for the third derivatives of z are known, then all the succeeding derivatives can be constructed. Taking their values for initial values $x = a$, $y = b$, we can construct a Cauchy integral; and the only unassigned (and therefore arbitrary) quantities, which occur in the expression of the integral, are the initial values of z, p, q, s. In other words, we should then expect an integral common to the two equations and involving four arbitrary constants.

Ex. Are the equations

$$r + sX = 0, \quad t + sY = 0,$$

compatible, where X is a function of x only, and Y is a function of y only?

In this case, the critical quantity is $XY - 1$, which does not vanish; consequently, the derivatives of the third order can be obtained. They are

$$\alpha = -sX', \quad \beta = 0, \quad \gamma = 0, \quad \delta = -sY';$$

and it is easy to verify that these values do satisfy the conditions. Consequently, the equations are compatible; and they possess an integral

$$\begin{aligned} z = a + bx + cy + Axy \\ - A\left(\frac{x^2}{2!}X_0 + \frac{x^3}{3!}X_0' + \frac{x^4}{4!}X_0'' + \dots\right) \\ - A\left(\frac{y^2}{2!}Y_0 + \frac{y^3}{3!}Y_0' + \frac{y^4}{4!}Y_0'' + \dots\right), \end{aligned}$$

involving the four arbitrary constants a, b, c, A, the quantities X_0, X_0', ..., Y_0, Y_0', ..., denoting the values of X, X', ..., Y, Y', ..., when $x = 0$, $y = 0$.

Next, suppose that the relation

$$1 - \frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s} = 0$$

is satisfied: then the two derivatives of θ and the two derivatives of ϕ, which involve the quantities α, β, γ, δ, are equivalent to only three equations involving these quantities, together with the equation

$$\frac{d\theta}{dy} - \frac{\partial\theta}{\partial s}\frac{d\phi}{dx} = 0.$$

A procedure, similar to that adopted in the case of the first hypothesis, will serve to settle the question as to whether the new equations

$$1 - \frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial s} = 0, \qquad \frac{d\theta}{dy} - \frac{\partial\theta}{\partial s}\frac{d\phi}{dx} = 0,$$

are compatible with one another and with the original equations: we shall assume that all the necessary conditions are satisfied, and that all the equations are therefore compatible with one another. There are various possibilities.

It may happen that each of the new equations is an equation of only the first order in the derivatives of z. They are compatible with one another and with the original equations: hence they determine p and q, and consequently all derivatives of z, in terms of x, y, z. The resulting primitive, common to all the equations, involves one arbitrary constant: this may be regarded as an arbitrarily assigned value of z for initial values of x and y.

It may happen that only one of the new equations is an equation of the first order, the other three equations being compatible with it and with one another. That equation can be regarded as determining (say) p in terms of x, y, z, q: the other equations, and their derivatives, determine all the derivatives of z in terms of these same quantities. The resulting primitive, common to all the equations, involves two arbitrary constants: they may be regarded as the values arbitrarily assigned to z and q for initial values of x and y.

It may happen that one or other of the new equations, while not an equation of the first order, is a new equation of the second order compatible with, yet algebraically independent of, the original equations: in that case, s is the only derivative of the second order

which it can involve. This last equation, in conjunction with the original equations, will serve to determine r, s, t, in terms of x, y, z, p, q: and then all the derivatives of z are expressible in terms of the same quantities. The resulting primitive, common to all the equations, involves three arbitrary constants: they may be regarded as the values arbitrarily assigned to z, p, q, for initial values of x and y.

Lastly, it may happen that each of the new equations is satisfied identically, so that the only surviving significant equations are the original two equations: such a pair of equations is said, after Lie*, to be *in involution*. They do not suffice for the determination of r, s, t, in terms of x, y, z, p, q: one of these three quantities can have an arbitrary initial value assigned to it, and the other two then are determinate. Again, there are only three independent derived equations in $\alpha, \beta, \gamma, \delta$, and they do not suffice for the determination of these four magnitudes in terms of x, y, z, p, q, r, s, t: one of these magnitudes can have an arbitrary initial value assigned to it, and the other three then are determinate. Similarly for the derivatives of the fourth order: denoting the four deduced equations of the third order by $A=0$, $B=0$, $C=0$, $D=0$, so that $C=0$ is a consequence of the other three, we have

$$\frac{dA}{dx}=0,$$

$$\frac{dA}{dy}=0=\frac{dB}{dx},$$

$$\frac{dB}{dy}=0,$$

$$\frac{dD}{dy}=0,$$

as four independent equations involving the five derivatives of the fourth order. We do not have

$$\frac{dC}{dx}=0, \quad \frac{dC}{dy}=0,$$

as new independent equations, because $C=0$ is a dependent equation; and, as

$$\frac{dD}{dx}=\frac{dC}{dy},$$

* *Leipz. Ber.*, t. XLVII (1895), p. 73.

we do not have

$$\frac{dD}{dx}=0,$$

as a new independent equation. The four independent equations do not suffice for the determination of the five derivatives in question: one of them can have an arbitrary initial value assigned to it, and the other four then are determinate. And so for all the orders in succession: each of them allows an arbitrary initial value. Hence, when we construct an integral by means of a doubly-infinite power-series, the integral so obtained will involve an unlimited number of constants*.

Ex. 1. Consider the equations †

$$f=r-q=0, \quad f'=t-p=0.$$

Thus

$$\frac{df}{dx}=-s, \quad \frac{df}{dy}=-t, \quad \frac{df'}{dx}=-r, \quad \frac{df'}{dy}=-s;$$

and so the equations for u are

$$\Delta_1=\frac{du}{dx}+s\frac{\partial u}{\partial r}+t\frac{\partial u}{\partial s}+r\frac{\partial u}{\partial t}=0,$$

$$\Delta_2=\frac{du}{dy}+t\frac{\partial u}{\partial r}+r\frac{\partial u}{\partial s}+s\frac{\partial u}{\partial t}=0.$$

The Jacobian condition of coexistence, viz.

$$(\Delta_1, \Delta_2)=0,$$

is satisfied identically: thus $\Delta_1=0$ and $\Delta_2=0$ are a complete Jacobian system. Accordingly, they have six common integrals: two of these are f and f': hence other four, algebraically independent of f and f', are required. When any one of the customary methods of integration is adopted, it leads to four integrals

$$\begin{aligned} u_1&=s-z,\\ u_2&=\tfrac{1}{3}(r+s+t)\,e^{-x-y},\\ u_3&=\tfrac{1}{3}(r+\omega^2 s+\omega t)\,e^{-\omega x-\omega^2 y},\\ u_4&=\tfrac{1}{3}(r+\omega s+\omega^2 t)\,e^{-\omega^2 x-\omega y},\end{aligned}$$

where ω is an imaginary cube-root of unity. Eliminating p, q, r, s, t between the six equations

$$f=0, \quad f'=0,$$

$$u_\mu=a_\mu, \qquad (\mu=1, 2, 3, 4),$$

* For further discussion of simultaneous equations in involution, reference may be made to Lie's memoir quoted on p. 295, and to chapter VI in Goursat's treatise quoted on p. 303.

† This example is given by Vályi (*l.c.*).

we have a primitive in the form

$$\begin{aligned} z+a_1 &= a_2 e^{x+y} + \omega a_3 e^{\omega x+\omega^2 y} + \omega^2 a_4 e^{\omega^2 x+\omega y} \\ &= c_2 e^{x+y} + c_3 e^{\omega x+\omega^2 y} + c_4 e^{\omega^2 x+\omega y}, \end{aligned}$$

on changing the constants.

Ex. 2. Obtain an integral common to the equations

$$r-t=0,$$

$$rt-s^2=a\,(x^2-y^2),$$

in the form

$$z=ax+\beta y+\gamma+b\,(x+y)^3+c\,(x-y)^3,$$

where

$$144bc=a.$$

Ex. 3. Obtain an integral common to the equations

$$r-t=0,$$

$$zs=pq+2c^2z^3,$$

in the form

$$z=\wp\{c\,(x+y)+\alpha\}-\wp\{c\,(x-y)+\beta\},$$

where α, β, and the two invariants of the elliptic functions are four arbitrary constants. Verify also that

$$z_1=4\frac{h^2}{c^2}\{e^{h(x+y)+\alpha}-e^{h(x+y)-\alpha}\}^{-2},$$

$$z_2=4\frac{k^2}{c^2}\{e^{k(x-y)+\beta}-e^{k(x-y)-\beta}\}^{-2},$$

where h, α, k, β are arbitrary constants, also are integrals; and further that, when $h=k$, then z_1-z_2 is an integral. (Bourlet.)

Ex. 4. Prove that the two equations

$$3r+s^3=0, \quad st=1,$$

possess a common primitive: and obtain this primitive in its most general form. (Goursat.)

Ex. 5. Prove that the equations

$$\frac{r}{u^2}=\frac{s}{-u}=t,$$

where u is a function of x, y, z, p, q, satisfying the equation

$$u\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}+(p+uq)\frac{\partial u}{\partial z}=0,$$

possess a primitive, representing developable surfaces: and investigate the properties of the surfaces which satisfy the equations. (Goursat.)

Ex. 6. Obtain a complete integral of the equations

$$\frac{r}{1+p^2}=\frac{s}{pq}=\frac{t}{1+q^2}$$

n the form
$$x^2+y^2+z^2+2a_1x+2a_2y+2a_3z=a_4.$$
Do the equations possess other integrals? (Goursat.)

Ex. 7. In connection with the two equations
$$\left.\begin{array}{r} r+f(x,\ y,\ z,\ p,\ q,\ s,\ t)=0 \\ u(x,\ y,\ z,\ p,\ q,\ s,\ t)=a \end{array}\right\},$$
where a is a constant, let quantities Δ, D_2, D_3, D_4 be defined as follows :—
$$\Delta=\left(\frac{\partial u}{\partial t}\right)^2-\frac{\partial f}{\partial s}\frac{\partial u}{\partial s}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial t}\left(\frac{\partial u}{\partial s}\right)^2,$$
$$-D_2=\frac{df}{dy}\left(\frac{\partial u}{\partial t}\right)^2-\frac{du}{dx}\left(\frac{\partial f}{\partial s}\frac{\partial u}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial u}{\partial s}\right)-\frac{du}{dy}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t},$$
$$-D_3=-\frac{df}{dy}\frac{\partial u}{\partial s}\frac{\partial u}{\partial t}+\frac{du}{dx}\frac{\partial u}{\partial t}+\frac{du}{dy}\frac{\partial f}{\partial t}\frac{\partial u}{\partial s},$$
$$-D_4=\frac{df}{dy}\left(\frac{\partial u}{\partial s}\right)^2-\frac{du}{dx}\frac{\partial u}{\partial s}+\frac{du}{dy}\left(\frac{\partial u}{\partial t}-\frac{\partial f}{\partial s}\frac{\partial u}{\partial s}\right),$$
where
$$\frac{d}{dx}=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q},$$
$$\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}:$$
prove that a quantity R exists, such that
$$\Delta\frac{dD_3}{dx}-D_3\frac{d\Delta}{dx}-\left(\Delta\frac{dD_2}{dy}-D_2\frac{d\Delta}{dy}\right)=R\frac{\partial u}{\partial t},$$
$$\Delta\frac{dD_4}{dx}-D_4\frac{d\Delta}{dr}-\left(\Delta\frac{dD_3}{dy}-D_3\frac{d\Delta}{dy}\right)=R\frac{\partial u}{\partial s}.$$

Shew that any integral u of the equation $R=0$, which does not make Δ vanish, leads to a complete integral of the equation $r+f=0$ involving five parameters.

Prove that the two equations
$$r+f=0,\quad u=a,$$
are a system in involution, if
$$\Delta=0,$$
and if, at the same time, another condition (which obtain) is satisfied: the integrals of the equation $r+f=0$ are then governed by the theorem in the text.

Discuss the integral or integrals of the equation $r+f=0$, if $\Delta=0$ (but not the other condition for involution) is satisfied. (König.)

Note. The memoir by König, from which the foregoing results are taken and which has been quoted already (p. 303), discusses also the case, when an equation of order higher than the second is compatible with an equation $r+f=0$.

HAMBURGER'S METHOD.

264. The method, usually associated with the name of Darboux, is not the only process of constructing equations of the second order (when this can be done) compatible with a given equation: an equivalent set of equations, in differential elements rather than in differential coefficients, is provided by Hamburger's application*, of his process (as explained in Chapter XI of the preceding volume) of solving a number of simultaneous equations of the first order in the same number of dependent variables, to the integration of equations of order higher than the first. In particular, consider a general equation

$$f = f(x, y, z, p, q, r, s, t) = 0;$$

the first stage of the problem is to determine two other equations

$$u = a, \quad v = b,$$

where u and v are functions of r, s, t, and of the other variables, such that u, v, f, being algebraically independent, provide values of r, s, t, which make the equations

$$\left.\begin{aligned} dp &= r\,dx + s\,dy \\ dq &= s\,dx + t\,dy \\ dz &= p\,dx + q\,dy \end{aligned}\right\}$$

an integrable system. With the preceding notation, we have

$$R\frac{\partial r}{\partial x} + S\frac{\partial s}{\partial x} + T\frac{\partial t}{\partial x} = -X,$$

$$R\frac{\partial r}{\partial y} + S\frac{\partial s}{\partial y} + T\frac{\partial t}{\partial y} = -Y;$$

and we also have

$$\begin{aligned} dr &= \frac{\partial r}{\partial x}dx + \frac{\partial s}{\partial x}dy, \\ ds &= \frac{\partial s}{\partial x}dx + \frac{\partial t}{\partial x}dy = \frac{\partial r}{\partial y}dx + \frac{\partial s}{\partial y}dy, \\ dt &= \qquad\qquad\qquad\qquad \frac{\partial s}{\partial y}dx + \frac{\partial t}{\partial y}dy. \end{aligned}$$

Following the method of dealing with simultaneous equations of the first order, we construct two linear combinations of these

* *Crelle*, t. XCIII (1882), pp. 188—214.

elations in differential elements, one of them involving derivatives f r, s, t, with regard to x only, and the other of them involving lerivatives of the same quantities with regard to y only: they are

$$\lambda_1 dr + \lambda_2 ds = \frac{\partial r}{\partial x}\lambda_1 dx + \frac{\partial s}{\partial x}(\lambda_2 dx + \lambda_1 dy) + \frac{\partial t}{\partial x}\lambda_2 dy,$$

$$\lambda_1 ds + \lambda_2 dt = \frac{\partial r}{\partial y}\lambda_1 dx + \frac{\partial s}{\partial y}(\lambda_2 dx + \lambda_1 dy) + \frac{\partial t}{\partial y}\lambda_2 dy,$$

vhatever be the values of λ_1 and λ_2. In connection with these elations, and having regard to the preceding complete derivatives f f with regard to x and to y, we construct the subsidiary quations

$$\frac{\lambda_1 dx}{R} = \frac{\lambda_2 dx + \lambda_1 dy}{S} = \frac{\lambda_2 dy}{T} = \frac{\lambda_1 dr + \lambda_2 ds}{-X} = \frac{\lambda_1 ds + \lambda_2 dt}{-Y}.$$

'he equality of the first three fractions determines values of $\frac{dy}{dx}$ nd of $\frac{\lambda_2}{\lambda_1}$ for the subsidiary system. Taking

$$dy = \mu\, dx,$$

re have

$$\lambda_1 (S - \mu R) = \lambda_2 R,$$

$$\lambda_2 (T - \mu S) = -\mu\lambda_1 T,$$

$$= -\mu\lambda_2 \frac{RT}{S - \mu R},$$

hich, on the removal of a non-zero factor $\lambda_2 S$, gives

$$\mu^2 R - \mu S + T = 0:$$

ıd then

$$\frac{\lambda_2}{\lambda_1} = \frac{S}{R} - \mu = \lambda,$$

here λ is the other root of the quadratic. Hence we have

$$\left.\begin{aligned} R\,dr + (S - \mu R)\,ds &= -X\,dx \\ R\,ds + (S - \mu R)\,dt &= -Y\,dx \\ dy &= \mu\,dx \\ dz &= (p + \mu q)\,dx \\ dp &= (r + \mu s)\,dx \\ dq &= (s + \mu t)\,dx \end{aligned}\right\},$$

where μ is a root of the quadratic

$$\mu^2 R - \mu S + T = 0.$$

Let $u =$ constant be an integral equivalent of this subsidiary system, so that $du = 0$ is a linear combination of the set of equations: then

$$\begin{aligned}\left(\frac{du}{dx} + \mu \frac{du}{dy}\right) dx + \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt \\ = \alpha \{R dr + (S - \mu R) ds + X dx\} \\ + \beta \{R ds + (S - \mu R) dt + Y dx\};\end{aligned}$$

and therefore

$$\left.\begin{aligned}\frac{du}{dx} + \mu \frac{du}{dy} &= \alpha X + \beta Y \\ \frac{\partial u}{\partial r} &= \alpha R \\ \frac{\partial u}{\partial s} &= \alpha (S - \mu R) + \beta R \\ \frac{\partial u}{\partial t} &= \beta (S - \mu R) = \beta \frac{T}{\mu}\end{aligned}\right\}.$$

When α and β are eliminated among these four equations, two relations survive; they are

$$\left.\begin{aligned}\lambda^2 \frac{\partial u}{\partial r} - \lambda \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0 \\ \frac{du}{dx} + \mu \frac{du}{dy} - \frac{X}{R} \frac{\partial u}{\partial r} - \mu \frac{Y}{T} \frac{\partial u}{\partial t} = 0\end{aligned}\right\},$$

which (§ 261) are the equations given by Darboux's method as characteristic of a quantity u, where

$$u = \text{constant}$$

is an equation of the second order that can be associated with the given equation $f = 0$. Also, when the quadratic

$$\mu^2 R - \mu S + T = 0$$

has unequal roots, we have the two sets of equations that occur in Darboux's method, on taking the values of μ in turn.

Remembering the relation between the subsidiary system in differential elements, which arises in Monge's method of constructing an intermediate integral of the first order (when it

exists), and the subsidiary system in differential coefficients, which arises in Boole's method of achieving the same aim, we see that there is a corresponding relation between the two sets of equations for constructing a compatible integral in the form of an equation of the second order. The integration of the equations

$$\left.\begin{aligned} \lambda^2 \frac{\partial u}{\partial r} - \lambda \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0 \\ \frac{du}{dx} + \mu \frac{du}{dy} - \frac{X}{R}\frac{\partial u}{\partial r} - \mu \frac{Y}{T}\frac{\partial u}{\partial t} = 0 \end{aligned}\right\}$$

is equivalent to the quadrature of the system

$$\left.\begin{aligned} \frac{dx}{1} = \frac{dy}{\mu} = \frac{dz}{p+\mu q} = \frac{dp}{r+\mu s} = \frac{dq}{s+\mu t} \\ = \frac{dr + \lambda ds}{-\dfrac{X}{R}} = \frac{ds + \lambda dt}{-\dfrac{Y}{R}} \end{aligned}\right\},$$

where λ and μ are the roots of the quadratic

$$\mu^2 R - \mu S + T = 0.$$

265. It was assumed (§ 260) that, if each of the two systems of equations possesses an integral involving r, s, t, the two integrals can be combined with $f=0$ to furnish values of r, s, and t, which make the quadratures possible: the assumption can be established as follows. Let μ_1 and μ_2 be the two roots of the quadratic; and let

$$v_1 = 0, \quad v_2 = 0,$$

be the integrals of the respective systems, such that these two equations and $f=0$ can be resolved so as to express r, s, t in terms of the other variables: then the Jacobian

$$J\left(\frac{f,\ v_1,\ v_2}{r,\ s,\ t}\right)$$

does not vanish. Writing

$$\frac{dv_1}{dx} = X_1, \quad \frac{\partial v_1}{\partial r} = R_1, \quad \frac{\partial v_1}{\partial s} = S_1, \quad \frac{\partial v_1}{\partial t} = T_1,$$

and so for derivatives of v_2, we have the equations

$$X + R\frac{dr}{dx} + S\frac{ds}{dx} + T\frac{dt}{dx} = 0,$$

$$X_1 + R_1\frac{dr}{dx} + S_1\frac{ds}{dx} + T_1\frac{dt}{dx} = 0,$$

satisfied identically when the resolved values of r, s, t are substituted in $f=0$, $v_1=0$; and therefore

$$X_1 R - R_1 X + (S_1 R - R_1 S)\frac{ds}{dx} + (T_1 R - R_1 T)\frac{dt}{dx} = 0.$$

Similarly,

$$Y_1 T - T_1 Y + (R_1 T - T_1 R)\frac{dr}{dy} + (S_1 T - T_1 S)\frac{ds}{dy} = 0.$$

But on account of the equations satisfied by v_1, we have

$$X_1 - R_1\frac{X}{R} = -\mu_1\left(Y_1 - \frac{Y}{T}T_1\right),$$

so that

$$RX_1 - R_1 X = -\frac{R}{T}\mu_1(TY_1 - YT_1)$$

$$= -\frac{1}{\mu_2}(TY_1 - YT_1);$$

consequently,

$$\mu_2(S_1 R - R_1 S)\frac{ds}{dx} + (R_1 T - T_1 R)\frac{dr}{dy}$$

$$+ \mu_2(T_1 R - R_1 T)\frac{dt}{dx} + (S_1 T - T_1 S)\frac{ds}{dy} = 0.$$

Now

$$\mu_2(S_1 R - R_1 S) = \mu_2 R\{S_1 - (\mu_1 + \mu_2)R_1\}$$

$$= \mu_2 R\left(\frac{1}{\mu_2}T_1 - \mu_1 R_1\right)$$

$$= RT_1 - TR_1,$$

so that the first line of the last equation is

$$(RT_1 - TR_1)\left(\frac{ds}{dx} - \frac{dr}{dy}\right);$$

and, similarly, the second line is found to be

$$(T_1 S - S_1 T)\left(\frac{dt}{dx} - \frac{ds}{dy}\right);$$

thus the equation is

$$(RT_1 - TR_1)\left(\frac{ds}{dx} - \frac{dr}{dy}\right) + (T_1 S - S_1 T)\left(\frac{dt}{dx} - \frac{ds}{dy}\right) = 0.$$

Similarly, we find

$$(RT_2 - TR_2)\left(\frac{ds}{dx} - \frac{dr}{dy}\right) + (T_2 S - S_2 T)\left(\frac{dt}{dx} - \frac{ds}{dy}\right) = 0.$$

Now

$$\begin{vmatrix} RT_1 - TR_1, & T_1S - S_1T \\ RT_2 - TR_2, & T_2S - S_2T \end{vmatrix} = -T \begin{vmatrix} R\,, & S\,, & T \\ R_1, & S_1, & T_1 \\ R_2, & S_2, & T_2 \end{vmatrix};$$

the Jacobian does not vanish and, without loss of generality, we may assume that T does not vanish; hence the two equations can hold only if

$$\frac{ds}{dx} - \frac{dr}{dy} = 0, \quad \frac{dt}{dx} - \frac{ds}{dy} = 0,$$

that is, the conditions of integrability are satisfied.

In this discussion, an assumption obviously is made that the roots of the quadratic differ from one another. When they are the same, so that there is only a single set of equations, and when that single set offers an integrable combination $v = 0$, we then should use the Vályi process for associating a third equation with $v = 0, f = 0$. The consideration of the matter will be resumed in the discussion of the characteristics (Chap. XX.).

The actual construction of the integrals of a subsidiary system, when they are possessed by it, and the use made of the integrals, correspond with the construction and the use in Ampère's method. Suppose that a subsidiary system has two integrals, say

$$v_1 = \text{constant}, \quad v_2 = \text{constant};$$

the equation to be associated with the original equation is of the form

$$v_2 = \phi(v_1),$$

where ϕ is an arbitrary function. Suppose also that the other subsidiary system has two integrals

$$w_1 = \text{constant}, \quad w_2 = \text{constant};$$

then, similarly, we have an equation

$$w_2 = \psi(w_1),$$

where ψ is arbitrary. The three equations

$$f = 0, \quad v_2 = \phi(v_1), \quad w_2 = \psi(w_1),$$

are used to effect the quadratures in

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy, \quad dz = p\,dx + q\,dy;$$

and frequently, it is convenient in practice to replace x and y by v_1 and w_1 as the independent variables.

Ex. 1. Let it be required to integrate the equation

$$r - qs + pt = 0.$$

With the preceding notation, we have

$$R = 1, \quad S = -q, \quad T = t;$$

thus λ and μ are the roots of the quadratic

$$\theta^2 + q\theta + p = 0,$$

so that

$$\lambda + \mu = -q, \quad \lambda\mu = p.$$

Also

$$X = rt - s^2, \quad Y = 0:$$

thus a subsidiary system, taken in connection with the differential elements, is

$$\frac{dp}{dx} = r + \mu s = (q + \mu)\, s - pt = -\lambda s - \lambda\mu t,$$

$$\frac{dq}{dx} = s + \mu t,$$

and therefore

$$\frac{dp}{dx} + \lambda \frac{dq}{dx} = 0.$$

Hence

$$\frac{dp}{dq} = -\lambda = \tfrac{1}{2} q + (\tfrac{1}{4} q^2 - p)^{\frac{1}{2}},$$

and therefore

$$\left(\frac{dp}{dq}\right)^2 - q \frac{dp}{dq} = -p,$$

the well-known Clairaut form: we therefore can write

$$\frac{dp}{dq} = \text{constant},$$

so that

$$\lambda = \text{constant}$$

for the system. Also, as $Y = 0$, we have

$$ds + \lambda\, dt = 0,$$

that is,

$$s + \lambda t = \text{constant};$$

and so an appropriate integral is given by

$$s + \lambda t = \phi(\lambda).$$

Similarly, from the other system, we have

$$s + \mu t = \psi(\mu),$$

where ϕ and ψ are arbitrary. Thus we have

$$p = \lambda\mu, \quad q = -\lambda - \mu,$$

$$t = \frac{\phi(\lambda) - \psi(\mu)}{\lambda - \mu},$$

$$s = \frac{-\mu\phi(\lambda) + \lambda\psi(\mu)}{\lambda - \mu},$$

$$r = qs - pt = \frac{\mu^2\phi(\lambda) - \lambda^2\psi(\mu)}{\lambda - \mu},$$

and therefore

$$rt-s^2=-\phi(\lambda)\,\psi(\mu).$$

The equations for quadrature are

$$r\,dx+s\,dy=dp=\mu\,d\lambda+\lambda\,d\mu,$$
$$s\,dx+t\,dy=dq=-d\lambda-d\mu;$$

hence

$$(rt-s^2)\,dx=(\mu t+s)\,d\lambda+(\lambda t+s)\,d\mu$$
$$=\psi(\mu)\,d\lambda+\phi(\lambda)\,d\mu,$$

that is,

$$-dx=\frac{d\lambda}{\phi(\lambda)}+\frac{d\mu}{\psi(\mu)}.$$

Similarly,

$$-dy=\frac{\lambda}{\phi(\lambda)}\,d\lambda+\frac{\mu}{\psi(\mu)}\,d\mu;$$

and therefore

$$dz=p\,dx+q\,dy$$
$$=\frac{\lambda^2}{\phi(\lambda)}\,d\lambda+\frac{\mu^2}{\psi(\mu)}\,d\mu.$$

To obtain explicit expressions for x, y, z, we take

$$\phi(\lambda)=\frac{1}{f'''(\lambda)},\quad \psi(\mu)=\frac{1}{g'''(\mu)},$$

where f and g are arbitrary; and we have

$$\left.\begin{aligned} z&=\lambda^2 f''(\lambda)-2\lambda f'(\lambda)+2f(\lambda)+\mu^2 g''(\mu)-2\mu g'(\mu)+2g(\mu)\\ -x&=f''(\lambda)+g''(\mu)\\ -y&=\lambda f''(\lambda)-f'(\lambda)+\mu g''(\mu)-g'(\mu)\end{aligned}\right\},$$

which constitute the primitive of the equation.

Ex. 2. Obtain the primitive of the equation

$$qr-ps+t=\frac{p^2-4q}{y+a},$$

in the form

$$-\frac{1}{y+a}=f''(a)+g''(\beta),$$

$$-\frac{x}{y+a}=af''(a)-f'(a)+\beta g''(\beta)-g'(\beta),$$

$$-z-\frac{x^2}{y+a}=a^2f''(a)-2af'(a)+2f(a)+\beta^2g''(\beta)-2\beta g'(\beta)+2g(\beta).$$

(De Boer.)

Ex. 3. Shew that the equation

$$r+t=zu,$$

where u is a function of x and y different from zero, cannot have an intermediate integral. Find the equation or equations to be satisfied by u, in order that two equations of the second order, of the Darboux type, may be compatible with the given equation; and, assuming the conditions satisfied, obtain the primitive.

Ex. 4. Obtain two equations of the second order, that are compatible with the equation

$$r=f(s),$$

in the form

$$\left.\begin{array}{r} t-g(s)=\phi(y) \\ y+xf'(s)=\psi(s) \end{array}\right\},$$

where

$$g'(s)f'(s)=1;$$

and construct the primitive. (Goursat.)

Ex. 5. When a surface is referred to its minimal lines as parametric curves, each of the coordinates of any point on the surface satisfies the equation

$$rt-s^2-cqr-apt=b(\lambda-pq)-acpq,$$

where the arc on the surface is given by

$$ds^2=4\lambda\,dx\,dy,$$

p, q, r, s, t are the first and the second derivatives of any one of the coordinates, and where

$$a=\frac{\partial(\log\lambda)}{\partial x},\quad c=\frac{\partial(\log\lambda)}{\partial y},\quad b=2\frac{\partial^2(\log\lambda)}{\partial x\,\partial y}:$$

so that the equations of all surfaces deformable into a given surface are thus provided *.

Prove that the differential equation possesses no intermediate integral of the first order: and find the equation which must be satisfied by λ, if equations of the second order exist that are compatible with, but are algebraically independent of, the given equation.

EQUATIONS $f(r, s, t)=0$ INTEGRABLE BY DARBOUX'S METHOD.

266. In an interesting memoir †, De Boer discusses equations of the form

$$f(r, s, t)=0,$$

which admit two compatible equations of the second order derivable through the two subsidiary systems of equations in Darboux's method. The following discussion differs in form from that which is given in the memoir quoted.

Let the given equation $f=0$ be resolved with regard to one of its arguments, say r, so that it has the form

$$r+g(s, t)=0.$$

* Darboux, *Théorie générale des surfaces*, t. III, p. 261.

† *Arch. Néerl.*, t. XXVII (1894), pp. 355—412.

Writing

$$\frac{\partial g}{\partial s} = S, \quad \frac{\partial g}{\partial t} = T,$$

the quadratic, which has λ and μ for its roots, is

$$\kappa^2 - \kappa S + T = 0:$$

these quantities λ and μ are functions of s and t only. The equations for the determination of u are, in general,

$$\lambda^2 \frac{\partial u}{\partial r} - \lambda \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0,$$

$$\frac{du}{dx} + \mu \frac{du}{dy} - \frac{X}{R} \frac{\partial u}{\partial r} - \mu \frac{Y}{T} \frac{\partial u}{\partial t} = 0.$$

In the present case,

$$X = 0, \quad Y = 0,$$

for f involves only r, s, t: also, if u contained r explicitly, that variable could be removed by substituting its value $-g(s, t)$. Hence we have equations for u in the form

$$\Delta_1(u) = \frac{\partial u}{\partial r} = 0,$$

$$\Delta_2(u) = \lambda \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0,$$

$$\Theta(u) = \frac{du}{dx} + \mu \frac{du}{dy} = 0.$$

Applying the Jacobian tests of coexistence, we must have

$$\Delta_3(u) = (\Delta_1, \Theta) = \frac{\partial u}{\partial p} = 0;$$

and then

$$\Delta_4(u) = (\Delta_3, \Theta) = \frac{\partial u}{\partial z} = 0.$$

Using $\Delta_3 = 0$ and $\Delta_4 = 0$, we can replace Θ by Θ', where

$$\Theta'(u) = \frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial y} + (s + \mu t) \frac{\partial u}{\partial q} = 0.$$

We have

$$(\Delta_1, \Theta') = 0, \quad (\Delta_1, \Delta_2) = 0, \quad (\Delta_1, \Delta_3) = 0, \quad (\Delta_1, \Delta_4) = 0;$$

$$(\Delta_2, \Delta_3) = 0, \quad (\Delta_2, \Delta_4) = 0, \quad (\Delta_3, \Delta_4) = 0;$$

$$(\Delta_3, \Theta') = 0, \quad (\Delta_4, \Theta') = 0;$$

and it remains to consider (Δ_2, Θ'). We have

$$(\Delta_2, \Theta') = \frac{\partial u}{\partial y}\Delta_2(\mu) + \{t\Delta_2(\mu) + \lambda - \mu\}\frac{\partial u}{\partial q}.$$

Let

$$t + \frac{\lambda - \mu}{\Delta_2(\mu)} = \theta;$$

and take

$$\Delta_5(u) = \frac{\partial u}{\partial y} + \theta\frac{\partial u}{\partial q} = 0,$$

$$\Delta_6(u) = \frac{\partial u}{\partial x} + (s + \mu t - \mu\theta)\frac{\partial u}{\partial q} = 0,$$

so that $(\Delta_2, \Theta') = 0$; and $\Theta' = 0$ can be omitted when $\Delta_5 = 0$ and $\Delta_6 = 0$ are retained. The complete Jacobian system is not to contain more than six members* if there is to be an equation of the type specified by Darboux: for the system must then possess two distinct integrals. We already have six equations, viz.

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0, \quad \Delta_4 = 0, \quad \Delta_5 = 0, \quad \Delta_6 = 0;$$

hence the equations

$$(\Delta_2, \Delta_5) = 0, \quad (\Delta_2, \Delta_6) = 0,$$

must be satisfied, all the other conditions for the system being actually satisfied. Consequently,

$$\Delta_2(\theta) = 0$$

from the former, and

$$(t - \theta)\Delta_2(\mu) - \mu\Delta_2(\theta) + \lambda - \mu = 0;$$

and both these conditions are satisfied by means of the single equation

$$\Delta_2(\theta) = 0,$$

that is,

$$\lambda\frac{\partial\theta}{\partial s} - \frac{\partial\theta}{\partial t} = 0,$$

or

$$\left(\lambda\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\lambda - \mu}{\lambda\dfrac{\partial\mu}{\partial s} - \dfrac{\partial\mu}{\partial t}}\right) = 1,$$

which is a necessary condition that the selected system should have an integral of the assigned type.

* This appears to differ from the earlier theory: the explanation is that $\Delta_1 = 0$ prevents the original equation from occurring as an integral.

Suppose this condition satisfied. As λ is a function of s and t only, let an integral of

$$\frac{ds}{\lambda} = dt$$

be given by

$$v = v(s, t) = \text{constant}.$$

The one integral of the Jacobian system is

$$v;$$

and another integral is given by

$$q - y\theta - x(s + \mu t - \mu\theta);$$

and therefore an equation that can coexist with the given equation is

$$q - y\theta - x(s + \mu t - \mu\theta) = \phi(v),$$

where ϕ is an arbitrary function.

In order that the same kind of equation, compatible with the original equation, may be provided by the alternative subsidiary system, it is obvious that the corresponding condition (obtained by the interchange of λ and μ) must be satisfied, that is,

$$\left(\mu\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\mu - \lambda}{\mu\dfrac{\partial\lambda}{\partial s} - \dfrac{\partial\lambda}{\partial t}}\right) = 1;$$

and then, if

$$\vartheta = t + \frac{\mu - \lambda}{\mu\dfrac{\partial\lambda}{\partial s} - \dfrac{\partial\lambda}{\partial t}},$$

the required equation to be associated with the given equation is

$$q - y\vartheta - x(s + \lambda t - \lambda\vartheta) = \psi(w),$$

where ψ is arbitrary, and where

$$w = w(s, t) = \text{constant}$$

is an integral of

$$\frac{ds}{\mu} = dt.$$

We now have three relations, theoretically expressing r, s, t in terms of the other variables in such a way that the equations

$$\left.\begin{aligned} dp &= r\,dx + s\,dy \\ dq &= s\,dx + t\,dy \\ dz &= p\,dx + q\,dy \end{aligned}\right\}$$

are a completely integrable system. In practice, and assuming the equation $f(r, s, t) = 0$ resolved with regard to r, it would obviously be convenient to make s and t (or v and w) the independent variables for the operative quadratures.

It therefore appears that the conditions

$$\left(\lambda \frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\lambda - \mu}{\lambda \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t}}\right) = 1,$$

$$\left(\mu \frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\mu - \lambda}{\mu \frac{\partial \lambda}{\partial s} - \frac{\partial \lambda}{\partial t}}\right) = 1,$$

secure the existence of two equations of the specified type which can be associated with $r + g(s, t) = 0$.

267. In order to discuss the two preceding conditions to be satisfied by the function $g(s, t)$, and in order both to abbreviate the notation and to simplify it, we replace s and t temporarily by x and y. Derivatives of g with regard to its arguments will be denoted by $p, q, r, s, t, \alpha, \beta, \gamma, \delta$: derivatives of λ and of μ with regard to x and y will be denoted by $\lambda_1, \lambda_2, \ldots$, so that

$$\frac{\partial \lambda}{\partial x} = \lambda_1, \quad \frac{\partial^2 \lambda}{\partial x^2} = \lambda_{11}, \quad \frac{\partial \lambda}{\partial y} = \lambda_2, \quad \frac{\partial^2 \lambda}{\partial x \partial y} = \lambda_{12}, \ \ldots,$$

and similarly for μ.

The first of the two conditions is

$$\frac{\lambda\lambda_1 - \lambda_2}{\lambda\mu_1 - \mu_2} - \frac{\lambda - \mu}{(\lambda\mu_1 - \mu_2)^2}\{\lambda^2\mu_{11} - 2\lambda\mu_{12} + \mu_{22} + \mu_1(\lambda\lambda_1 - \lambda_2)\} = 2,$$

which is easily reduced to

$$(\lambda\lambda_1 - \lambda_2)(\mu\mu_1 - \mu_2) - 2(\lambda\mu_1 - \mu_2)^2 = (\lambda - \mu)(\lambda^2\mu_{11} - 2\lambda\mu_{12} + \mu_{22});$$

and the second of the two conditions similarly reduces to

$$(\mu\mu_1 - \mu_2)(\lambda\lambda_1 - \lambda_2) - 2(\mu\lambda_1 - \lambda_2)^2 = (\mu - \lambda)(\mu^2\lambda_{11} - 2\mu\lambda_{12} + \lambda_{22}).$$

Now

$$\lambda + \mu = S = p, \quad \lambda\mu = T = q;$$

hence

$$\lambda_1 + \mu_1 = r,$$
$$\lambda_2 + \mu_2 = s = \mu\lambda_1 + \lambda\mu_1,$$
$$t = \mu\lambda_2 + \lambda\mu_2,$$

and therefore

$$\left.\begin{aligned}(\lambda-\mu)\lambda_1 &= \lambda r - s\\ (\lambda-\mu)\mu_1 &= -\mu r + s\end{aligned}\right\},\qquad \left.\begin{aligned}(\lambda-\mu)\lambda_2 &= \lambda s - t\\ (\lambda-\mu)\mu_2 &= -\mu s + t\end{aligned}\right\}.$$

As

$$\lambda\mu_1 - \mu_2 = -(\mu\lambda_1 - \lambda_2),$$

the two left-hand sides of the reduced conditions are the same: their common value is

$$-\frac{A}{(\lambda-\mu)^2},$$

where

$$\begin{aligned}A &= (\lambda^2 r - 2\lambda s + t)(\mu^2 r - 2\mu s + t) + 2\{\lambda\mu r - (\lambda+\mu)s + t\}^2\\ &= 3(qr - ps + t)^2 + (rt - s^2)(p^2 - 4q).\end{aligned}$$

Subtracting the two equations in their reduced forms (and assuming, as has been done throughout, that $\lambda - \mu$ is not zero), we have

$$\lambda^2\mu_{11} + \mu^2\lambda_{11} - 2(\lambda\mu_{12} + \mu\lambda_{12}) + \mu_{22} + \lambda_{22} = 0.$$

When we construct the symmetrical combinations in this equation and substitute, it is found (after some reduction) that this equation takes the form

$$q\alpha - p\beta + \gamma = 2\frac{\lambda_1 - \mu_1}{\lambda - \mu}(qr - ps + t).$$

A first integral of this equation can be at once obtained in the form

$$qr - ps + t = (\lambda - \mu)^2 Y,$$

where, so far, Y is an arbitrary function of y.

Again, adding the equations, we have

$$-\frac{2A}{(\lambda-\mu)^2} = (\lambda-\mu)\{\lambda^2\mu_{11} - \mu^2\lambda_{11} - 2(\lambda\mu_{12} - \mu\lambda_{12}) + \mu_{22} - \lambda_{22}\};$$

constructing the combinations on the right-hand side, substituting, and reducing, we find

$$-\frac{2A}{(\lambda-\mu)^2} = -p(q\alpha - p\beta + \gamma) + 2(q\beta - p\gamma + \delta) + 2U,$$

where

$$U = \frac{qr - ps + t}{(\lambda-\mu)^2}\{(p^2 - 2q)r - 2ps + 2t\}.$$

Noting that

$$q\alpha - p\beta + \gamma = \frac{\partial}{\partial x}(qr - ps + t),$$

$$q\beta - p\gamma + \delta = \frac{\partial}{\partial y}(qr - ps + t) - (rt - s^2),$$

substituting, and reducing, we find that the equation is satisfied, provided

$$Y^2 + Y' = 0,$$

and therefore

$$Y = \frac{1}{y+a},$$

where a is a constant. Also

$$(\lambda - \mu)^2 = p^2 - 4q;$$

and therefore the equation* for the determination of g is

$$qr - ps + t = \frac{p^2 - 4q}{y+a}.$$

The equations, constituting the primitive of this partial equation, have already been given (§ 265, Ex. 2). Taking account of the facts, that we are seeking equations of the form

$$r + g(s, t) = 0,$$

that in the differential equation thus obtained x and y have replaced s and t, and that g is the dependent variable which can therefore be replaced by $-r$, we infer that any differential equation of the second order, such as to admit of two equations of the second order compatible with itself and with one another, is given by the system

$$-\frac{1}{t+a} = f''(\alpha) + g''(\beta),$$

$$-\frac{s}{t+a} = \alpha f''(\alpha) - f'(\alpha) + \beta g''(\beta) - g'(\beta),$$

$$r - \frac{s^2}{t+a} = \alpha^2 f''(\alpha) - 2\alpha f'(\alpha) + 2f(\alpha) + \beta^2 g''(\beta) - 2\beta g'(\beta) + 2g(\beta).$$

Ex. 1. For the preceding equations, prove that

$$\alpha = x + \lambda y, \quad \beta = x + \mu y;$$

deduce the values of p and q, and integrate the equation.

* The case considered by Goursat, t. II, p. 132, is obtained by making Y vanish through an infinite value of a.

Ex. 2. In the preceding investigation, it has been assumed that the critical quadratic

$$\kappa^2 - \kappa S + T = 0$$

has unequal roots, so that there are two subsidiary systems. Discuss the case when the quadratic has equal roots.

Ex. 3. Determine the form of the function k, if the equation

$$r + k(t) = 0$$

is integrable by Darboux's method: and integrate the equation.

In this case,

$$S = 0, \quad T = k'(t);$$

and the equation giving λ and μ is

$$\kappa^2 + k'(t) = 0.$$

Thus

$$\lambda = -\mu,$$

and neither of them involves s. We can proceed, either from the general result just given, or from the original conditions in § 266. It is easy to see that the two conditions are equivalent to one only, viz.

$$\frac{\partial}{\partial t}\left(\frac{2\mu}{\frac{\partial \mu}{\partial t}}\right) = -1,$$

so that

$$\mu (t + a)^2 = 3b,$$

where a and b are constants. Hence

$$\begin{aligned} k'(t) &= -\mu^2 \\ &= -\frac{9b^2}{(t+a)^4}, \end{aligned}$$

and therefore

$$k(t) = -c + \frac{3b^2}{(t+a)^3},$$

where c is an arbitrary constant. Thus the original differential equation is

$$(r + c)(t + a)^3 = 3b^2.$$

The general investigation gives assistance towards the construction of the primitive. We have

$$w = \text{constant}$$

as an integral of

$$\frac{ds}{\mu} - dt = 0,$$

that is, we can take

$$w = s + \frac{3b}{t+a};$$

and similarly, we have

$$v = \text{constant}$$

as an integral of

$$\frac{ds}{\lambda} - dt = 0,$$

that is, we can take

$$v = s - \frac{3b}{t+a}.$$

Hence

$$s = \tfrac{1}{2}(w+v), \quad \frac{3b}{t+a} = \tfrac{1}{2}(w-v);$$

and

$$r + c = \frac{3b^2}{(t+a)^3} = \frac{1}{72b}(w-v)^3.$$

Again,

$$\theta = t + \frac{\lambda - \mu}{\lambda \dfrac{\partial\mu}{\partial s} - \dfrac{\partial\mu}{\partial t}} = t + \frac{2\mu}{\dfrac{\partial\mu}{\partial t}} = -a;$$

and, similarly,

$$\vartheta = -a.$$

The two integral equations that can be associated with the given equation are

$$q + ay - xw = V'',$$
$$q + ay - xv = W'',$$

where V and W are arbitrary functions of v and of w respectively; and therefore

$$x = \frac{V'' - W''}{v - w},$$

$$q + ay = \frac{vV'' - wW''}{v - w}.$$

Also,

$$dq = s\,dx + t\,dy,$$

so that

$$dy = \frac{1}{t+a}\{d(q+ay) - s\,dx\},$$

which, on substitution and reduction, gives

$$y = \frac{1}{12b}\{(w-v)(V'' + W'') + 2(V' - W')\}.$$

Further,

$$dp = r\,dx + s\,dy,$$

so that

$$d(p - rx - sy) = -x\,dr - y\,ds;$$

on substitution and reduction, we have

$$p - rx - sy = \frac{1}{12b}\{(v-w)(V' + W') - 2V + 2W\}.$$

We thus have x, y, p, q expressed in terms of v and w; another quadrature, effected on

$$dz = p\,dx + q\,dy$$

after substitution, gives the value of z.

The result agrees with the result given by De Boer.

Ex. 4. Discuss the case when $b=0$. (De Boer.)

Compatible Equations of Higher Orders derivable by Darboux's Method.

268. Should it be found that neither of the subsidiary systems, constructed with a view to the formation of an equation of the second order to be associated with the original equation, can provide such an equation, then we proceed to use the method for the construction of an equation or equations of higher order which can be associated with

$$f = f(x, y, z, p, q, r, s, t) = 0.$$

As the present argument follows the earlier argument closely, here it will be made quite brief: and it will be restricted to the consideration of equations of the third order.

The derivatives of z of the third order will, as before, be denoted by α, β, γ, δ: those of the fourth order by π, ρ, σ, τ, υ, where

$$\pi = \frac{\partial^4 z}{\partial x^4}, \quad \rho = \frac{\partial^4 z}{\partial x^3 \partial y}, \quad \sigma = \frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \tau = \frac{\partial^4 z}{\partial x \partial y^3}, \quad \upsilon = \frac{\partial^4 z}{\partial y^4}.$$

We have

$$\begin{aligned} 0 &= \frac{d^2 f}{dx^2} + \frac{\partial f}{\partial r}\pi + \frac{\partial f}{\partial s}\rho + \frac{\partial f}{\partial t}\sigma, \\ 0 &= \frac{d^2 f}{dx\,dy} \qquad + \frac{\partial f}{\partial r}\rho + \frac{\partial f}{\partial s}\sigma + \frac{\partial f}{\partial t}\tau, \\ 0 &= \frac{d^2 f}{dy^2} \qquad\qquad + \frac{\partial f}{\partial r}\sigma + \frac{\partial f}{\partial s}\tau + \frac{\partial f}{\partial t}\upsilon, \end{aligned}$$

where $\dfrac{d^2 f}{dx^2}$ includes the complete second derivative of f with regard to x except the terms involving the fourth derivatives of z, and similarly for $\dfrac{d^2 f}{dx\,dy}$, $\dfrac{d^2 f}{dy^2}$. Let

$$u = u(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0$$

be an equation of the third order which is compatible with the given equation of the second order: then, taking

$$\begin{aligned} \frac{d}{dx} &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q} + \alpha\frac{\partial}{\partial r} + \beta\frac{\partial}{\partial s} + \gamma\frac{\partial}{\partial t}, \\ \frac{d}{dy} &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q} + \beta\frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s} + \delta\frac{\partial}{\partial t}, \end{aligned}$$

we have

$$0 = \frac{du}{dx} + \frac{\partial u}{\partial \alpha}\pi + \frac{\partial u}{\partial \beta}\rho + \frac{\partial u}{\partial \gamma}\sigma + \frac{\partial u}{\partial \delta}\tau,$$

$$0 = \frac{du}{dy} + \frac{\partial u}{\partial \alpha}\rho + \frac{\partial u}{\partial \beta}\sigma + \frac{\partial u}{\partial \gamma}\tau + \frac{\partial u}{\partial \delta}\upsilon.$$

Thus there are five equations for the determination of values of $\pi, \rho, \sigma, \tau, \upsilon$; as before, because

$$u = 0, \quad f = 0,$$

are compatible with one another and are not independent of one another, the values provided for the five derivatives by the five derived equations must not be determinate: and therefore

$$\left\| \begin{matrix} \dfrac{du}{dx}, & \dfrac{\partial u}{\partial \alpha}, & \dfrac{\partial u}{\partial \beta}, & \dfrac{\partial u}{\partial \gamma}, & \dfrac{\partial u}{\partial \delta}, & 0 \\ \dfrac{du}{dy}, & 0, & \dfrac{\partial u}{\partial \alpha}, & \dfrac{\partial u}{\partial \beta}, & \dfrac{\partial u}{\partial \gamma}, & \dfrac{\partial u}{\partial \delta} \\ \dfrac{d^2 f}{dx^2}, & \dfrac{\partial f}{\partial r}, & \dfrac{\partial f}{\partial s}, & \dfrac{\partial f}{\partial t}, & 0, & 0 \\ \dfrac{d^2 f}{dx\,dy}, & 0, & \dfrac{\partial f}{\partial r}, & \dfrac{\partial f}{\partial s}, & \dfrac{\partial f}{\partial t}, & 0 \\ \dfrac{d^2 f}{dy^2}, & 0, & 0, & \dfrac{\partial f}{\partial r}, & \dfrac{\partial f}{\partial s}, & \dfrac{\partial f}{\partial t} \end{matrix} \right\| = 0.$$

These are equivalent to two independent relations among the derivatives of u.

The two independent relations are resolved into equations that are linear in the derivatives of u; and the process is the same as in § 261, leading here also in general to a couple of subsidiary systems. Let λ and μ be the roots of the critical quadratic

$$\frac{\partial f}{\partial t} - \theta \frac{\partial f}{\partial s} + \theta^2 \frac{\partial f}{\partial r} = 0;$$

when the equations for u are resolved, we have

$$\left.\begin{aligned} &\frac{\partial u}{\partial \delta} - \lambda \frac{\partial u}{\partial \gamma} + \lambda^2 \frac{\partial u}{\partial \beta} - \lambda^3 \frac{\partial u}{\partial \alpha} = 0 \\ &\frac{du}{dx} + \mu \frac{du}{dy} - \frac{\dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial r}} \frac{d^2 f}{dx^2} - \frac{\dfrac{\partial u}{\partial \beta} - \lambda \dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial r}} \frac{d^2 f}{dx\,dy} - \mu \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial t}} \frac{d^2 f}{dy^2} = 0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} &\frac{\partial u}{\partial \delta} - \mu \frac{\partial u}{\partial \gamma} + \mu^2 \frac{\partial u}{\partial \beta} - \mu^3 \frac{\partial u}{\partial \alpha} = 0 \\ &\frac{du}{dx} + \lambda \frac{du}{dy} - \frac{\dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial r}} \frac{d^2 f}{dx^2} - \frac{\dfrac{\partial u}{\partial \beta} - \mu \dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial r}} \frac{d^2 f}{dx\,dy} - \lambda \frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial t}} \frac{d^2 f}{dy^2} = 0 \end{aligned}\right\},$$

as the two subsidiary systems for the determination of the quantity u. The form of the second equation in each system can be modified.

If the method is to be effective in the sense designed by Darboux, the first subsidiary system must have two independent integrals u_1 and u_2, and the second subsidiary system must likewise have two independent integrals v_1 and v_2. When these requirements are satisfied, then

$$\phi(u_1, u_2) = 0, \quad \psi(v_1, v_2) = 0,$$

are two equations of the third order compatible with

$$f = 0,$$

whatever be the arbitrary functions ϕ and ψ. Also, we have

$$\frac{df}{dx} + \alpha \frac{\partial f}{\partial r} + \beta \frac{\partial f}{\partial s} + \gamma \frac{\partial f}{\partial t} = 0,$$

$$\frac{df}{dy} + \beta \frac{\partial f}{\partial r} + \gamma \frac{\partial f}{\partial s} + \delta \frac{\partial f}{\partial t} = 0;$$

these two equations, together with $\phi = 0$ and $\psi = 0$, suffice for the determination of $\alpha, \beta, \gamma, \delta$, in terms of the variables that occur in f. Their values are substituted in the first three relations of the set

$$\left.\begin{aligned} dr &= \alpha\,dx + \beta\,dy \\ ds &= \beta\,dx + \gamma\,dy \\ dt &= \gamma\,dx + \delta\,dy \\ dp &= r\,dx + s\,dy \\ dq &= s\,dx + t\,dy \\ dz &= p\,dx + q\,dy \end{aligned}\right\}.$$

The set then becomes an exactly integrable system: quadratures lead to the primitive, which obviously will contain two arbitrary functions. And, as before, it may be convenient to change the independent variables in the quadratures: thus it may be a

practical shortening of the calculations to select u_2 and v_2 for this purpose.

Just as in the case of the Monge method and the Boole method for the construction of an intermediate integral, when the integration of a set of equations in differential elements was equivalent to the integration of a system of equations in differential coefficients of the first order, and (§§ 259, 264) was similarly the case in the construction of a compatible equation of the second order, so here also it is possible to construct a compatible equation of the third order by means of a set of equations in differential elements. The integration of the system of equations

$$\left.\begin{aligned} &\frac{\partial u}{\partial \delta} - \lambda \frac{\partial u}{\partial \gamma} + \lambda^2 \frac{\partial u}{\partial \beta} - \lambda^3 \frac{\partial u}{\partial \alpha} = 0 \\ &\frac{du}{dx} + \mu \frac{du}{dy} - \frac{\frac{\partial u}{\partial \alpha}}{\frac{\partial f}{\partial r}} \frac{d^2 f}{dx^2} - \frac{\frac{\partial u}{\partial \beta} - \lambda \frac{\partial u}{\partial \alpha}}{\frac{\partial f}{\partial r}} \frac{d^2 f}{dx\,dy} - \mu \frac{\frac{\partial u}{\partial \delta}}{\frac{\partial f}{\partial t}} \frac{d^2 f}{dy^2} = 0 \end{aligned}\right\},$$

is equivalent to the integration of the set of equations

$$\left.\begin{aligned} \frac{dx}{1} = \frac{dy}{\mu} &= \frac{dz}{p + \mu q} = \frac{dp}{r + \mu s} = \frac{dq}{s + \mu t} \\ &= \frac{dr}{\alpha + \mu\beta} = \frac{ds}{\beta + \mu\gamma} = \frac{dt}{\gamma + \mu\delta} \\ &= \frac{d\alpha + \lambda d\beta}{-\frac{\frac{d^2 f}{dx^2}}{\frac{\partial f}{\partial r}}} = \frac{d\beta + \lambda d\gamma}{-\frac{\frac{d^2 f}{dx\,dy}}{\frac{\partial f}{\partial r}}} = \frac{d\gamma + \lambda d\delta}{-\frac{\frac{d^2 f}{dy^2}}{\frac{\partial f}{\partial r}}} \end{aligned}\right\},$$

λ and μ being the roots of the critical quadratic: and similarly for the other system.

The equations in the differential elements here have their obviously simplest form. The equations, which involve the derivatives of u, are capable of a variety of forms: in particular, it is easy to verify that the second equation as given is equivalent to (and can be replaced by) the equation

$$\left(\frac{du}{dx} + \mu \frac{du}{dy}\right) \frac{\partial f}{\partial r} = \left(\frac{d^2 f}{dx^2} - \lambda \frac{d^2 f}{dx\,dy} + \lambda^2 \frac{d^2 f}{dy^2}\right) \frac{\partial u}{\partial \alpha}$$
$$+ \left(\frac{d^2 f}{dx\,dy} - \lambda \frac{d^2 f}{dy^2}\right) \frac{\partial u}{\partial \beta} + \frac{d^2 f}{dy^2} \frac{\partial u}{\partial \gamma},$$

which, though containing more terms than the other form, is often more convenient in practice.

The subsidiary system of the two initial equations for u contains twelve independent variables. It must be satisfied by three integrals

$$f, \quad \frac{df}{dx} + \alpha \frac{\partial f}{\partial r} + \beta \frac{\partial f}{\partial s} + \gamma \frac{\partial f}{\partial t}, \quad \frac{df}{dy} + \beta \frac{\partial f}{\partial r} + \gamma \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t};$$

if it is to possess an integral of the type required by Darboux, it must possess two integrals independent of the three just mentioned. Accordingly, when it is made a complete Jacobian system, that system will consist of seven members.

Ex. 1. Consider the equation

$$r - t - 4\frac{p}{x} = 0,$$

which has no intermediate integral and does not admit of a compatible equation of the second order of Darboux's type. It can be integrated, after transformation of the independent variables, by the Laplace method; but here it will be considered as an illustration of the Darboux method so as, if possible, to obtain equations of the third order with which it is compatible.

The critical quadratic is

$$\theta^2 - 1 = 0,$$

so that there are two subsidiary sets of equations, given by the two assignments of the roots. We have

$$\begin{aligned}
\frac{\partial f}{\partial r} &= 1, \\
\frac{d^2 f}{dx^2} &= -4\frac{\alpha}{x} + 8\frac{r}{x^2} - 8\frac{p}{x^3}, \\
\frac{d^2 f}{dx\,dy} &= -4\frac{\beta}{x} + 4\frac{s}{x^2}, \\
\frac{d^2 f}{dy^2} &= -4\frac{\gamma}{x}.
\end{aligned}$$

The subsidiary system, given by $\lambda = 1$ and $\mu = -1$, is

$$\theta_1(u) = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} = 0,$$

$$\begin{aligned}
\theta_2(u) = \frac{du}{dx} - \frac{du}{dy} &+ \left(4\frac{\alpha - \beta + \gamma}{x} - 4\frac{2r - s}{x^2} + 8\frac{p}{x^3}\right)\frac{\partial u}{\partial \alpha} \\
&+ \left(4\frac{\beta - \gamma}{x} - 4\frac{s}{x^2}\right)\frac{\partial u}{\partial \beta} + 4\frac{r}{x}\frac{\partial u}{\partial \gamma} = 0.
\end{aligned}$$

We have

$$(\theta_1, \theta_2) = \frac{4}{x}\theta_3(u) = 0,$$

where
$$\theta_3(u)=3\frac{\partial u}{\partial \alpha}-2\frac{\partial u}{\partial \beta}+\frac{\partial u}{\partial \gamma}+\tfrac{1}{2}x\left(\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)=0.$$
We also have
$$(\theta_3,\ \theta_2)=18\frac{\partial u}{\partial \alpha}-10\frac{\partial u}{\partial \beta}+4\frac{\partial u}{\partial \gamma}+x\left(\tfrac{9}{2}\frac{\partial u}{\partial r}-\tfrac{7}{2}\frac{\partial u}{\partial s}+\tfrac{1}{2}\frac{\partial u}{\partial t}\right)+x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0.$$
From the last equation, combined with
$$\theta_1(u)=0,\quad \theta_3(u)=0,$$
we can express $\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma}$ in terms of the other derivatives: the results are
$$\Delta_1(u)=\frac{\partial u}{\partial \alpha}-\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(3\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-5\frac{\partial u}{\partial t}\right)+\tfrac{1}{2}x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0,$$
$$\Delta_2(u)=\frac{\partial u}{\partial \beta}-3\frac{\partial u}{\partial \delta}+x\left(\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-3\frac{\partial u}{\partial t}\right)+x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0,$$
$$\Delta_3(u)=\frac{\partial u}{\partial \gamma}-3\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(\frac{\partial u}{\partial r}+3\frac{\partial u}{\partial s}-7\frac{\partial u}{\partial t}\right)+\tfrac{1}{2}x^2\left(\frac{\partial u}{\partial p}-\frac{\partial u}{\partial q}\right)=0.$$
Let these values of $\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma}$ be substituted in $\theta_2(u)$, and let the resulting form be denoted by $\theta_2'(u)$: then
$$\theta_2'(u)=0.$$
Now
$$(\Delta_1,\ \Delta_2)=0,\quad (\Delta_1,\ \Delta_3)=0,\quad (\Delta_2,\ \Delta_3)=0;$$
and
$$(\Delta_1,\ \theta_2')=-\theta_4,\quad (\Delta_2,\ \theta_2')=-2\theta_4,\quad (\Delta_3,\ \theta_2')=-\theta_4,$$
where
$$\theta_4(u)=\frac{\partial u}{\partial r}-3\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial p}-3x\frac{\partial u}{\partial q}-x^2\frac{\partial u}{\partial z}=0.$$
Further,
$$(\theta_4,\ \theta_2')=6\theta_5,$$
where
$$\theta_5(u)=\frac{\partial u}{\partial q}+x\frac{\partial u}{\partial z}=0;$$
and
$$(\theta_5,\ \theta_2')=-\frac{\partial u}{\partial z}=0.$$

Combining these equations so as to have simple forms, we take them to be
$$\nabla_1(u)=\frac{\partial u}{\partial z}=0,$$
$$\nabla_2(u)=\frac{\partial u}{\partial q}=0,$$
$$\nabla_3(u)=\frac{\partial u}{\partial r}-3\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial p}=0,$$
$$\nabla_4(u)=\frac{\partial u}{\partial \alpha}-\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)=0,$$
$$\nabla_5(u)=\frac{\partial u}{\partial \beta}-3\frac{\partial u}{\partial \delta}+x\frac{\partial u}{\partial s}=0,$$
$$\nabla_6(u)=\frac{\partial u}{\partial \gamma}-3\frac{\partial u}{\partial \delta}+\tfrac{1}{4}x\left(-\frac{\partial u}{\partial r}+3\frac{\partial u}{\partial s}-\frac{\partial u}{\partial t}\right)=0,$$

$$\nabla_7(u)=\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t}\left\{-a+\beta+\gamma-\delta+\frac{1}{x}(5r-4s)-\frac{2}{x^2}p\right\}$$
$$+\frac{\partial u}{\partial r}\left(\frac{r}{x}-\frac{2p}{x^2}\right)+\frac{\partial u}{\partial s}\left\{-a-2\beta-\gamma+\frac{1}{x}(2r+3s)-\frac{2}{x^2}p\right\}$$
$$+\frac{\partial u}{\partial \delta}\left\{\frac{4}{x}(a+2\beta+\gamma)-\frac{8}{x^2}(r+s)+\frac{8}{x^3}p\right\}=0.$$

This is a complete Jacobian system: hence it possesses five simultaneous integrals. Three of these five are known, being

$$r-t-4\frac{p}{x},$$
$$a-\gamma-4\frac{r}{x}+4\frac{p}{x^2},$$
$$\beta-\delta-4\frac{s}{x},$$

all of which vanish. Two others, when these results are used, are found to be

$$\frac{1}{x^2}(a+3\beta+3\gamma+\delta),\quad x+y;$$

hence the subsidiary system provides an equation

$$a+3\beta+3\gamma+\delta=x^2\phi(x+y),$$

ϕ being an arbitrary function: this equation is compatible with the original equation.

Similarly, the subsidiary system, given by taking $\lambda=-1$ and $\mu=1$, provides an equation

$$a-3\beta+3\gamma-\delta=x^2\psi(y-x),$$

ψ being an arbitrary function: this equation is compatible with the original equation.

These two equations, together with

$$a-\gamma=4\frac{r}{x}-4\frac{p}{x^2},\quad \beta-\delta=4\frac{s}{x},$$

give values of a, β, γ, δ, in terms of the other quantities. Substituting them in the differential relations

$$dr=a\,dx+\beta\,dy,\quad ds=\beta\,dx+\gamma\,dy,\quad dt=\gamma\,dx+\delta\,dy,$$

effecting the quadratures, and substituting the deduced values of r and s in

$$dp=r\,dx+s\,dy,\quad dq=s\,dx+t\,dy,$$

and, lastly, substituting the deduced values of p and q in

$$dz=p\,dx+q\,dy,$$

a quadrature leads to the value

$$z=x^2(f''-g'')-3x(f'+g')+3(f-g),$$

where

$$\phi(x+y)=8\frac{\partial^5 f(x+y)}{\partial x^5},\quad \psi(y-x)=8\frac{\partial^5 g(y-x)}{\partial y^5}.$$

Ex. 2. Integrate the equations:—

$$\text{(i)} \quad r - t - \frac{1}{x}(p - q) = 0;$$

$$\text{(ii)} \quad x^2 r + 2xys + y^2(1 - x^2)t = 0;$$

neither of which possesses an intermediate integral.

Ex. 3. Obtain an integral of the equation

$$u^2 r - t = 0,$$

where u is a function of x and y, satisfying a relation

$$uy + x = f(u),$$

and f is any functional form. (Winckler.)

Ex. 4. Solve the equation

$$r + t + \frac{8z}{(1 + x^2 + y^2)^2} = 0,$$

obtaining the primitive in the form

$$z = f'(u) + g'(v) - \frac{2}{1 + x^2 + y^2}\{vf(u) + ug(v)\},$$

where $u = x + iy$, $v = x - iy$. (Schwarz.)

Ex. 5. Obtain two equations of the third order, which are compatible with (but are not mere derivatives of) the equation

$$x^{\frac{4}{3}} r - t = 0. \qquad \text{(Winckler.)}$$

Ex. 6. Shew that the equation

$$r - t + zf(x) = 0$$

possesses two compatible equations of the second order, if

$$\frac{d}{dx}\left(\frac{1}{f}\frac{df}{dx}\right) + f = 0;$$

and find the equation that must be satisfied by f, if there are two compatible equations of the third order.

Ex. 7. When an equation

$$r + g(s, t) = 0$$

admits two equations of the third order, compatible with itself and algebraically independent of its derivatives with regard to x and to y, in the forms

$$\phi(u_1, u_2) = 0, \quad \psi(v_1, v_2) = 0,$$

the quantities u_1 and u_2 are integrals of one subsidiary system, and the quantities v_1 and v_2 are integrals of the other subsidiary system.

Obtain the conditions, analogous to those in § 266, in order that each of the subsidiary systems may possess two integrals which are not immediately derivable from the given equation; and, by means of these equations, find the suitable forms of the original equation.

In particular, obtain the equations

$$r = f(t),$$

which have the required property.

CHAPTER XIX.

Generalisation of Integrals.

The present chapter is devoted to the problem of connecting the general primitive of an equation of the second order with a primitive, that is either complete or incomplete in the aggregate of parameters which it contains. The problem is of the utmost importance in the case of equations of the first order: on that account, Lagrange attempted it for an equation of the second order, using his method of the variation of parameters for this purpose. Having included too many parameters, he did not attain to a satisfactory result except in special cases; consequently, he set the method aside. Later, Imschenetsky limited the number of parameters and made less restricted conditions in order to secure the generalisation: he obtains a generalisation that is important and, within the limits of analysis which can be effected, is practicable.

The original equation and the generalising equation are, in many instances, connected with one another by means of contact transformations.

269. It was seen in the case of partial differential equations of the first order that, when a complete integral is known, it can be adapted to the derivation of other classes of integrals: the method used for this purpose is the variation of parameters. In the case of partial differential equations of the second order, integrals have been obtained containing a number of arbitrary constants; thus there were complete integrals (§ 180) containing five arbitrary constants, and there were integrals (§ 241) for special types of equations containing three arbitrary constants; and other instances have occurred. Such integrals are not necessarily particular forms of the general integral: and it is natural to inquire whether the method of variation of parameters, applied to such integrals, will lead to the general integral or to any other

classes of integrals. For the purposes of the present discussion, we shall assume that there are only two independent variables.

Following Lagrange*, by whom the question was first considered, we begin with a complete integral in the form

$$f(x, y, z, a_1, a_2, a_3, a_4, a_5) = 0,$$

where a_1, a_2, a_3, a_4, a_5 are arbitrary constants: the elimination of these five constants, among the six equations

$$f = 0, \quad p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} = 0, \quad q\frac{\partial f}{\partial z} + \frac{\partial f}{\partial y} = 0,$$

$$r\frac{\partial f}{\partial z} + p^2\frac{\partial^2 f}{\partial z^2} + 2p\frac{\partial^2 f}{\partial x\partial z} + \frac{\partial^2 f}{\partial x^2} = 0,$$

$$s\frac{\partial f}{\partial z} + pq\frac{\partial^2 f}{\partial z^2} + q\frac{\partial^2 f}{\partial x\partial z} + p\frac{\partial^2 f}{\partial y\partial z} + \frac{\partial^2 f}{\partial x\partial y} = 0,$$

$$t\frac{\partial f}{\partial z} + q^2\frac{\partial^2 f}{\partial z^2} + 2q\frac{\partial^2 f}{\partial y\partial z} + \frac{\partial^2 f}{\partial y^2} = 0,$$

leads in general† to a single differential equation

$$\phi(x, y, z, p, q, r, s, t) = 0,$$

which accordingly has $f = 0$ for a complete integral.

On the analogy of partial differential equations of the first order, we attempt to generalise the complete integral by making the five parameters functions of the variables: this is done most directly by making a_1 and a_2 functions of x and y, and a_3, a_4, a_5 functions of a_1 and a_2. The functions are to be determined by the condition, that the forms of z, p, q, r, s (and therefore, owing to the differential equation, the form of t also) are left unaltered by the change: the passage from $f = 0$ to the differential equation will then be the same as before. Writing

$$\left.\begin{aligned} \frac{d}{da_1} &= \frac{\partial}{\partial a_1} + \frac{\partial a_3}{\partial a_1}\frac{\partial}{\partial a_3} + \frac{\partial a_4}{\partial a_1}\frac{\partial}{\partial a_4} + \frac{\partial a_5}{\partial a_1}\frac{\partial}{\partial a_5} \\ \frac{d}{da_2} &= \frac{\partial}{\partial a_2} + \frac{\partial a_3}{\partial a_2}\frac{\partial}{\partial a_3} + \frac{\partial a_4}{\partial a_2}\frac{\partial}{\partial a_4} + \frac{\partial a_5}{\partial a_2}\frac{\partial}{\partial a_5} \end{aligned}\right\},$$

* *Œuvres complètes*, t. IV, pp. 5—108.

† The conditions are the non-evanescence of certain Jacobians of the left-hand members of the six equations with regard to the constants: the detailed examination of their forms can be omitted, as not pertinent to the immediate discussion.

we see that the values of p and of q are unaltered if

$$\frac{\partial a_1}{\partial x}\frac{df}{da_1}+\frac{\partial a_2}{\partial x}\frac{df}{da_2}=0,$$

$$\frac{\partial a_1}{\partial y}\frac{df}{da_1}+\frac{\partial a_2}{\partial y}\frac{df}{da_2}=0;$$

and therefore, as no limitation upon full generality is imposed by assuming a_1 and a_2 to be independent functions of x and y, we have

$$\frac{df}{da_1}=0,\quad \frac{df}{da_2}=0.$$

When these are satisfied, the values of p and q are given by

$$p\frac{\partial f}{\partial z}+\frac{\partial f}{\partial x}=0,\quad q\frac{\partial f}{\partial z}+\frac{\partial f}{\partial y}=0.$$

Differentiating the first of these equations with respect to x and to y, and introducing the condition that the second derivatives of z are to be the same as before, we find

$$\left(p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}\right)\frac{\partial a_1}{\partial x}+\left(p\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial x}\right)\frac{\partial a_2}{\partial x}=0,$$

$$\left(p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}\right)\frac{\partial a_1}{\partial y}+\left(p\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial x}\right)\frac{\partial a_2}{\partial y}=0;$$

consequently,

$$p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}=0,\quad p\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial x}=0.$$

Similarly treating the second of the equations, we have

$$q\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial y}=0,\quad q\frac{\partial^2 f}{\partial a_2\partial z}+\frac{\partial^2 f}{\partial a_2\partial y}=0.$$

But the equation

$$\frac{df}{da_1}=0$$

is satisfied identically when the proper values of z and the parameters are substituted: hence

$$p\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial x}+\frac{d^2 f}{da_1^2}\frac{\partial a_1}{\partial x}+\frac{d^2 f}{da_1 da_2}\frac{\partial a_2}{\partial x}=0,$$

$$q\frac{\partial^2 f}{\partial a_1\partial z}+\frac{\partial^2 f}{\partial a_1\partial y}+\frac{d^2 f}{da_1^2}\frac{\partial a_1}{\partial y}+\frac{d^2 f}{da_1 da_2}\frac{\partial a_2}{\partial y}=0,$$

and therefore

$$\frac{d^2 f}{da_1^2}\frac{\partial a_1}{\partial x}+\frac{d^2 f}{da_1 da_2}\frac{\partial a_2}{\partial x}=0,$$

$$\frac{d^2 f}{da_1^2}\frac{\partial a_1}{\partial y}+\frac{d^2 f}{da_1 da_2}\frac{\partial a_2}{\partial y}=0.$$

These equations give

$$\frac{d^2 f}{da_1^2}=0,\quad \frac{d^2 f}{da_1 da_2}=0,$$

because a_1 and a_2 are independent functions of x and y. Similarly, the equation

$$\frac{df}{da_2}=0$$

leads to the relations

$$\frac{d^2 f}{da_1 da_2}=0,\quad \frac{d^2 f}{da_2^2}=0.$$

We thus have six equations in all, viz.

$$f=0,$$

$$\frac{df}{da_1}=0,\quad \frac{df}{da_2}=0,$$

$$\frac{d^2 f}{da_1^2}=0,\quad \frac{d^2 f}{da_1 da_2}=0,\quad \frac{d^2 f}{da_2^2}=0,$$

which are free from p, q, r, s, t; the second and the third contain first derivatives of a_3, a_4, a_5 with regard to a_1 and a_2, and the last three contain second derivatives of the same quantities. Now let x, y, z be eliminated among the six equations: the resulting eliminant is composed of three simultaneous equations of the second order in three dependent variables. The problem, thus provided for the determination of a_3, a_4, a_5 in terms of a_1 and a_2, is more difficult than the original problem, which is the solution of a single equation of the second order in a single dependent variable. Consequently, the derivation of further integrals from the complete integral cannot be regarded as generally possible if attempted by the indicated process.

It is possible that the method may be effective in particular cases: but the course of the analysis must be different. Thus Lagrange takes the equation

$$t=m,$$

where m is a constant (which can be made unity without loss of generality). Obviously the equation

$$z = a_1 + a_2 x + a_3 y + a_4 xy + a_5 (x^2 + my^2),$$

provides an integral; and then

$$p = a_2 + a_4 y + 2a_5 x,$$
$$q = a_3 + a_4 x + 2ma_5 y.$$

Varying the parameters, and keeping the values of r, s, t unaltered, we have

$$0 = da_1 + x\,da_2 + y\,da_3 + xy\,da_4 + (x^2 + my^2)\,da_5,$$
$$0 = da_2 + y\,da_4 + 2x\,da_5,$$
$$0 = da_3 + x\,da_4 + 2my\,da_5.$$

The last two can be replaced by

$$da_3 + m^{\frac{1}{2}} da_2 + (x + m^{\frac{1}{2}} y)(da_4 + 2m^{\frac{1}{2}} da_5) = 0,$$
$$da_3 - m^{\frac{1}{2}} da_2 + (x - m^{\frac{1}{2}} y)(da_5 - 2m^{\frac{1}{2}} du_5) = 0;$$

and the first of them can then be regarded as giving da_1. The first modified equation shews that $da_3 + m^{\frac{1}{2}} da_2$ and $da_4 + 2m^{\frac{1}{2}} da_5$ vanish together, so that $a_3 + m^{\frac{1}{2}} a_2$ and $a_4 + 2m^{\frac{1}{2}} a_5$ are constant together: hence, taking account of their generally variable values, we can write

$$a_3 + m^{\frac{1}{2}} a_2 = \phi\,(a_4 + 2m^{\frac{1}{2}} a_5),$$

where ϕ is any functional form; and then

$$x + m^{\frac{1}{2}} y + \phi'\,(a_4 + 2m^{\frac{1}{2}} a_5) = 0.$$

Similarly, the second modified equation leads to the relations

$$a_3 - m^{\frac{1}{2}} a_2 = \psi\,(a_4 - 2m^{\frac{1}{2}} a_5),$$
$$x - m^{\frac{1}{2}} y + \psi'\,(a_4 - 2m^{\frac{1}{2}} a_5) = 0,$$

where ψ is any functional form. Writing

$$x + m^{\frac{1}{2}} y = u, \quad x - m^{\frac{1}{2}} y = v,$$

and inverting the functional forms ϕ' and ψ', we have

$$a_4 + 2m^{\frac{1}{2}} a_5 = g\,(u), \quad a_4 - 2m^{\frac{1}{2}} a_5 = h\,(v);$$

and then

$$da_3 + m^{\frac{1}{2}} da_2 = -ug'\,(u)\,du,$$

so that

$$a_3 + m^{\frac{1}{2}} a_2 = -\int ug'\,(u)\,du.$$

Similarly,

$$a_3 - m^{\frac{1}{2}} a_2 = -\int vh'\,(v)\,dv.$$

For a_1, we have

$$\begin{aligned}-da_1 = &-\frac{1}{4m^{\frac{1}{2}}}(u+v)\{ug'(u)\,du - vh'(v)\,dv\}\\ &-\frac{1}{4m^{\frac{1}{2}}}(u-v)\{ug'(u)\,du + vh'(v)\,dv\}\\ &+\frac{1}{8m^{\frac{1}{2}}}(u^2-v^2)\{g'(u)\,du + \ h'(v)\,dv\}\\ &+\frac{1}{8m^{\frac{1}{2}}}(u^2+v^2)\{g'(u)\,du - \ h'(v)\,dv\}\\ = &-\frac{1}{4m^{\frac{1}{2}}}\{u^2g'(u)\,du - v^2h'(v)\,dv\},\end{aligned}$$

so that

$$a_1 = \frac{1}{4m^{\frac{1}{2}}}\int u^2g'(u)\,du - \frac{1}{4m^{\frac{1}{2}}}\int v^2h'(v)\,dv.$$

With the values of a_1, a_2, a_3, a_4, a_5 thus obtained, z becomes

$$\begin{aligned}z = &\frac{1}{4m^{\frac{1}{2}}}\left\{\int u^2g'(u)\,du - 2u\int ug'(u)\,du + u^2g(u)\right\}\\ &-\frac{1}{4m^{\frac{1}{2}}}\left\{\int v^2h'(v)\,dv - 2v\int vh'(v)\,dv + v^2h(v)\right\};\end{aligned}$$

writing

$$g(u) = 2m^{\frac{1}{2}}G''(u), \quad h(v) = -2m^{\frac{1}{2}}H''(v),$$

and effecting the quadratures, we find

$$\begin{aligned}z &= G(u) + H(v)\\ &= G(x + m^{\frac{1}{2}}y) + H(x - m^{\frac{1}{2}}y).\end{aligned}$$

All the conditions are satisfied by keeping ϕ and ψ arbitrary: hence g and h are arbitrary, and therefore also G and H are arbitrary functions.

IMSCHENETSKY'S GENERALISATION.

270. In the preceding example, we have obtained the general integral; but it is manifest that, in the process, the effectiveness depends on the peculiar simplicity of the equations. In general, as already stated, the method of variation of parameters, when applied to equations of the second order, requires the solution of a problem distinctly more difficult than the original problem: on this account, Lagrange described* the method as more curious than useful.

* *L.c.*, p. 101.

Bour applied the method to a number of cases in which the difficulties were overcome: and he placed on record* his opinion that the method would yet be developed. It was reserved for Imschenetsky† to achieve a real generalisation of an integral by the Lagrangian method of variation of parameters, the equation being of the form considered by Monge, Ampère, and Boole.

The real difficulty in the generalisation, which was attempted by Lagrange, lies in the necessity of determining three out of five parameters by means of partial equations of the second order. In the generalisation which was achieved by Imschenetsky, what is required is the determination of one parameter in terms of other two by means of a single equation of the second order: but, instead of using an integral (of the type called complete) involving five arbitrary parameters, he makes an integral, which involves three such parameters, the foundation of the structure of other integrals. It is of course no longer possible to assume that, in the variation of the parameters, each of the derivatives p, q, r, s, t remains unaltered in form: for the immediate purpose, and in the absence of assigned initial conditions, it is sufficient that the partial equation of the second order shall be satisfied. This is precisely the requirement which, in the last resort, is adopted in Imschenetsky's method, the forms of p and q being kept unaltered.

Accordingly, let it be assumed that an integral of the equation of the second order has been obtained in a form

$$z = f(x, y, a, b, c),$$

involving three arbitrary constants: the process by which the integral has been obtained is immaterial. When this value of z is substituted in the equation, the latter is satisfied identically.

Now let a and b be chosen to be independent functions of x and y, subject to the condition that p and q have the same forms as when a and b are parametric; and let c then be chosen such a function of x and y (therefore, also, of a and b) that the differential equation is satisfied. When we write

$$\frac{df}{da} = \frac{\partial f}{\partial a} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial a}, \quad \frac{df}{db} = \frac{\partial f}{\partial b} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial b},$$

* *Journ. de l'Éc. Polyt.*, Cah. XXXIX (1862), p. 191.

† *Grunert's Archiv*, t. LIV (1872), ch. IV.

the forms of p and q are unaltered in the changed circumstances, provided

$$\frac{df}{da}\frac{\partial a}{\partial x}+\frac{df}{db}\frac{\partial b}{\partial x}=0,\quad \frac{df}{da}\frac{\partial a}{\partial y}+\frac{df}{db}\frac{\partial b}{\partial y}=0:$$

hence, as a and b are independent functions of x and y, we must have

$$\frac{df}{da}=0,\quad \frac{df}{db}=0,$$

the values of p and q still being given in the forms

$$p=\frac{\partial f}{\partial x},\quad q=\frac{\partial f}{\partial y}.$$

Again,

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 f}{\partial x^2}+\frac{dp}{da}\frac{\partial a}{\partial x}+\frac{dp}{db}\frac{\partial b}{\partial x}=r+h,$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=\frac{\partial^2 f}{\partial x\partial y}+\frac{dp}{da}\frac{\partial a}{\partial y}+\frac{dp}{db}\frac{\partial b}{\partial y}\\&=\frac{\partial^2 f}{\partial x\partial y}+\frac{dq}{da}\frac{\partial a}{\partial x}+\frac{dq}{db}\frac{\partial b}{\partial x}=s+k,\end{aligned}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 f}{\partial y^2}+\frac{dq}{da}\frac{\partial a}{\partial y}+\frac{dq}{db}\frac{\partial b}{\partial y}=t+l,$$

where r, s, t are the second derivatives of f which satisfy the original differential equation.

Also, because the proper values of a and b, as functions of x and y, satisfy the equations

$$\frac{df}{da}=0,\quad \frac{df}{db}=0,$$

identically, we have

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)+\frac{d^2 f}{da^2}\frac{\partial a}{\partial x}+\frac{d^2 f}{da\,db}\frac{\partial b}{\partial x}=0,$$

and three similar equations. When the proper value of c, as a function of a and b, is substituted in f, the latter becomes a function of x, y, a, b only: and the partial derivative of the modified function with regard to a is the quantity denoted by $\frac{df}{da}$. Hence

$$\frac{\partial}{\partial x}\left(\frac{df}{da}\right)=\frac{d}{da}\left(\frac{\partial f}{\partial x}\right)=\frac{dp}{da},$$

and therefore

$$\frac{d^2 f}{da^2}\frac{\partial a}{\partial x} + \frac{d^2 f}{da\,db}\frac{\partial b}{\partial x} = -\frac{dp}{da};$$

similarly,

$$\frac{d^2 f}{da\,db}\frac{\partial a}{\partial x} + \frac{d^2 f}{db^2}\frac{\partial b}{\partial x} = -\frac{dp}{db},$$

$$\frac{d^2 f}{da^2}\frac{\partial a}{\partial y} + \frac{d^2 f}{da\,db}\frac{\partial b}{\partial y} = -\frac{dq}{da},$$

$$\frac{d^2 f}{da\,db}\frac{\partial a}{\partial y} + \frac{d^2 f}{db^2}\frac{\partial b}{\partial y} = -\frac{dq}{db}.$$

These equations enable us to express the derivatives of a and of b with respect to x and y in terms of $\frac{dp}{da}, \frac{dp}{db}, \frac{dq}{da}, \frac{dq}{db}$; and they obviously verify the relation

$$\frac{dp}{da}\frac{\partial a}{\partial y} + \frac{dp}{db}\frac{\partial b}{\partial y} = \frac{dq}{da}\frac{\partial a}{\partial x} + \frac{dq}{db}\frac{\partial b}{\partial x}.$$

We find

$$h\Delta = \left(\frac{d^2 f}{da^2},\ -\frac{d^2 f}{da\,db},\ \frac{d^2 f}{db^2}\right)\left(\frac{dp}{db},\ \frac{dp}{da}\right)^2,$$

$$k\Delta = \left(\frac{d^2 f}{da^2},\ -\frac{d^2 f}{da\,db},\ \frac{d^2 f}{db^2}\right)\left(\frac{dp}{db},\ \frac{dp}{da}\right)\left(\frac{dq}{db},\ \frac{dq}{da}\right),$$

$$l\Delta = \left(\frac{d^2 f}{da^2},\ -\frac{d^2 f}{da\,db},\ \frac{d^2 f}{db^2}\right)\left(\frac{dq}{db},\ \frac{dq}{da}\right)^2,$$

where

$$\Delta = \left(\frac{d^2 f}{da\,db}\right)^2 - \frac{d^2 f}{da^2}\frac{d^2 f}{db^2};$$

also

$$(k^2 - hl)\,\Delta = \left(\frac{dp}{da}\frac{dq}{db} - \frac{dp}{db}\frac{dq}{da}\right)^2.$$

Suppose that the differential equation is

$$U\left\{\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2\right\} + R\frac{\partial^2 z}{\partial x^2} + 2S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} = V,$$

where U, R, S, T, V do not involve second derivatives; then it is satisfied by

$$z = f(x, y, a, b, c),$$

whether a, b, c be parametric or variable, and therefore

$$U(rt - s^2) + Rr + 2Ss + Tt = V,$$

$$U\{(r+h)(t+l) - (s+k)^2\} + R(r+h) + 2S(s+k) + T(t+l) = V.$$

Subtracting, we have

$$(R + Ut)h + 2(S - Us)k + (T + Ur)l = U(k^2 - hl):$$

when the preceding values of h, k, l are substituted in this equation, it takes the form

$$R_1 \frac{d^2f}{da^2} - 2S_1 \frac{d^2f}{da\,db} + T_1 \frac{d^2f}{db^2} = V_1,$$

where

$$R_1 = \left(R + Ut,\ S - Us,\ T + Ur \between \frac{dp}{db},\ \frac{dq}{db}\right)^2,$$

$$S_1 = \left(R + Ut,\ S - Us,\ T + Ur \between \frac{dp}{db},\ \frac{dq}{db} \between \frac{dp}{da},\ \frac{dq}{da}\right),$$

$$T_1 = \left(R + Ut,\ S - Us,\ T + Ur \between \frac{dp}{da},\ \frac{dq}{da}\right)^2,$$

$$V_1 = U\left(\frac{dp}{da}\frac{dq}{db} - \frac{dp}{db}\frac{dq}{da}\right)^2,$$

the quantities r, s, and t, in these expressions being the second derivatives of $f(x, y, a, b, c)$, when a, b, c are parametric. Now

$$\frac{d^2f}{da^2} = \frac{\partial^2 f}{\partial a^2} + 2\frac{\partial^2 f}{\partial a \partial c}\frac{\partial c}{\partial a} + \frac{\partial^2 f}{\partial c^2}\left(\frac{\partial c}{\partial a}\right)^2 + \frac{\partial f}{\partial c}\frac{\partial^2 c}{\partial a^2},$$

and so for the others; so that the new equation is linear in the second derivatives of c, and the coefficients of these derivatives involve x, y, z, a, b, c, $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$. The equations

$$z - f = 0, \quad \frac{df}{da} = 0, \quad \frac{df}{db} = 0,$$

determine x, y, and z as functions of a, b, c, $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$: when their values are substituted in the coefficients, the new equation takes the form

$$A\frac{\partial^2 c}{\partial a^2} + 2H\frac{\partial^2 c}{\partial a \partial b} + B\frac{\partial^2 c}{\partial b^2} = F,$$

where A, H, B, F are functions of a, b, c, $\frac{\partial c}{\partial a}$, $\frac{\partial c}{\partial b}$ only, and do not involve second derivatives of c.

When this equation has been integrated, expressing c as a function of a and b, then the original equation possesses an

integral, which results from the elimination of a, b, c among the equations

$$z=f(x, y, a, b, c),$$
$$c=\theta(a, b),$$
$$0=\frac{\partial f}{\partial a}+\frac{\partial f}{\partial c}\frac{\partial \theta}{\partial a},$$
$$0=\frac{\partial f}{\partial b}+\frac{\partial f}{\partial c}\frac{\partial \theta}{\partial b}.$$

The possibility of the generalisation thus depends on the integration of the new equation for c: and the form of that equation is affected by the form of f. In practice, it would therefore usually be convenient to take simple forms of f where choice can be exercised; it is unnecessary to aim at securing generality in the form of f, because that generality can be secured through the form of θ when the new equation can be completely integrated.

There are various ways in which an integral involving three parameters can be obtained. Sometimes it is possible to write down such an integral almost by inspection. Again, when the subsidiary systems in the methods of Monge and of Boole, and in the method of Ampère, possess three integrable combinations, in the forms

$$u=a, \quad v=b, \quad w=c,$$

the elimination of p and q leads to an integral involving three parameters: in this particular case, the critical quadratic must have equal roots. Again, if either of the subsidiary systems in question admits only one integrable combination in a form

$$u=a,$$

where u involves p or q or both, this equation, regarded as of the first order and integrated by Charpit's method, will lead to an integral involving three parameters. Further, if each of the subsidiary systems admits only a single integrable combination in a form

$$u=a, \quad v=b,$$

and if these equations can be resolved for p and q, the substitution of these values in

$$dz=p\,dx+q\,dy,$$

followed by a quadrature, gives an integral with the desired three parameters. As will be seen later (§§ 271, 274), some of these

possibilities (and they are not exhaustive) are bound up with the form of the equation for the determination of c in terms of a and b.

Ex. 1. Consider the equation

$$(q+yt)(r+1)=(ys-p-x)s.$$

When we proceed to integrate it by Ampère's method, we find that one of the subsidiary systems admits of the integrable combinations

$$\frac{p+x}{y}=\text{constant},\quad qy=\text{constant},$$

and that the other of the subsidiary systems admits of the integrable combinations

$$p+x=\text{constant},\quad qy-x(p+x)=\text{constant}.$$

In order to construct some integral of the original equation, involving three arbitrary constants, we take

$$qy=a,\quad p+x=b;$$

and then, as

$$dz=p\,dx+q\,dy$$

$$=(b-x)\,dx+\frac{a}{y}\,dy,$$

we have

$$z=c+bx-\tfrac{1}{2}x^2+a\log y.$$

This is the integral to be generalised. We have, in the notation of the text,

$$R=q,\quad S=\tfrac{1}{2}(p+x),\quad T=y,\quad U=y;$$

also

$$p=b-x,\quad q=\frac{a}{y},\quad r=-1,\quad s=0,\quad t=-\frac{a}{y^2},$$

so that

$$\frac{dp}{da}=0,\quad \frac{dp}{db}=1,$$

$$\frac{dq}{da}=\frac{1}{y},\quad \frac{dq}{db}=0.$$

Thus

$$R+Ut=0,\quad S-Us=\tfrac{1}{2}(p+x)=\tfrac{1}{2}b,\quad T+Ur=0;$$

and therefore

$$R_1=0,\quad S_1=\tfrac{1}{2}\frac{b}{y},\quad T_1=0,\quad V_1=\frac{1}{y}.$$

Moreover,

$$\frac{d^2f}{da^2}=\frac{\partial^2 c}{\partial a^2},\quad \frac{d^2f}{da\,db}=\frac{\partial^2 c}{\partial a\,\partial b},\quad \frac{d^2f}{db^2}=\frac{\partial^2 c}{\partial b^2};$$

consequently, the equation for c is

$$b\frac{\partial^2 c}{\partial a\,\partial b}=-1.$$

Hence

$$\frac{\partial c}{\partial a}=\phi'(a)-\log b,$$

$$c=\phi(a)+\psi(b)-a\log b.$$

Consequently,

$$z=\phi(a)+\psi(b)+bx-\tfrac{1}{2}x^2+a\log\frac{y}{b};$$

and the other equations are

$$0=\phi'(a)+\log\frac{y}{b},$$

$$0=x-\frac{a}{b}+\psi'(b).$$

From the last equation but one, we have

$$\phi'(a)=\log\frac{b}{y},$$

and therefore

$$\phi''(a)=\frac{y}{b}\frac{\partial}{\partial a}\left(\frac{b}{y}\right).$$

Let

$$\phi(a)-a\phi'(a)=-\theta\left(\frac{b}{y}\right),$$

the left-hand side being manifestly some function of $\frac{b}{y}$; then

$$a\phi''(a)=\theta'\left(\frac{b}{y}\right)\frac{\partial}{\partial a}\left(\frac{b}{y}\right),$$

that is,

$$\frac{a}{b}=\frac{1}{y}\theta'\left(\frac{b}{y}\right);$$

hence the equations are

$$\left.\begin{aligned} z&=bx-\tfrac{1}{2}x^2+\psi(b)-\theta\left(\frac{b}{y}\right)\\ 0&=x+\psi'(b)-\frac{1}{y}\theta'\left(\frac{b}{y}\right)\end{aligned}\right\},$$

which constitute a general primitive for the equation.

Ex. 2. The equation

$$r+t=0$$

possesses an integral

$$z=c+ax+by:$$

prove that the equation for c is only a transformation of the original equation, and deduce the customary primitive.

Ex. 3. The equation

$$r+2(q-x)s+(q-x)^2t=q$$

possesses two integrals

$$z=ax+by+\tfrac{1}{2}bx^2+c,$$
$$z=ax+by+\tfrac{1}{2}bx(x-b)+c;$$

generalise each of these.

Ex. 4. The equation

$$r-t=2\frac{p}{x}$$

admits an integral

$$z=c+ay+bx^3:$$

generalise it, so as to obtain the primitive

$$z=\theta(y+x)+\psi(y-x)-x\theta'(y+x)+x\psi'(y-x).$$

271. For the purpose of Imschenetsky's generalisation, it is necessary to have an integral of the given differential equation involving three arbitrary parameters; in order to complete the generalisation, it is necessary to obtain the primitive of the linear equation of the second order satisfied by c. If the three-constant integral has been obtained without the use of any systematic method, say as by mere inspection, the equation for c has no special properties or form. If, however, that integral has been obtained through one of the subsidiary systems in Ampère's method, it is possible to recognise an *a priori* limitation upon the form of the equation satisfied by c. For example, suppose that the subsidiary system associated with the argument α offers an integrable combination of the form

$$\begin{aligned} u(x, y, z, p, q) &= \text{constant} \\ &= \alpha, \end{aligned}$$

in accordance with the Ampère process: and let this equation, of the first order, be integrated by any of the methods leading to a complete integral, which will have a form

$$z = f(x, y, \alpha, a, c).$$

When Imschenetsky's generalising process is applied to this integral so as to determine c in terms of a and α, the equation for the determination of c is linear in

$$\frac{\partial^2 c}{\partial \alpha^2}, \quad \frac{\partial^2 c}{\partial a \partial \alpha}, \quad \frac{\partial^2 c}{\partial a^2}.$$

The arbitrary functions, which occur in the general integral of the original equation, are introduced by the arbitrary functions, which occur in the value of c. Now in one of the arbitrary functions in the required general integral, the argument is known (from the theory of Ampère's method) to be α; hence α must be the argument in one of the arbitrary functions occurring in the completed expression for c. Thus the equation determining c must be satisfied by an expression containing an arbitrary function of α together with, it may be, some of the derivatives of this function. In order that this may be the case, the term in $\dfrac{\partial^2 c}{\partial \alpha^2}$ must be absent from the equation: otherwise, the equation could not be satisfied by such a value of c, for that term would introduce derivatives of

the arbitrary function of order higher than those introduced by any other term.

When there are two subsidiary systems, and when each of them admits an integrable combination of the form

$$u(x, y, z, p, q) = \text{constant} = \alpha,$$
$$v(x, y, z, p, q) = \text{constant} = \beta,$$

respectively, where (by Ampère's theory) α and β are the arguments of the arbitrary functions in the general primitive, we resolve these two equations for p and q, substitute the resolved values in

$$dz = p\,dx + q\,dy,$$

and effect the quadrature: when the equation thus obtained is resolved with regard to z, it becomes

$$z = h(x, y, \alpha, \beta, c).$$

The Imschenetsky method can be applied to generalise this integral: an argument, similar to that in the preceding case, shews that the equation which determines c as a function of α and β is of the form

$$\frac{\partial^2 c}{\partial\alpha\partial\beta} = C,$$

where C is a function of α, β, c, $\dfrac{\partial c}{\partial\alpha}$, $\dfrac{\partial c}{\partial\beta}$ at the utmost.

Lastly, in the case of equations having only a single subsidiary system, so that the arbitrary functions in the general primitive have one and the same argument α, suppose that there is an integrable combination

$$g(x, y, z, p, q) = \text{constant}$$
$$= \alpha.$$

Let this be integrated, by Charpit's method or otherwise, leading to a complete integral

$$z = k(x, y, \alpha, a, c).$$

When this integral is generalised by the Imschenetsky process, so as to make c a function of a and α, then the equation of the second order determining c is similarly proved to have the form

$$\frac{\partial^2 c}{\partial a} = K,$$

where K is a function of a, α, c, and $\dfrac{\partial c}{\partial a}$, but does not involve $\dfrac{\partial c}{\partial \alpha}$. The absence of the term in $\dfrac{\partial^2 c}{\partial a \partial \alpha}$ is due to the fact that there is only a single argument in the arbitrary functions; in consequence, such a term would introduce a derivative of higher order than any introduced by other terms.

Hence when the equation in three parameters has been obtained, wholly or partly, from the integrable combinations of the subsidiary systems, the generalising equations are of the forms

$$2H \frac{\partial^2 c}{\partial a \partial b} + B \frac{\partial^2 c}{\partial b^2} = F,$$

with which may be associated

$$A \frac{\partial^2 c}{\partial a^2} + 2H \frac{\partial^2 c}{\partial a \partial b} = F,$$

as arising through the use of the alternative subsidiary system;

$$\frac{\partial^2 c}{\partial a \partial b} = C;$$

$$\frac{\partial^2 c}{\partial a^2} = K;$$

in the respective cases: and F, C, K vanish, if the combination $rt - s^2$ does not occur in the original equation, for then $U = 0$.

Ex. 1. Integrate the equations:—

(i) $x^4 r - 4x^2 qs + 4q^2 t + 2px^3 = 0$;

(ii) $x^2 r + 2x^2 s + \left(x^2 - \dfrac{b^2}{x^2 q^2}\right) t - 2z = 0$;

(iii) $r + 2qs + (q^2 - x^2) t - q = 0$;

(iv) $x^4 r - 4x^2 qs + 3qt + 2px^3 = 0$;

(v) $zs + \dfrac{z}{p^2} t + pq = 0.$ (Ampère; Imschenetsky.)

Ex. 2. The equation

$$a \frac{x^2}{y^2} r + b \frac{y^2}{x^2} t + (lx + my + nxy)(rt - s^2 + 1)$$
$$= \frac{1}{x^2 y^2} (lx + my + nxy)(z - px - qy + xy)^2$$

has an integral

$$z = ax + by + cxy;$$

obtain the primitive. (Imschenetsky.)

Ex. 3. Verify that the equation

$$rt - s^2 = (1 + p^2 + q^2)^2$$

is satisfied by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 1;$$

shew how to generalise this integral.

(The differential equation is the equation of surfaces with constant curvature, all the quantities in the equations quoted being real when the curvature is positive. For a full discussion of the properties, reference should be made to the treatises on differential geometry by Darboux and by Bianchi, where full citations of the original authorities will be found).

Generalisation of an Intermediate Integral.

272. The preceding discussion relates to the generalisation of a primitive of the equation, when the primitive is not complete. Similarly, it is possible to generalise a complete intermediate integral. Let

$$u(x, y, z, p, q, a, b) = 0$$

be such an integral, so that the differential equation of the second order is the result of eliminating a and b between the equations

$$u = 0, \quad \frac{du}{dx} = 0, \quad \frac{du}{dy} = 0.$$

The eliminant manifestly will be the same if a and b, instead of remaining parametric, are replaced by functions of x and y, such that

$$\frac{\partial u}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial x} = 0,$$

$$\frac{\partial u}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial y} = 0.$$

It might be possible that the equations

$$\frac{\partial u}{\partial a} = 0, \quad \frac{\partial u}{\partial b} = 0,$$

should satisfy all the conditions necessary for coexistence with $u = 0$ and with the equation of the second order: if these conditions are satisfied, the result of eliminating a and b between

$$u = 0, \quad \frac{\partial u}{\partial a} = 0, \quad \frac{\partial u}{\partial b} = 0,$$

would be a special or singular intermediate integral: the equations of the second order would be a very limited class. The alternative is that the relation

$$\frac{\partial (a, b)}{\partial (x, y)} = 0$$

should be satisfied. If this is satisfied identically (and we shall neglect all other cases), a functional relation exists between a and b: let it be

$$b = \phi (a).$$

Also we have

$$\frac{\partial u}{\partial a} da + \frac{\partial u}{\partial b} db = 0,$$

that is,

$$\frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \phi' (a) = 0.$$

The three equations

$$\left.\begin{aligned} u = 0, \quad b = \phi (a) \\ \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \phi' (a) = 0 \end{aligned}\right\}$$

constitute a generalisation of the complete intermediate integral.

Ex. 1. The equation

$$z - \frac{q^2 r - 2pqs + p^2 t}{rt - s^2} + \frac{(sp - rq)(sq - tp)}{(rt - s^2)^2} = 0$$

has a complete intermediate integral

$$z + cp + aq + ac = 0:$$

a generalised form is given by

$$\left.\begin{aligned} z + \ aq \ + (p + a) \phi (a) = 0 \\ q + \phi (a) + (p + a) \phi' (a) = 0 \end{aligned}\right\}.$$

Ex. 2. The equation

$$p + q = \lambda + \mu (x + y),$$

where λ and μ are arbitrary constants, is a complete intermediate integral of

$$r = t.$$

Generalising it, we take

$$\lambda = \phi (\mu),$$

and then

$$0 = \phi' (\mu) + x + y,$$

that is, μ is a function of $x + y$: changing the functions, we find

$$p + q = 2F' (x + y).$$

Similarly, from

$$p - q = \alpha + \beta (x - y),$$

where α and β are arbitrary, we find

$$p - q = 2G' (x - y).$$

Hence

$$p = F'(x+y) + G'(x-y),$$
$$q = F'(x+y) - G'(x-y):$$

substituting in

$$dz = p\,dx + q\,dy,$$

we have the customary general primitive

$$z = F(x+y) + G(x-y).$$

IMSCHENETSKY'S METHOD APPLIED TO LAPLACE'S LINEAR EQUATION.

273. It might be possible to generalise an incomplete primitive of an equation not of the form considered by Imschenetsky; but the analysis connected with even so simple an equation as

$$rt = 1,$$

having an incomplete primitive

$$z = ax + by + \tfrac{1}{2}x^2e^c + \tfrac{1}{2}y^2e^{-c},$$

is enough to suggest that the process would usually be impracticable.

There is, however, one class of equations, which are formally included among those considered and which yet provide little towards the construction of an incomplete primitive: it is the class of Laplace's linear equations

$$s + Ap + Bq + Cz = 0,$$

where A, B, C are functions of x and y only. For general values of A, B, and C, this equation does not possess an intermediate integral: hence there is no simplification to be expected *a priori* in the form of the generalising equation. In order to apply the Imschenetsky method, we take a primitive

$$z = az_1 + bz_2 + cz_3 = f,$$

where initially a, b, c are arbitrary parameters, and z_1, z_2, z_3 obviously are particular integrals of the equation. Applying the detailed results of the method, we have

$$\frac{df}{da} = z_1 + z_3\frac{\partial c}{\partial a} = 0,$$

$$\frac{df}{db} = z_2 + z_3\frac{\partial c}{\partial b} = 0;$$

so that x and y are functions of $\frac{\partial c}{\partial a}$ and $\frac{\partial c}{\partial b}$ only.

With the earlier notation, we have

$$R=0, \quad S=\tfrac{1}{2}, \quad T=0, \quad U=0, \quad V=-(Ap+Bq+Cz);$$

hence

$$R_1=\frac{dp}{db}\frac{dq}{db},$$

$$2S_1=\frac{dp}{db}\frac{dq}{da}+\frac{dp}{da}\frac{dq}{db},$$

$$T_1=\frac{dp}{da}\frac{dq}{da},$$

$$V_1=0,$$

and the equation for c is

$$R_1\frac{d^2f}{da^2}-2S_1\frac{d^2f}{da\,db}+T_1\frac{d^2f}{db^2}=V_1.$$

But

$$\frac{d^2f}{da^2}=z_3\frac{\partial^2c}{\partial a^2}, \quad \frac{d^2f}{da\,db}=z_3\frac{\partial^2c}{\partial a\,\partial b}, \quad \frac{d^2f}{db^2}=z_3\frac{\partial^2c}{\partial b^2};$$

and therefore the equation for c is

$$\frac{dp}{db}\frac{dq}{db}\frac{\partial^2c}{\partial a^2}-\left(\frac{dp}{db}\frac{dq}{da}+\frac{dp}{da}\frac{dq}{db}\right)\frac{\partial^2c}{\partial a\,\partial b}+\frac{dp}{da}\frac{dq}{da}\frac{\partial^2c}{\partial b^2}=0.$$

Also,

$$\frac{dp}{da}=p_1+p_3\frac{\partial c}{\partial a}, \quad \frac{dp}{db}=p_2+p_3\frac{\partial c}{\partial b},$$

$$\frac{dq}{da}=q_1+q_3\frac{\partial c}{\partial a}, \quad \frac{dq}{db}=q_2+q_3\frac{\partial c}{\partial b};$$

hence, when z_1, z_2, z_3 are known, the coefficients for the differential equation can be regarded as known. It is clear that these coefficients are expressible in terms of the derivatives $\frac{\partial c}{\partial a}, \frac{\partial c}{\partial b}$ alone: and so the differential equation for the determination of c does not explicitly involve a, b, or c.

When this equation is integrated, so as to give c in terms of a and b in a form

$$c=\theta(a, b),$$

then the general primitive of the Laplace's equation is given by eliminating a and b between the equations

$$\left.\begin{aligned} z &= z_1 a + z_2 b + z_3\theta(a,\ b) \\ 0 &= z_1 \quad + z_3\frac{\partial\theta}{\partial a} \\ 0 &= z_2 \quad + z_3\frac{\partial\theta}{\partial b} \end{aligned}\right\}.$$

It seems obvious that, the simpler the forms of z_1, z_2, z_3 initially chosen, the less complicated will usually be the details of the generalising analysis.

Ex. Consider the equation

$$s-\frac{p+q}{x+y}=0.$$

We can take

$$z_1=xy,\quad z_2=x-y,\quad z_3=1;$$

so that

$$z=axy+b(x-y)+c,$$

and then

$$xy=-\frac{\partial c}{\partial a},$$

$$x-y=-\frac{\partial c}{\partial b}.$$

Also

$$\frac{dp}{da}=y,\quad \frac{dp}{db}=1,$$

$$\frac{dq}{da}=x,\quad \frac{dq}{db}=-1;$$

substituting these values in the equation for c, we find

$$-\frac{\partial^2 c}{\partial a^2}-(x-y)\frac{\partial^2 c}{\partial a\,\partial b}+xy\frac{\partial^2 c}{\partial b^2}=0,$$

that is,

$$\frac{\partial^2 c}{\partial a^2}-\frac{\partial c}{\partial b}\frac{\partial^2 c}{\partial a\,\partial b}+\frac{\partial c}{\partial a}\frac{\partial^2 c}{\partial b^2}=0.$$

A primitive of this equation has already (§ 265, Ex. 1) been given: it is constituted by the three equations

$$\left.\begin{aligned} c&=\lambda^2 f''(\lambda)-2\lambda f'(\lambda)+2f(\lambda)+\mu^2 g''(\mu)-2\mu g'(\mu)+2g(\mu) \\ -a&=f''(\lambda)+g''(\mu) \\ -b&=\lambda f''(\lambda)-f'(\lambda)+\mu g''(\mu)-g'(\mu) \end{aligned}\right\}.$$

The quantities λ and μ must be identified: we have

$$\begin{aligned}\lambda^2 f'''d\lambda+\mu^2 g''d\mu &\\ &= dc\\ &= \frac{\partial c}{\partial a}da+\frac{\partial c}{\partial b}db\\ &= -\frac{\partial c}{\partial a}(f'''d\lambda+g''d\mu)-\frac{\partial c}{\partial b}(\lambda f'''d\lambda+\mu g''d\mu)\,;\end{aligned}$$

and therefore

$$\lambda^2=-\frac{\partial c}{\partial a}-\lambda\frac{\partial c}{\partial b}=xy+\lambda(x-y),$$

$$\mu^2=-\frac{\partial c}{\partial a}-\mu\frac{\partial c}{\partial b}=xy+\mu(x-y).$$

Hence we can take

$$\lambda=x,\quad \mu=-y.$$

Writing

$$f(\lambda)=X,\quad g(\mu)=Y,$$

we have

$$\begin{aligned}c&=x^2X''-2xX'+2X+y^2Y''-2yY'+2Y,\\ -a&=X''+Y'',\\ -b&=xX''-X'-yY''+Y'.\end{aligned}$$

Substituting these values of a, b, c in the equation

$$z=axy+b(x-y)+c,$$

and reducing, we find

$$z=2X+2Y-(x+y)(X'+Y'),$$

which is the general primitive of the original equation.

274. The transformation adopted in § 273 and applied to the preceding example, is easily seen there to be a contact-transformation between the two sets of variables: in fact, we have

$$\begin{aligned}z&=az_1+bz_2+cz_3\\ &=z_3\left(c-a\frac{\partial c}{\partial a}-b\frac{\partial c}{\partial b}\right);\end{aligned}$$

and, in the example, z_3 is unity. This property can be secured in general by an appropriate initial modification of the Laplace equation.

Let z_3 be a particular integral of the equation, and write $z=z_3Z$, so that, if

$$Z_1=\frac{z_1}{z_3},\quad Z_2=\frac{z_2}{z_3},$$

then Z_1, Z_2, 1 are three particular integrals of the transformed equation which is easily found to be

$$S+A'P+B'Q=0,$$

where

$$A' = A + \frac{q_3}{z_3}, \quad B' = B + \frac{p_3}{z_3}.$$

Accordingly, as the term in Z has been removed, we may suppose that our equation initially has the form

$$s + Ap + Bq = 0.$$

We then take

$$z = c + az_1 + bz_2;$$

and the other equations then become

$$0 = \frac{\partial c}{\partial a} + z_1, \quad 0 = \frac{\partial c}{\partial b} + z_2,$$

so that

$$z = c - a\frac{\partial c}{\partial a} - b\frac{\partial c}{\partial b},$$

being the Legendrian contact-transformation.

The direct construction of the generalising equation is simple. Denoting the second derivatives of c with regard to a and b by ρ, σ, τ respectively, and writing

$$\frac{\partial a}{\partial x} = a_x, \quad \frac{\partial a}{\partial y} = a_y,$$

and similarly for b, we have

$$\left.\begin{aligned} -p_1 &= \rho a_x + \sigma b_x \\ -q_1 &= \rho a_y + \sigma b_y \end{aligned}\right\}, \quad \left.\begin{aligned} -p_2 &= \sigma a_x + \tau b_x \\ -q_2 &= \sigma a_y + \tau b_y \end{aligned}\right\}$$

as derivatives of the equations

$$-z_1 = \frac{\partial c}{\partial a}, \quad -z_2 = \frac{\partial c}{\partial b};$$

hence

$$\left.\begin{aligned} (\rho\tau - \sigma^2)\, a_x &= -p_1\tau + p_2\sigma \\ (\rho\tau - \sigma^2)\, a_y &= -q_1\tau + q_2\sigma \\ (\rho\tau - \sigma^2)\, b_x &= p_1\sigma - p_2\rho \\ (\rho\tau - \sigma^2)\, b_y &= q_1\sigma - q_2\rho \end{aligned}\right\}.$$

Again, from the equation

$$z = c + az_1 + bz_2,$$

we have

$$p = ap_1 + bp_2,$$

$$q = aq_1 + bq_2,$$

the terms that arise through the variations of a, b, c vanishing; and then

$$\begin{aligned} s &= as_1 + bs_2 + p_1 a_y + p_2 b_y \\ &= as_1 + bs_2 - \frac{1}{\rho\tau - \sigma^2}\{p_1 q_1 \tau - (p_1 q_2 + p_2 q_1)\sigma + p_2 q_2 \rho\}. \end{aligned}$$

Also,

$$\begin{aligned} s &= -Ap - Bq \\ &= -a(Ap_1 + Bq_1) - b(Ap_2 + Bq_2) \\ &= as_1 + bs_2; \end{aligned}$$

and therefore

$$p_1 q_1 \tau - (p_1 q_2 + p_2 q_1)\sigma + p_2 q_2 \rho = 0.$$

Moreover,

$$p_1 = \frac{dp}{da}, \quad p_2 = \frac{dp}{db},$$

from the value of p; and similarly

$$q_1 = \frac{dq}{da}, \quad q_2 = \frac{dq}{db},$$

from the value of q. Consequently,

$$\frac{dp}{db}\frac{dq}{db}\frac{\partial^2 c}{\partial a^2} - \left(\frac{dp}{da}\frac{dq}{db} + \frac{dp}{db}\frac{dq}{da}\right)\frac{\partial^2 c}{\partial a\,\partial b} + \frac{dp}{da}\frac{dq}{da}\frac{\partial^2 c}{\partial b^2} = 0,$$

agreeing with the earlier form obtained for the equation which is to determine c.

275. A different way of proceeding is as follows*. Let the equation be taken in the form

$$s + \alpha p + \beta q = 0,$$

where α and β are functions of x and y only, so that $z = 1$ is an integral of the equation. Let z_1 and z_2 be two other integrals of the equation, and assume that

$$z = u + vz_1 + wz_2,$$

where the new unknown quantities u, v, w are to be determined by the condition that the forms of p and q (and therefore, owing to the equation, the form of s also) are the same, when u, v, w are variable, as they would be if u, v, w were parametric. Then

$$\begin{aligned} p &= vp_1 + wp_2, \\ q &= vq_1 + wq_2, \end{aligned}$$

* In connection with this investigation, a memoir by R. Liouville, *Journ. de l'Éc. Polyt.*, Cah. LVI (1886), pp. 7—62, may be consulted.

provided

$$\frac{\partial u}{\partial x} + z_1 \frac{\partial v}{\partial x} + z_2 \frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial u}{\partial y} + z_1 \frac{\partial v}{\partial y} + z_2 \frac{\partial w}{\partial y} = 0:$$

also

$$s = vs_1 + ws_2,$$

provided

$$p_1 \frac{\partial v}{\partial y} + p_2 \frac{\partial w}{\partial y} = 0,$$

because s can be derived from p, and

$$q_1 \frac{\partial v}{\partial x} + q_2 \frac{\partial w}{\partial x} = 0,$$

because s can be derived from q. Hence

$$\frac{\partial v}{\partial x} = -\frac{q_2}{q_1}\frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial y} = -\frac{p_2}{p_1}\frac{\partial w}{\partial y},$$

$$\frac{\partial u}{\partial x} = \left(z_1 \frac{q_2}{q_1} - z_2\right)\frac{\partial w}{\partial x}, \quad \frac{\partial u}{\partial y} = \left(z_1 \frac{p_2}{p_1} - z_2\right)\frac{\partial w}{\partial y};$$

and therefore

$$\frac{\partial}{\partial y}\left(\frac{q_2}{q_1}\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{p_2}{p_1}\frac{\partial w}{\partial y}\right),$$

$$\frac{\partial}{\partial y}\left\{\left(z_1 \frac{q_2}{q_1} - z_2\right)\frac{\partial w}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\left(z_1 \frac{p_2}{p_1} - z_2\right)\frac{\partial w}{\partial y}\right\},$$

apparently two distinct equations of the second order satisfied by w. It is easy, however, to see that the second equation (on the removal of a non-vanishing factor z_1) becomes the same as the first; in fact, they both reduce to the single equation

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{p_2}{q_2}\frac{q_1 t_2 - q_2 t_1}{p_1 q_2 - p_2 q_1}\frac{\partial w}{\partial x} - \frac{q_2}{p_2}\frac{p_1 r_2 - p_2 r_1}{p_1 q_2 - p_2 q_1}\frac{\partial w}{\partial y} = 0,$$

which may be written

$$\frac{\partial^2 w}{\partial x \partial y} + \alpha' \frac{\partial w}{\partial x} + \beta' \frac{\partial w}{\partial y} = 0.$$

When the value of w is known, then

$$v = -\int\left(\frac{q_2}{q_1}\frac{\partial w}{\partial x} dx + \frac{p_2}{p_1}\frac{\partial w}{\partial y} dy\right),$$

so that v is obtainable by quadrature: and then

$$u = -\int (z_1 dv + z_2 dw),$$

so that u also is obtainable by quadrature.

Ex. 1. Let the method be applied to the equation

$$s = \frac{p+q}{x+y}.$$

Particular integrals are given by

$$z_1 = x - y, \quad z_2 = xy;$$

we therefore take

$$z = u + (x-y)v + xyw.$$

The equations satisfied by u, v, w are

$$\frac{\partial u}{\partial x} + (x-y)\frac{\partial v}{\partial x} + xy\frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial u}{\partial y} + (x-y)\frac{\partial v}{\partial y} + xy\frac{\partial w}{\partial y} = 0,$$

$$-\frac{\partial v}{\partial x} + x\frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial v}{\partial y} + y\frac{\partial w}{\partial y} = 0.$$

From these we have

$$\frac{\partial v}{\partial x} = x\frac{\partial w}{\partial x}, \quad \frac{\partial u}{\partial x} = -x^2\frac{\partial w}{\partial x},$$

$$\frac{\partial v}{\partial y} = -y\frac{\partial w}{\partial y}, \quad \frac{\partial u}{\partial y} = -y^2\frac{\partial w}{\partial y};$$

and therefore

$$\frac{\partial}{\partial y}\left(x\frac{\partial w}{\partial x}\right) = -\frac{\partial}{\partial x}\left(y\frac{\partial w}{\partial y}\right),$$

$$\frac{\partial}{\partial y}\left(x^2\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(y^2\frac{\partial w}{\partial y}\right),$$

both satisfied in virtue of

$$\frac{\partial^2 w}{\partial x \partial y} = 0;$$

and therefore

$$w = X'' + Y'',$$

where X and Y are arbitrary functions of x and of y respectively. Then

$$\begin{aligned} v &= \int (xX'''dx - yY'''dy) \\ &= xX'' - X' - yY'' + Y', \\ u &= -\int (x^2X'''dx + y^2Y'''dy) \\ &= -x^2X'' + 2xX' - 2X - y^2Y'' + 2yY' - 2Y; \end{aligned}$$

consequently,

$$\begin{aligned} z &= u + (x-y)v + xyw \\ &= -2X - 2Y + (x+y)(X' + Y'). \end{aligned}$$

Ex. 2. Apply the preceding method to the equation

$$s=m\frac{p+q}{x+y},$$

where m is a constant; and using the integrals

$$z_1=x-y, \quad z_2=x^m y^m,$$

prove that the equation for w is

$$\frac{\partial^2 w}{\partial x'\,\partial y'}=\frac{m-1}{x'+y'}\left(\frac{\partial w}{\partial x'}+\frac{\partial w}{\partial y'}\right),$$

where $xx'=1$, $yy'=1$. Hence integrate the equation

$$s=2\frac{p+q}{x+y};$$

and shew how the property can be used to connect the integration of the two equations

$$s=m\frac{p+q}{x+y}, \quad s=(m-2)\frac{p+q}{x+y}.$$

Ex. 3. Shew that, if z_1, z_2, z_3 denote three linearly independent integrals of the equation

$$s+ap+bq+cz=0,$$

where a, b, c are functions of x and y alone, and if three quantities u, v, w are introduced such that

$$z=uz_1+vz_2+wz_3$$

is another integral, which keeps the same forms for p, q, s, whether u, v, w be variable or parametric, then w satisfies an equation

$$\begin{vmatrix} z_1, & p_1, & q_1 \\ z_2, & p_2, & q_2 \\ z_3, & p_3, & q_3 \end{vmatrix}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\begin{vmatrix} z_1, & p_1 \\ z_2, & p_2 \end{vmatrix}}{\begin{vmatrix} z_1, & q_1 \\ z_2, & q_2 \end{vmatrix}}\begin{vmatrix} z_1, & q_1, & t_1 \\ z_2, & q_2, & t_2 \\ z_3, & q_3, & t_3 \end{vmatrix}\frac{\partial w}{\partial x}$$
$$-\frac{\begin{vmatrix} z_1, & q_1 \\ z_2, & q_2 \end{vmatrix}}{\begin{vmatrix} z_1, & p_1 \\ z_2, & p_2 \end{vmatrix}}\begin{vmatrix} z_1, & p_1, & r_1 \\ z_2, & p_2, & r_2 \\ z_3, & p_3, & r_3 \end{vmatrix}\frac{\partial w}{\partial y}=0.$$

Shew how to determine u and v when w is known: and prove that u and v satisfy equations of the same form as the equation satisfied by w. Prove also that, if the original differential equation be of rank n in either of the variables x and y, then the equation for w is of rank $n+1$.

Ex. 4. Prove that, if z_1, z_2, z_3 denote three linearly independent integrals of Laplace's linear equation

$$s+ap+bq+cz=0,$$

no relation, which is homogeneous and of the second order, can subsist among z_1, z_2, z_3 alone.

CHAPTER XX.

Characteristics of Equations of the Second Order: Intermediate Integrals.

The theory of characteristics led to an effective method of integration when applied to equations of the first order; it can be applied also to equations of the second order and, in the application, it indicates the geometrical significance of the possession of intermediate integrals. The following account of these characteristics and of the construction of equations that possess intermediate integrals, not necessarily of the type considered by Monge, is based mainly upon Goursat's memoir*.

276. In order to solve Cauchy's problem for an equation of the second order

$$f(x, y, z, p, q, r, s, t) = 0,$$

in the most general form of that problem, which assigns general functions of x and y as values of z and p when x and y are connected by a given relation $g(x, y) = 0$, we can proceed as follows. Along the curve representing the relation, we have

$$\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy = 0,$$

as a property of the curve: also, because the value of z, say $\phi(x, y)$, is given along the curve, the quantity

$$pdx + qdy, \ = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy,$$

for the elements dx, dy along the curve, is known. Hence, as p is given along the curve, so also q is known along the curve; and therefore Cauchy's problem amounts to the determination of a

* *Acta Math.*, t. XIX (1895), pp. 285—340. Reference also should be made to vol. I, ch. IV, of his treatise already (p. 7) quoted.

surface, satisfying the differential equation, passing through a given curve in the plane of x and y, and touching a given developable surface* along the curve.

Accordingly, at points on the curve, the values of x, y, z, p, q in connection with Cauchy's problem are expressible in terms of a single parameter u: and the values of r, s, t are given by the three equations

$$f = 0,$$
$$r\,dx + s\,dy = dp,$$
$$s\,dx + t\,dy = dq.$$

These three equations will suffice for the determination of r, s, t in terms of u, unless the Jacobian of the three equations with respect to the three variables vanishes, that is, unless

$$\begin{vmatrix} R, & S, & T \\ dx, & dy, & 0 \\ 0, & dx, & dy \end{vmatrix} = 0,$$

where

$$R = \frac{\partial f}{\partial r}, \quad S = \frac{\partial f}{\partial s}, \quad T = \frac{\partial f}{\partial t}.$$

When expanded, this equation is

$$R\left(\frac{dy}{dx}\right)^2 - S\frac{dy}{dx} + T = 0.$$

In the first place, suppose that (save possibly at isolated points, and these we neglect) the last equation is not satisfied: then the three equations determine one, or more than one, set of values of r, s, t in terms of u. We select any one such set, and proceed to consider the derivatives of the third order along the curve.

Denoting these as before by $\alpha, \beta, \gamma, \delta$, and writing

$$\left.\begin{aligned} X &= \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q} \\ Y &= \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q} \end{aligned}\right\},$$

* The values of p and q determine the tangent plane: along the given curve, they are functions of a single variable, so that the equation of the tangent planes contains only a single parameter: they thus are enveloped by a developable surface.

we have

$$\begin{aligned} dr &= \alpha\, dx + \beta\, dy, \\ ds &= \beta\, dx + \gamma\, dy, \\ dt &= \gamma\, dx + \delta\, dy, \end{aligned}$$

along the curve, and

$$\begin{aligned} X + R\alpha + S\beta + T\gamma = 0, \\ Y + R\beta + S\gamma + T\delta = 0, \end{aligned}$$

always. Thus there are five equations involving the four quantities $\alpha, \beta, \gamma, \delta$: so that one of them must be dependent upon the others, or there must be a linear relation among them. It is easily obtained; for, on multiplying the fourth by dx, the fifth by dy, and using the first three, we have

$$X\,dx + Y\,dy + R\,dr + S\,ds + T\,dt = 0,$$

that is,

$$df = 0,$$

which is satisfied in virtue of $f = 0$: so that there are only four independent equations. Also, denoting the value of $\dfrac{dy}{dx}$ along the curve by μ, and eliminating the quantities β and γ from the fourth equation by the first two, we have

$$X + \alpha\left(R - \frac{1}{\mu}S + \frac{1}{\mu^2}T\right) + \frac{dr}{dy}S + \left(\frac{ds}{dy} - \frac{1}{\mu}\frac{dr}{dy}\right)T = 0.$$

The coefficient of α does not vanish, by hypothesis: and therefore α is determinate at the point on the curve in the Cauchy problem. Similarly, the values of β, γ, δ are determinate there.

Similarly for the derivatives of all the orders in succession: each of them is determinate at the point on the curve, as associated with the assigned initial conditions: the only requirement is that

$$R\mu^2 - S\mu + T$$

does not vanish generally along the curve.

Accordingly, let the function z be developed in a power-series in $x - a$, $y - b$, where a, b is a point on the curve; as all the derivatives of z at a, b are known, all the coefficients in the series are known. Under certain conditions, which do not substantially concern us here, the series can be proved to converge: the function which it represents is an integral of the equation: and we merely obtain Cauchy's theorem again.

277. In the next place, consider a curve C for which the equation

$$R\left(\frac{dy}{dx}\right)^2 - S\frac{dy}{dx} + T = 0$$

is everywhere satisfied. Reviewing the past analysis, we see that the three equations, which involve r, s, t, do not suffice for the determination of those three quantities: one of the quantities can be taken arbitrarily, and then the other two are determinate. Similarly, the equations involving α, β, γ, δ do not suffice for the determination of those four quantities: one of them can be taken arbitrarily, and then the other three are determinate. The same holds for all the other orders in succession. Now suppose that

$$z = \phi(x, y)$$

is an integral of the differential equation: it represents a surface. At all points on this surface, z and all its derivatives are functions of x and y, and so also are R, S, T: hence the equation

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$$

defines two families of curves upon the surface, except when the relation

$$S^2 - 4RT = 0$$

is satisfied identically, in which case it defines only a single family. When there are two families of such curves, then one curve of each family (and, in general, only one curve of each family) passes through a point on the surface: and the directions of the two curves through the point, one from each family, are different from one another unless the point lies upon the locus

$$S^2 - 4RT = 0, \quad z = \phi(x, y).$$

Consider now the equations that are satisfied along C. Everywhere upon the surface we have

$$\begin{aligned} X + R\alpha + S\beta + T\gamma &= 0, \\ \alpha\,dx + \beta\,dy &= dr, \\ \beta\,dx + \gamma\,dy &= ds, \\ \gamma\,dx + \delta\,dy &= dt, \end{aligned}$$

and therefore everywhere on the surface, we have

$$X + \alpha\left(R - \frac{1}{\mu}S + \frac{1}{\mu^2}T\right) + \frac{dr}{dy}S + \left(\frac{ds}{dy} - \frac{1}{\mu}\frac{dr}{dy}\right)T = 0,$$

μ being $\dfrac{dy}{dx}$ and, for full variation of values of dx and dy, giving all directions through the point. In particular, select a direction giving a curve C through the point, so that

$$R\mu^2 - S\mu + T = 0;$$

the coefficient of α vanishes, and the coefficient of $\dfrac{dr}{dy}$ is $S - \dfrac{1}{\mu}T$, that is, it is $R\mu$; hence

$$X + \frac{ds}{dy}T + R\mu\frac{dr}{dy} = 0,$$

which can be written in the form

$$X\,dxdy + Rdrdy + Tdsdx = 0.$$

Similarly, from

$$Y + R\beta + S\gamma + T\delta = 0,$$

we find

$$Y\,dxdy + Rdsdy + Tdtdx = 0.$$

It therefore follows that the equations

$$\left.\begin{array}{c} f = 0 \\ dz = pdx + qdy, \quad dp = rdx + sdy, \quad dq = sdx + tdy \\ Rdy^2 - Sdxdy + Tdx^2 = 0 \\ X\,dxdy + Rdrdy + Tdsdx = 0 \\ Y\,dxdy + Rdsdy + Tdt\,dx = 0 \end{array}\right\}$$

are satisfied along the curve C. Apparently there are seven equations: but the relation

$$df = 0$$

is satisfied in virtue of the last six equations, so that $f = 0$ can be regarded as an equation not independent of the last six equations.

The aggregate of these equations determines a *characteristic* of the equation

$$f = 0.$$

They involve eight quantities x, y, z, p, q, r, s, t, and no quantities of order higher than the second: and being ordinary relations among differential elements, they determine seven of the quantities in terms of the eighth. But the number of independent equations is only six: one of the seven quantities can be arbitrarily assigned, and the other six are then determinate, their expressions (and

therefore also the characteristic) being affected by the form of the arbitrary assignment.

Hence every integral of the differential equation, when that integral is regarded as a surface, is a locus of characteristic curves; and a given differential equation usually has two systems of characteristics though, for equations of specialised form, the two systems may coalesce into a single system.

Let x be selected as the independent variable for a characteristic of $f=0$: and let λ, μ be the roots of the quadratic

$$R\theta^2 - S\theta + T = 0,$$

so that

$$S = R(\lambda + \mu), \quad T = R\lambda\mu.$$

Then the equations of one characteristic, after a slight transformation, become

$$\left.\begin{aligned} \frac{dy}{dx} &= \lambda \\ \frac{dz}{dx} &= p + \lambda q \\ \frac{dp}{dx} &= r + \lambda s \\ \frac{dq}{dx} &= s + \lambda t \\ \frac{dr}{dx} + \mu\frac{ds}{dx} &= -\frac{X}{R} \\ \frac{ds}{dx} + \mu\frac{dt}{dx} &= -\frac{Y}{R} \end{aligned}\right\};$$

the similar equations of the other characteristic are given by the interchange of λ and μ in the preceding equations.

278. In particular, let the equation be

$$rt - s^2 + Ar + 2Bs + Ct = D;$$

so that

$$R = A + t, \quad S = 2(B - s), \quad T = C + r,$$

A, B, C, D not involving derivatives of the second order: the quadratic in $dy : dx$ becomes

$$(A + t)\,dy^2 - 2(B - s)\,dx\,dy + (C + r)\,dx^2 = 0,$$

that is,

$$A\,dy^2 - 2B\,dx\,dy + C\,dx^2 + dx\,dp + dy\,dq = 0.$$

Again, substituting in the original equation the values

$$r = \frac{dp}{dx} - \lambda s, \quad t = \frac{1}{\lambda}\left(\frac{dq}{dx} - s\right),$$

given in the equations of the characteristic, and taking account of the differential relation just stated, together with

$$\frac{dy}{dx} = \lambda,$$

we find

$$A\,dp\,dy + C\,dq\,dx + dp\,dq - D\,dx\,dy = 0.$$

These two equations are the equations that occur in the subsidiary systems of Monge and of Ampère. Using the modified forms there adopted, we denote by ρ and σ the roots of the quadratic

$$m^2 + 2mB + AC + D = 0,$$

which has equal roots only when the quadratic

$$R\theta^2 - S\theta + T = 0$$

also has equal roots; and we find that the foregoing two equations lead to the pair

$$\left.\begin{aligned} dp + C\,dx + \rho\,dy &= 0 \\ dq + \sigma\,dx + A\,dy &= 0 \end{aligned}\right\},$$

and to another pair obtained by interchanging ρ and σ. Three other equations of the characteristic become

$$dz - p\,dx - q\,dy = 0,$$

$$dp - r\,dx - s\,dy = 0,$$

$$dq - s\,dx - t\,dy = 0;$$

and the remaining two are the same as before.

Now three equations of this set, viz.

$$\left.\begin{aligned} dp + C\,dx + \rho\,dy &= 0 \\ dq + \sigma\,dx + A\,dy &= 0 \\ dz - p\,dx - q\,dy &= 0 \end{aligned}\right\},$$

involve only x, y, z, p, q: any set of values, which can satisfy these three equations, determines a *characteristic of the first order*. It is sufficient to have a set of values satisfying the equations, and there is no necessity to have an intermediate integral: but if the given differential does possess an intermediate integral in any

form, that integral will, of itself, determine a characteristic of the first order.

The aggregate of all the equations determines *characteristics of the second order.* The relations of the characteristics of the two orders and, in particular, of an intermediate integral to the characteristics of the second order, appear as follows.

We know that, in connection with a given equation, there are six independent equations in the system which determines the characteristics. When there is a characteristic of the first order, it is determined by three of these equations; and therefore three equations remain for the characteristics of the second order, which accordingly are sufficient for the determination of r, s, t without limitations or conditions. It therefore follows that the characteristics of the second order include those (if any) of the first order.

But, further, let the quantities x, y, z, p, q, as connected with a characteristic of the first order when it exists, be expressed in terms of a parameter α; this evidently is possible, in connection with Ampère's theory. We have

$$r\frac{dx}{d\alpha} = \frac{dp}{d\alpha} - s\frac{dy}{d\alpha},$$

$$\frac{dr}{d\alpha}\left(\frac{dx}{d\alpha}\right)^2 = \frac{dx}{d\alpha}\left(\frac{d^2p}{d\alpha^2} - \frac{ds}{d\alpha}\frac{dy}{d\alpha} - s\frac{d^2y}{d\alpha^2}\right) - \left(\frac{dp}{d\alpha} - s\frac{dy}{d\alpha}\right)\frac{d^2x}{d\alpha^2},$$

$$t\frac{dy}{d\alpha} = \frac{dq}{d\alpha} - s\frac{dx}{d\alpha};$$

and the equation

$$X dx dy + R dr dy + T ds dx = 0$$

becomes

$$X\frac{dx}{d\alpha}\frac{dy}{d\alpha} + R\frac{dr}{d\alpha}\frac{dy}{d\alpha} + T\frac{ds}{d\alpha}\frac{dx}{d\alpha} = 0.$$

Replacing X, R, T by their values, and substituting for r, $\frac{dr}{d\alpha}$, t from the preceding relations, we find

$$\begin{aligned}\frac{ds}{d\alpha} &= \frac{A_1 s^2 + A_2 s + A_3}{A_4 s + A_5} \\ &= g(s, \alpha),\end{aligned}$$

where A_1, A_2, A_3, A_4, A_5 are functions of α. The integral of this equation involves an arbitrary constant: it determines s, and the

preceding relations give the values of r and t. Hence, when an equation

$$rt - s^2 + Ar + 2Bs + Ct = D$$

possesses a characteristic of the first order, the equations of the characteristic of the second order, which includes that of the first order, involve an arbitrary constant.

279. Returning to the consideration of the general case, we have seen that an integral system passes through the curve C along which the equation

$$Rdy^2 - Sdxdy + Tdx^2 = 0$$

is everywhere satisfied. There then is a want of determinateness in the equations

$$f=0, \quad dp - rdx - sdy = 0, \quad dq - sdx - tdy = 0,$$

as regards the derivation of sets of values of r, s, t: instead of assuming r arbitrarily, so that s and t can then be regarded as determinate, we derive from the three equations an infinitude of values, and we can regard them as a continuous system. In this aspect, we have an infinitude of integral surfaces which are themselves a continuous system; and as these surfaces have the same values of x, y, z, p, q along the curve C, they touch one another along the curve. Hence, when we substitute values of r and t, derived from

$$dp - rdx - sdy = 0, \quad dq - sdx - tdy = 0,$$

and expressed in terms of s, in the equation $f=0$, the changed form of the last equation must, quà equation in s, become evanescent: the coefficients of all powers of s must therefore vanish. There will thus result a number of simultaneous relations in x, y, z, p, q, dx, dy, dp, dq, which are homogeneous in the last four quantities. As the modified equation would otherwise have determined s, it would have contained at least two terms; so that there cannot be fewer than two conditions. On the other hand, the relations are homogeneous in dx, dy, dp, dq: hence there cannot be more than three independent relations.

Among the several cases, which thus are possibilities for an equation $f=0$, the following may be noted.

(i) The relations may constitute an inconsistent system, which accordingly cannot be satisfied. The equation does not then possess an infinitude of integral surfaces having contact of the first order along the curve C. The quantities r, s, t are determinate; and two integral surfaces have contact of the second order along the characteristics.

(ii) The relations may be such as to yield an equation or equations involving x, y, z, p, q only.

(iii) The relations may be such as not to yield any equation free from differential elements: the preceding explanations shew that the number of independent relations in the set is either two or three.

GEOMETRICAL INTERPRETATION.

280. The difference between the various cases can be illustrated through a geometrical interpretation introduced by Goursat. He regards r, s, t as the coordinates of a point in space, and x, y, z, p, q as parametric quantities: then the equation $f=0$ represents a surface. Denoting current coordinates by ρ, σ, τ, we have the tangent plane to the surface $f=0$ at the point r, s, t given by

$$(\rho - r) R + (\sigma - s) S + (\tau - t) T = 0;$$

and a parallel plane through the origin is

$$\rho R + \sigma S + \tau T = 0.$$

The original equation has two characteristics in general: but they coalesce into one if

$$S^2 = 4RT.$$

The envelope of the second plane, when this condition is satisfied, is

$$\rho\tau - \sigma^2 = 0,$$

which is a cone: so that the two characteristics coalesce into one, when the tangent planes of the surface $f=0$ are parallel to the tangent planes of the cone

$$rt - s^2 = 0.$$

Again, when we take

$$\frac{dp}{dx} = \xi, \quad \frac{dq}{dx} = \eta, \quad \frac{dy}{dx} = m,$$

so that ξ, η, m are parametric quantities, the equations

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

become

$$r = -ms + \xi, \quad s = -mt + \eta,$$

which are the equations of a line parallel to

$$r = -ms, \quad s = -mt,$$

and the latter is a generator of the cone

$$rt - s^2 = 0.$$

To find the intersections of the line with the surface $f = 0$, we substitute

$$r = -ms + \xi, \quad t = -\frac{1}{m}(s - \eta),$$

in the equation. In general, we have a set of values of s thus given: but if it happens that the equation is evanescent after the substitution, the line lies entirely in the surface $f = 0$, which therefore possesses generators parallel to those of the cone $rt - s^2 = 0$.

When the surface $f = 0$ is perfectly unconditioned, it obviously will not possess this special property: we have the first of the preceding cases (§ 279).

When the surface $f = 0$ is not quite arbitrary, but is such that certain conditions among its parameters x, y, z, p, q are satisfied, the special property can be possessed: we have the second of the preceding cases.

When the surface $f = 0$ is not quite arbitrary, the property may be possessed for appropriate values, or sets of values, of m, ξ, η. We then have the third of the preceding cases: it contains a couple of sub-cases.

In the first of these sub-cases, there are two relations between x, y, z, p, q, ξ, η, m. Hence there is a simple infinitude of sets of values for ξ, η, m, so that the surface contains a simple infinitude of generators: and these are parallel to the generators of the cone

$$rt - s^2 = 0,$$

which therefore is an asymptotic cone of the ruled surface (or scroll) represented by $f = 0$.

In the second of the sub-cases, there are three relations between $x, y, z, p, q, \xi, \eta, m$. We then have a finite number of sets of values for ξ, η, m, so that the surface represented by $f=0$ contains only a finite number of straight lines; it is not a ruled surface.

When the first sub-case arises, in which there are two relations, these may be taken in the form

$$G_1(x, y, z, p, q, \xi, \eta, m) = 0,$$
$$G_2(x, y, z, p, q, \xi, \eta, m) = 0;$$

owing to them, the equations

$$f = 0, \quad r = -ms + \xi, \quad s = -mt + \eta,$$

do not determine r, s, t definitely. If then functions x, y, z, p, q of a single variable can be so chosen that the equations

$$G_1 = 0, \quad G_2 = 0, \quad dz = pdx + qdy,$$

hold, r, s, t are not determinate. In these circumstances, two integrals of the differential equation can have the same values of x, y, z, p, q, but r, s, t will not necessarily be the same; the contact of the integrals along the characteristic cannot generally be of the second order but is generally of the first order.

Although the equations are satisfied for one of the roots m of the quadratic

$$m^2R - mS + T = 0,$$

they are not usually satisfied for the other root: then, for that other root, r, s, t are determinate; and so two integrals, having contact of only the first order along one characteristic, will usually have contact of the second order along the other. If, however, the quadratic has equal roots, there is only one characteristic: and the contact of two integrals is only of the first order.

In both of these cases, the surface represented by $f = 0$ has an infinitude of generators parallel to those of the cone $rt - s^2 = 0$: in the former, the surface is a scroll; in the latter, it is developable.

Classification of Equations according to Characteristics.

281. Returning now to the differential equations of the second order, we can classify them according to their characteristics.

One class of equations is composed of those which possess two different characteristics of the second order. Two integrals (when x, y, z are regarded as coordinates) have contact of the second order at least along both of the characteristics.

Another class of equations is composed of those which, when x, y, z, p, q are regarded as parametric, represent scrolls (ruled undevelopable surfaces) having $rt - s^2 = 0$ for an asymptotic cone. There are two distinct characteristics; two integrals have contact of only the first order along one of the characteristics and contact of the second order along the other.

Another class of equations is composed of those which, when x, y, z, p, q are regarded as parametric, represent developable surfaces having their tangent planes parallel to the tangent planes of the cone $rt - s^2 = 0$. There is a single characteristic: two integrals have contact of only the first order along that characteristic.

Another class of equations is composed of those which are linear in r, s, t, $rt - s^2$, of the form

$$f = Ar + 2Bs + Ct + K(rt - s^2) - D = 0.$$

When K is not zero, the scroll $f = 0$ has generators parallel to those of $rt - s^2 = 0$. Usually $f = 0$ has two systems of generators distinct from one another; there then are two systems of characteristics, and two integrals have contact of only the first order along each of them. But when the relation

$$B^2 - AC = DK$$

is satisfied, so that $f = 0$ has the form

$$(Kr + C)(Kt + A) = (Ks - B)^2,$$

and the surface is a cone, the same as $rt - s^2 = 0$ and similarly placed, there is only one system of generators: there is a single characteristic, and two integrals have contact of only the first order along it.

When K is zero, the surface $f=0$ is a plane: through any point of it, there are generally two (real or imaginary) straight lines in it parallel to generators of the cone $rt-s^2=0$; in that case, there are two characteristics, and two integrals have contact of only the first order along each of them. But if the plane is parallel to a tangent plane to the cone, then through a point in it only one line can be drawn parallel to a generator; there is a single characteristic, and two integrals have contact of only the first order along that characteristic.

It thus appears that, if a relation exists between x, y, z, p, q free from differential elements, the differential equation possesses at least one characteristic of the first order: though, conversely, it is not the fact that, even if the equation possesses a characteristic of the first order, some relation exists between x, y, z, p, q, which is free from differential elements. When such a relation does exist in a form, represented by

$$u(x, y, z, p, q)=0,$$

so that all its integrals possessing any arbitrary element (that is, integrals other than singular or special) satisfy the equation of the second order, it is called an intermediate integral. Thus the investigation of the characteristics of the first order involves the construction of intermediate integrals, if any such exist.

A method has already been given (in Chapter XVI) for the construction of intermediate integrals, if any, of a propounded equation of the second order: and that method is effective for any such equation, for it gives the tests that are necessary and sufficient to secure the existence of the intermediate integral. We need not, therefore, deal further with this question of constructing the intermediate integrals (if any) of a given equation.

Equations having Intermediate Integrals.

282. But the association of intermediate integrals with characteristics of the first order suggests an inquiry into the classes of equations that do possess intermediate integrals. After the classification of equations of the second order which has been adopted, it manifestly is unnecessary to consider any of the

equations in the first of the selected classes in § 281: and, after the full discussion of equations of the form

$$rt - s^2 + Ar + 2Bs + Ct = D,$$
$$Ar + 2Bs + Ct = D,$$

given in Chapter XVI, it is unnecessary to give further consideration to any of the equations in the last of those selected classes. We need only therefore consider equations of those classes which possess a characteristic of the first order.

There are two modes of constructing such equations: and, when regard is paid to the association of intermediate integrals with characteristics of the first order, the two modes are equivalent to one another.

A characteristic of the first order, if it exists, arises by assigning the conditions that the equations

$$f = 0, \quad dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

must be inadequate for the precise determination of r, s, t. The number of conditions, independent of one another, may be two or may be three; if, taken concurrently with the relation

$$dz = p\,dx + q\,dy,$$

they are satisfied by any relation

$$u(x, y, z, p, q) = 0,$$

which is independent of differential elements, that relation is an intermediate integral. Now the equations of the characteristic of $u = 0$, regarded as an equation of the first order, are (§ 94)

$$\frac{dx}{u_p} = \frac{dy}{u_q} = \frac{dz}{pu_p + qu_q} = \frac{dp}{-u_x} = \frac{dq}{-u_y},$$

where

$$u_x = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}, \quad u_y = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z},$$

and u_p, u_q are the derivatives of u with regard to p and q: and this characteristic must be the characteristic of f which is of the first order. The equations

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy,$$

are transformed by the coexistent equations of the characteristic of u into the equations

$$-u_x = ru_p + su_q, \quad -u_y = su_p + tu_q:$$

and the conditions now are that these two equations, together with $f=0$, must be inadequate for the precise determination of r, s, t.

The latter process is the alternative method of proceeding to an intermediate integral, and it is the method adopted in §§ 238—244. In order to be effective, there must be either two relations or three relations, involving the derivatives of u and independent of one another: and only those equations of the second order can have intermediate integrals when the application of the process leads to two relations or to three relations which, independent of one another, can be satisfied by a common value of u.

After these explanations, we shall adopt the latter process so far as the analysis is concerned: we shall therefore make the equations

$$f=0, \quad u_x + ru_p + su_q = 0, \quad u_y + su_p + tu_q = 0,$$

inadequate for the precise determination of r, s, t: and this indeterminateness will require either two independent relations or three independent relations.

First Case.

283. In the first place, suppose that the requirement for indeterminateness in the equations, so far as r, s, t are concerned, leads to a couple of algebraically independent conditions. Let these conditions be resolved for u_x and u_y, so that they have a form

$$\left.\begin{array}{l} u_x + g(x, y, z, p, q, u_p, u_q) = 0 \\ u_y + h(x, y, z, p, q, u_p, u_q) = 0 \end{array}\right\},$$

where the functions g and h are homogeneous, of order unity, in u_p and u_q. We require integrals u of these two simultaneous partial equations of the first order: as the number of variables, which can occur in u, is five, the number of independent integrals can be three, two, one, or none.

We proceed by the Jacobian method. It is convenient to change the notation: we take

$$x, y, z, p, q = x_1, x_2, x_3, x_4, x_5,$$

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial p}, \frac{\partial u}{\partial q} = p_1, p_2, p_3, p_4, p_5:$$

and then the equations are

$$F_1 = p_1 + x_4 p_3 + g(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5) = 0,$$
$$F_2 = p_2 + x_5 p_3 + h(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5) = 0.$$

The Poisson-Jacobi condition of coexistence, which is $(F_1, F_2) = 0$, must be satisfied: expressed in full, it is

$$\frac{\partial h}{\partial x_1} - \frac{\partial g}{\partial x_2} + x_4 \frac{\partial h}{\partial x_3} - x_5 \frac{\partial g}{\partial x_3}$$
$$+ \frac{\partial g}{\partial p_4}\frac{\partial h}{\partial x_4} - \left(p_3 + \frac{\partial g}{\partial x_4}\right)\frac{\partial h}{\partial p_4} + \frac{\partial g}{\partial p_5}\left(p_3 + \frac{\partial h}{\partial x_5}\right) - \frac{\partial h}{\partial p_5}\frac{\partial g}{\partial x_5} = 0.$$

Now this equation cannot be satisfied in virtue of $F_1 = 0$ or $F_2 = 0$, for it contains neither p_1 nor p_2: hence either it is an identity on account of the forms of g and h, or it is a new equation.

When it is an identity, the equations $F_1 = 0$ and $F_2 = 0$ are a complete Jacobian system; and then they possess three common integrals independent of one another. In that event, noting that neither g nor h contains p_3, we have

$$\frac{\partial g}{\partial p_5} = \frac{\partial h}{\partial p_4},$$

$$\frac{\partial h}{\partial x_1} - \frac{\partial g}{\partial x_2} + x_4 \frac{\partial h}{\partial x_3} - x_5 \frac{\partial g}{\partial x_3} + J\left(\frac{g,\ h}{p_4,\ x_4}\right) + J\left(\frac{g,\ h}{p_5,\ x_5}\right) = 0.$$

The former of these shews that there is some function F, such that

$$g = \frac{\partial F}{\partial p_4},\quad h = \frac{\partial F}{\partial p_5};$$

and F consists of two parts, one of them homogeneous of the second order in p_4 and p_5, the other of them not involving p_4 or p_5. Writing

$$p_5 = m p_4,$$

we can take

$$F = p_4^2 \psi(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ m) + G$$
$$= p_4^2 \psi + G,$$

where G does not involve p_4 or m; and then

$$g(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5) = 2p_4\psi - p_5 \frac{\partial \psi}{\partial m},$$

$$h(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5) = p_4 \frac{\partial \psi}{\partial m}.$$

The equations of the characteristic of u are

$$\frac{dx}{p_4} = \frac{dy}{p_5} = \frac{dp}{-p_1 - x_4 p_3} = \frac{dq}{-p_2 - x_5 p_3}$$
$$= \frac{dp}{2p_4\psi - p_5 \dfrac{\partial\psi}{\partial m}} = \frac{dq}{p_4 \dfrac{\partial\psi}{\partial m}},$$

so that

$$dy = m\,dx,$$
$$dp = 2\psi\,dx - \frac{\partial\psi}{\partial m}\,dy,$$
$$dq = \frac{\partial\psi}{\partial m}\,dx.$$

In order to obtain the differential equation, we have

$$dp = r\,dx + s\,dy = (r + sm)\,dx,$$
$$dq = s\,dx + t\,dy = (s + tm)\,dx,$$

and therefore

$$r + sm = 2\psi - m\frac{\partial\psi}{\partial m},$$
$$s + tm = \frac{\partial\psi}{\partial m}.$$

When we eliminate m, we have the required equation: or, what is the same thing, *the required equation is given by the elimination of m between the equations*

$$\left.\begin{aligned} r + 2sm + tm^2 &= 2\psi \\ s + tm &= \frac{\partial\psi}{\partial m} \end{aligned}\right\}.$$

(These equations, on the geometrical illustration of § 280, represent a developable surface: the result may be compared with the results before obtained). The function ψ does not involve $r, s,$ or t: and it has to satisfy a condition represented by the second relation between g and h. When the forms obtained for g and h are substituted in that relation, it takes the form*

$$2\frac{\partial^2\psi}{\partial m^2}\left(m\frac{\partial\psi}{\partial x_4} - \frac{\partial\psi}{\partial x_5}\right) + \frac{\partial^2\psi}{\partial m\,\partial x_5}\frac{\partial\psi}{\partial m} + \frac{\partial^2\psi}{\partial m\,\partial x_4}\left(2\psi - m\frac{\partial\psi}{\partial m}\right)$$
$$+ (x_4 + mx_5)\frac{\partial^2\psi}{\partial m\,\partial x_3} + m\frac{\partial^2\psi}{\partial m\,\partial x_2} + \frac{\partial^2\psi}{\partial m\,\partial x_1} - 2\frac{\partial\psi}{\partial m}\frac{\partial\psi}{\partial x_4} - 2x_5\frac{\partial\psi}{\partial x_3} - 2\frac{\partial\psi}{\partial x_2} = 0,$$

* The equation differs from Goursat's form (*Acta Math.*, t. XIX, p. 322) by the omission of one term.

or, on restoring the old variables, it is

$$2\frac{\partial^2\psi}{\partial m^2}\left(m\frac{\partial\psi}{\partial p}-\frac{\partial\psi}{\partial q}\right)+\frac{\partial^2\psi}{\partial m\partial q}\frac{\partial\psi}{\partial m}+\frac{\partial^2\psi}{\partial m\partial p}\left(2\psi-m\frac{\partial\psi}{\partial m}\right)$$

$$+(p+mq)\frac{\partial^2\psi}{\partial m\partial z}+m\frac{\partial^2\psi}{\partial m\partial y}+\frac{\partial^2\psi}{\partial m\partial x}-2\frac{\partial\psi}{\partial m}\frac{\partial\psi}{\partial p}-2q\frac{\partial\psi}{\partial z}-2\frac{\partial\psi}{\partial y}=0.$$

Any integral of this equation gives a possible form for ψ, and so determines a differential equation of the second order.

In this case, the equations $F_1=0$ and $F_2=0$ have three common integrals functionally independent of one another: let them be θ_1, θ_2, θ_3. We resolve the equations

$$F_1=0,\quad F_2=0,\quad \theta_1=a',\quad \theta_2=b',\quad \theta_3=c',$$

for p_1, p_2, p_3, p_4, p_5: we substitute in

$$du=\sum_{n=1}^{5} p_n dx_n,$$

integrate, and divide out by a homogeneous constant: and we then have a relation

$$u(x,\ y,\ z,\ p,\ q,\ a,\ b,\ c)=0,$$

which is an intermediate integral of the constructed differential equation of the second order. One constant, say c, is additive.

284. The primitive of the differential equation of the second order is obtained simply, on the basis of a general proposition due to Goursat. Let

$$u=0$$

denote the intermediate integral; and let a, b denote the two constants in u which are not merely additive: then Goursat's theorem is that, *if p and q be eliminated between the equations*

$$u=0,\quad \frac{\partial u}{\partial a}=\alpha,\quad \frac{\partial u}{\partial b}=\beta,$$

where α and β are arbitrary constants, the eliminant is a primitive of the equation of the second order. The proof is as follows. We have

$$F_1=p_1+x_4p_3+g(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5)=0,$$
$$F_2=p_2+x_5p_3+h(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ p_4,\ p_5)=0;$$

so that

$$\frac{\partial p_1}{\partial a} + x_4 \frac{\partial p_3}{\partial a} + \frac{\partial g}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial g}{\partial p_5}\frac{\partial p_5}{\partial a} = 0,$$

$$\frac{\partial p_2}{\partial a} + x_5 \frac{\partial p_3}{\partial a} + \frac{\partial h}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial h}{\partial p_5}\frac{\partial p_5}{\partial a} = 0.$$

Now let the Poisson-Jacobi combinant of u and $\frac{\partial u}{\partial a}$ be constructed: thus

$$\begin{aligned}
\left[u,\ \frac{\partial u}{\partial a}\right] &= (p_1 + x_4 p_3)\frac{\partial p_4}{\partial a} - p_4\left(\frac{\partial p_1}{\partial a} + x_4\frac{\partial p_3}{\partial a}\right) \\
&\qquad + (p_2 + x_5 p_3)\frac{\partial p_5}{\partial a} - p_5\left(\frac{\partial p_2}{\partial a} + x_5\frac{\partial p_3}{\partial a}\right) \\
&= -g\frac{\partial p_4}{\partial a} + p_4\left(\frac{\partial g}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial g}{\partial p_5}\frac{\partial p_5}{\partial a}\right) \\
&\qquad - h\frac{\partial p_5}{\partial a} + p_5\left(\frac{\partial h}{\partial p_4}\frac{\partial p_4}{\partial a} + \frac{\partial h}{\partial p_5}\frac{\partial p_5}{\partial a}\right) \\
&= \frac{\partial p_4}{\partial a}\left(-g + p_4\frac{\partial g}{\partial p_4} + p_5\frac{\partial h}{\partial p_4}\right) + \frac{\partial p_5}{\partial a}\left(-h + p_4\frac{\partial g}{\partial p_5} + p_5\frac{\partial h}{\partial p_5}\right) \\
&= \frac{\partial p_4}{\partial a}\left(-g + p_4\frac{\partial g}{\partial p_4} + p_5\frac{\partial g}{\partial p_5}\right) + \frac{\partial p_5}{\partial a}\left(-h + p_4\frac{\partial h}{\partial p_4} + p_5\frac{\partial h}{\partial p_5}\right),
\end{aligned}$$

because of the relation

$$\frac{\partial g}{\partial p_5} = \frac{\partial h}{\partial p_4}.$$

Also g and h are homogeneous of the first order in p_4 and p_5, so that

$$g = p_4\frac{\partial g}{\partial p_4} + p_5\frac{\partial g}{\partial p_5}, \quad h = p_4\frac{\partial h}{\partial p_4} + p_5\frac{\partial h}{\partial p_5};$$

consequently,

$$\left[u,\ \frac{\partial u}{\partial a}\right] = 0.$$

Similarly, we can prove that

$$\left[u,\ \frac{\partial u}{\partial b}\right] = 0,$$

$$\left[\frac{\partial u}{\partial a},\ \frac{\partial u}{\partial b}\right] = 0:$$

hence the equations

$$u = 0, \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta,$$

coexist. The elimination of p and q between them leads to a relation between $x, y, z, a, b, c, \alpha, \beta$, which is consistent with all of them: it is a complete primitive.

Goursat also shews that the general primitive of the equation of the second order can be deduced from a knowledge of the intermediate integral. Denoting the two non-additive constants in u by a and b as before, consider the equations

$$u=0, \quad b=\phi(a), \quad \frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\phi'(a)=0.$$

Resolving the last two equations for a and b, and substituting the deduced values in $u, \frac{\partial u}{\partial a}, \frac{\partial u}{\partial b}$, let us denote the results by u_1, u_2, u_3 respectively. Then

$$\begin{aligned}\frac{\partial u_1}{\partial x_n} &= \frac{\partial u}{\partial x_n}+\left\{\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b}\phi'(a)\right\}\frac{\partial a}{\partial x_n}\\ &= \frac{\partial u}{\partial x_n},\\ \frac{\partial u_2}{\partial x_n} &= \frac{\partial^2 u}{\partial x_n \partial a}+\left\{\frac{\partial^2 u}{\partial a^2}+\frac{\partial^2 u}{\partial a \partial b}\phi'(a)\right\}\frac{\partial a}{\partial x_n},\end{aligned}$$

for $n=1, 2, 3, 4, 5$; also

$$0=\frac{\partial^2 u}{\partial x_n \partial a}+\frac{\partial^2 u}{\partial x_n \partial b}\phi'(a)+\left\{\frac{\partial^2 u}{\partial a^2}+2\frac{\partial^2 u}{\partial a \partial b}\phi'(a)+\frac{\partial^2 u}{\partial b^2}\phi'^2(a)+\frac{\partial u}{\partial b}\phi''(a)\right\}\frac{\partial a}{\partial x_n}.$$

When the value of $\frac{\partial a}{\partial x_n}$ given by the last equation is substituted in $\frac{\partial u_2}{\partial x_n}$, we have

$$\frac{\partial u_2}{\partial x_n}=\lambda\frac{\partial^2 u}{\partial x_n \partial a}+\mu\frac{\partial^2 u}{\partial x_n \partial b},$$

where λ and μ do not change for the different values of n. Hence, as $\frac{\partial u_1}{\partial x_n}=\frac{\partial u}{\partial x_n}$, we have

$$\begin{aligned}[u_1, u_2] &= \lambda\left[u_1, \frac{\partial u}{\partial a}\right]+\mu\left[u_1, \frac{\partial u}{\partial b}\right]\\ &= \lambda\left[u, \frac{\partial u}{\partial a}\right]+\mu\left[u, \frac{\partial u}{\partial b}\right]\\ &= 0;\end{aligned}$$

and, similarly,

$$[u_1, u_3]=0,$$
$$[u_2, u_3]=0.$$

Thus we obtain a primitive by the elimination of p and q between the three equations

$$u_1 = 0, \quad u_2 = \alpha, \quad u_3 = \beta,$$

or (what is the same thing) by the elimination of p, q, a, b between the five equations

$$u = 0, \quad \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b}\phi'(a) = 0, \quad b = \phi(a), \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta.$$

Eliminating p and q first, we have relations

$$\theta(x, y, z, a, b, \alpha, \beta) = 0,$$
$$b = \phi(a), \quad \alpha + \beta\phi'(a) = 0,$$

from which to eliminate a and b. Replacing α by a new constant α', such that $\alpha' + \beta\phi'(a) = 0$, we have a primitive given by

$$\theta\{x, y, z, a, \phi(a), -\beta\phi'(a), \beta\} = 0;$$

and when we replace β by $\chi(a)$, where χ is an arbitrary function, the general primitive of the constructed equation of the second order is given by

$$\left.\begin{aligned} \theta\{x, y, z, a, \phi(a), -\chi(a)\phi'(a), \chi(a)\} = 0 \\ \frac{\partial\theta}{\partial a} + \frac{\partial\theta}{\partial\phi}\phi'(a) + \frac{\partial\theta}{\partial\phi'}\phi''(a) + \frac{\partial\theta}{\partial\chi}\chi'(a) = 0 \end{aligned}\right\},$$

involving two arbitrary functions.

285. The equation for ψ is of the second order, being linear in the derivatives of that order; but the construction of the quantity ψ is made difficult, on account of the number of independent variables with respect to which the derivatives are taken. In the absence of the knowledge of the most general integral, we cannot construct the aggregate of equations of the type under consideration: but, by obtaining special values of ψ satisfying the equation, we can construct special equations or even special classes of equations. One or two examples will suffice.

Ex. 1. It is obvious that the equation for ψ is satisfied by taking ψ to be a function of m only: ψ can be an arbitrary function of m. The differential equation then occurs as the eliminant of

$$\left.\begin{aligned} r + 2sm + tm^2 = 2\psi \\ s + tm = \frac{\partial\psi}{\partial m} \end{aligned}\right\},$$

or, what is the same thing, it is obtained by equating to zero the m-discriminant of

$$r+2sm+tm^2-2\psi.$$

The equations, satisfied by the quantity u connected with an intermediate integral, are

$$F_1=p_1+x_4p_3+2p_4\psi\left(\frac{p_5}{p_4}\right)-p_5\psi'\left(\frac{p_5}{p_4}\right)=0,$$

$$F_2=p_2+x_5p_3+\ p_4\psi'\left(\frac{p_5}{p_4}\right)=0;$$

and these two equations are a complete system. Obviously

$$(F_1,\ p_r)=0,\quad (F_2,\ p_r)=0,$$

for $r=1, 2, 3$; so that p_1, p_2, p_3 can be taken as the three integrals that are independent of one another. We then have

$$p_1=a,\quad p_2=b,\quad p_3=c,\quad F_1=0,\quad F_2=0;$$

thus

$$p_4(2\psi-m\psi')=-a-cx_4,$$
$$p_4\psi'=-b-cx_5,$$

and so m is determined by the relation

$$2\frac{\psi}{\psi'}-m=\frac{a+cx_4}{b+cx_5}=\mu,$$

say. Now

$$du=\sum_{n=1}^{5}p_ndx_n,$$

and therefore

$$d(ax_1+bx_2+cx_3-u)=\frac{b+cx_5}{\psi'}(dx_4+mdx_5).$$

Changing the independent variables on the right-hand side to m and x_5, we have

$$a+cx_4=(b+cx_5)\mu,$$

and therefore

$$cdx_4=c\mu dx_5+(b+cx_5)\mu'dm;$$

also

$$\mu+m=2\frac{\psi}{\psi'}.$$

Consequently,

$$d(ax_1+bx_2+cx_3-u)=\frac{b+cx_5}{\psi'}\left\{(\mu+m)dx_5+\frac{b+cx_5}{c}\mu'dm\right\}$$
$$=\frac{1}{c}d\left\{(b+cx_5)^2\frac{\psi}{\psi'^2}\right\},$$

on reduction: and therefore

$$ax_1+bx_2+cx_3-u=\frac{\psi}{c\psi'^2}(b+cx_5)^2+A.$$

Now the intermediate integral is $u=0$; hence, resuming the variables x, y, z, p, q, and making c equal to unity (which involves no loss of generality), we have

$$z+ax+by-(b+q)^2\frac{\psi}{\psi'^2}-k=0,$$

where k is an arbitrary additive constant: the argument m of ψ, which is an arbitrary function, is given by

$$2\frac{\psi}{\psi'}-m=\frac{p+a}{q+b}.$$

The equation thus obtained, account being taken of the value of m, is an intermediate integral of the equation of the second order, represented by

$$\text{Discr.}_m\,(r+2sm+tm^2-2\psi)=0.$$

The equation, which determines m in connection with the intermediate integral, shews that m is a function of p and q: hence, when we construct the equations of the characteristic of

$$z+ax+by-(b+q)^2\frac{\psi}{\psi'^2}-k=0,$$

one of them is

$$\frac{-dp}{a+p}=\frac{-dq}{b+q}.$$

An integral of this equation is

$$\frac{p+a}{q+b}=\gamma,$$

where γ is a constant: hence, for the primitive, m is a constant. Combining this equation with the intermediate integral, we have

$$p+a=\gamma\,(z+ax+by-k)^{\frac{1}{2}}\,\psi'\psi^{-\frac{1}{2}},$$
$$q+b=\ (z+ax+by-k)^{\frac{1}{2}}\,\psi'\psi^{-\frac{1}{2}};$$

substituting these values of p and q in the relation

$$dz=p\,dx+q\,dy,$$

and effecting the quadrature, we have

$$2\,(z+ax+by-k)^{\frac{1}{2}}=(\gamma x+y+\delta)\,\psi'\psi^{-\frac{1}{2}},$$

where the argument m of ψ now is given by

$$2\frac{\psi}{\psi'}-m=\gamma.$$

This equation is a complete primitive, involving five arbitrary constants a, b, k, γ, δ.

To obtain the general primitive, let

$$p+a=v\,(2\psi-m\psi'),$$

where v is a new magnitude: then

$$q+b=v\psi',$$

and

$$u=z+ax+by-k-v^2\psi=0.$$

Now

$$1=\frac{\partial v}{\partial a}(2\psi-m\psi')+v(\psi'-m\psi'')\frac{\partial m}{\partial a},$$

$$0=\frac{\partial v}{\partial a}\psi'+v\psi''\frac{\partial m}{\partial a}:$$

hence

$$\frac{\partial u}{\partial a}=x-2v\frac{\partial v}{\partial a}\psi-v^2\psi'\frac{\partial m}{\partial a}$$

$$=x-v;$$

and

$$\frac{\partial u}{\partial b}=y-vm,$$

similarly obtained. Accordingly, after the general theory, we eliminate m and v between the equations

$$z+ax+by-k-v^2\psi=0,$$

$$x-\ v\ =\alpha,$$

$$y-mv=\beta,$$

leading to a relation

$$\theta(x, y, z, a, b, \alpha, \beta)=0.$$

The general primitive is given by

$$\left.\begin{aligned}\theta\{x, y, z, a, \phi(a), -\phi'(a)\chi(a), \chi(a)\}=0\\ \frac{d\theta}{da}=0\end{aligned}\right\}.$$

Ex. 2. It is natural to enquire whether ψ can have the form

$$\psi=Am^2+2Bm+C,$$

where A, B, C involve only x, y, z, p, q. If it is possible, the differential equation of the second order is the eliminant of

$$r+2sm+tm^2=2Am^2+4Bm+2C,$$

$$s+tm=2Am+2B,$$

that is, it is

$$(r-2C)(t-2A)=(s-2B)^2.$$

On the present hypothesis, there are to be three integrals of the subsidiary system: we, therefore, have the problem of § 241.

The preceding analysis shews that the possibility will be realised, if ψ satisfies the differential equation of the second order. That this may be the case, we find (on actual substitution of the supposed value of ψ) that the relation

$$\Theta_1m^3+\Theta_2m^2+\Theta_3m+\Theta_4=0$$

must be satisfied identically, where Θ_1, Θ_2, Θ_3, Θ_4 do not contain m; and therefore we must have

$$\Theta_1=0,\quad \Theta_2=0,\quad \Theta_3=0,\quad \Theta_4=0.$$

On effecting the calculations, it appears that the equations

$$\Theta_1=0,\quad \Theta_2=0,$$

must be satisfied, both identically: the equation $\Theta_3=0$ becomes

$$\left(\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+2B\frac{\partial}{\partial q}+2C\frac{\partial}{\partial p}\right)A=\left(\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+2A\frac{\partial}{\partial q}+2B\frac{\partial}{\partial p}\right)B,$$

which, on noting that $2C$, $2B$, $2A=R$, S, T, is the old relation

$$\Delta R=\Delta' S;$$

and the equation $\Theta_4=0$ similarly leads to the old relation

$$\Delta S=\Delta' T.$$

The equation, subject to these conditions, has already been fully discussed.

Ex. 3. Obtain the conditions which must be satisfied, in order that ψ may (if possible) have either of the forms

$$\frac{Am+B}{Cm+D},\quad Am^3+3Bm^2+3Cm+D,$$

where A, B, C, D are functions of x, y, z, p, q only.

Ex. 4. Discuss the equations that arise when ψ has either of the following forms, each of which satisfies the equation of the second order:

(i), $\psi=am^2p^n$, where a and n are arbitrary;

(ii), $\psi=am+a^2f(q)+b$, where a and b are arbitrary, and f is an arbitrary function.

286. We now pass to the consideration of the equations

$$F_1=0,\quad F_2=0,$$

when the condition $(F_1, F_2)=0$, which must be satisfied and cannot be satisfied in virtue of either of the preceding equations, is not an identity; consequently, it is a new equation, say

$$F_3=Ap_3+B=0,$$

where

$$A=\frac{\partial g}{\partial p_5}-\frac{\partial h}{\partial p_4},$$

and

$$B=\frac{\partial h}{\partial x_1}-\frac{\partial g}{\partial x_2}+x_4\frac{\partial h}{\partial x_3}-x_5\frac{\partial g}{\partial x_3}+J\left(\frac{g,\ h}{p_4,\ x_4}\right)+J\left(\frac{g,\ h}{p_5,\ x_5}\right).$$

If the number of independent integrals of the original system $F_1=0$, $F_2=0$, is to be two, then $F_1=0$, $F_2=0$, $F_3=0$, must be a complete Jacobian system.

If, in these circumstances, it were possible that A should vanish identically, the equation $F_3=0$ would become

$$B=0.$$

Now B is homogeneous of the first order in p_4 and p_5: hence it can be resolved into a number of equations of the type

$$p_4 + \mu p_5 = 0,$$

where μ is a function of x_1, x_2, x_3, x_4, x_5; and the set of equations would be

$$\begin{aligned} F_1 &= p_1 + x_4 p_3 + p_5 G = 0, \\ F_2 &= p_2 + x_5 p_3 + p_5 H = 0, \\ F_3 &= \qquad\quad p_4 + p_5 \mu = 0, \end{aligned}$$

where G, H, μ are functions of x_1, x_2, x_3, x_4, x_5 only. Also

$$(F_1, F_3) = p_5 \left(\frac{\partial \mu}{\partial x_1} + x_4 \frac{\partial \mu}{\partial x_3} + G\frac{\partial \mu}{\partial x_5}\right) - \left(p_3 + p_5 \frac{\partial G}{\partial x_4}\right) - \mu p_5 \frac{\partial G}{\partial x_5}.$$

The right-hand side must vanish: it cannot vanish in virtue of $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, because p_3 occurs only in the term $-p_3$, while p_1 and p_2 do not occur; and it does not vanish identically. Hence it is a new equation, and therefore the set of three equations is not complete. It therefore follows that, when $(F_1, F_2) = 0$ provides a new equation, the quantity A is not zero.

Two cases occur for consideration when A is not zero, according as B does or does not vanish.

Second Case.

287. When B vanishes, the equation $F_3 = 0$ becomes $p_3 = 0$: when this is used to modify the other equations, the system is

$$\begin{aligned} F_1 &= p_1 + p_4 G\ (x_1, x_2, x_3, x_4, x_5, m) = 0, \\ F_2 &= p_2 + p_4 H (x_1, x_2, x_3, x_4, x_5, m) = 0, \\ F_3 &= p_3 \qquad\qquad\qquad\qquad\qquad\quad = 0, \end{aligned}$$

where

$$p_5 = m p_4.$$

As the system is to be complete, the necessary conditions must be satisfied. From $(F_1, F_3) = 0$, we have

$$\frac{\partial G}{\partial x_3} = 0;$$

from $(F_2, F_3) = 0$, we have

$$\frac{\partial H}{\partial x_3} = 0;$$

and from $(F_1, F_2) = 0$, we have (after an easy reduction)

$$\frac{\partial H}{\partial x_1} - \frac{\partial G}{\partial x_2} + G\frac{\partial H}{\partial x_4} - H\frac{\partial G}{\partial x_4} + m\frac{\partial(G, H)}{\partial(x_4, m)} - \frac{\partial(G, H)}{\partial(x_5, m)} = 0.$$

It follows that both F and G are explicitly free from x_3: and the last equation shews, that either G or H may be arbitrarily assumed (subject to the non-occurrence of x_3), and that the other is determinable as an integral of an equation of the first order, though the explicit form of this integral will be affected by the form adopted for the other quantity. Thus, assuming G assigned, the equation is

$$\frac{\partial H}{\partial x_1} + \left(G - m\frac{\partial G}{\partial m}\right)\frac{\partial H}{\partial x_4} + \frac{\partial G}{\partial m}\frac{\partial H}{\partial x_5} + \left(m\frac{\partial G}{\partial x_4} - \frac{\partial G}{\partial x_5}\right)\frac{\partial H}{\partial m}$$
$$= H\frac{\partial G}{\partial x_4} + \frac{\partial G}{\partial x_2};$$

for the determination of H, we should require integrals of the system

$$dx_1 = \frac{dx_4}{G - m\dfrac{\partial G}{\partial m}} = \frac{dx_5}{\dfrac{\partial G}{\partial m}} = \frac{dm}{m\dfrac{\partial G}{\partial x_4} - \dfrac{\partial G}{\partial x_5}} = \frac{dH}{H\dfrac{\partial G}{\partial x_4} + \dfrac{\partial G}{\partial x_2}}.$$

Ex. 1. A set of cases is provided by taking

$$G = \frac{\partial \phi}{\partial x_1}, \quad H = \frac{\partial \phi}{\partial x_2},$$

where ϕ is any function of x_1, x_2, m. The equations are

$$F_3 = p_3 = 0,$$
$$F_1 = p_1 + p_4\frac{\partial \phi}{\partial x_1} = 0,$$
$$F_2 = p_2 + p_4\frac{\partial \phi}{\partial x_2} = 0.$$

Evidently

$$(F_r, p_4) = 0, \quad (F_r, p_5) = 0,$$

for $r = 1, 2, 3$; hence p_4 and p_5 are two independent integrals of the system. Accordingly, we take

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 0, \quad p_4 = a, \quad p_5 = b,$$

where a and b are constants; and then

$$du = -a\frac{\partial \phi\left(x_1, x_2, \dfrac{b}{a}\right)}{\partial x_1}dx_1 - a\frac{\partial \phi\left(x_1, x_2, \dfrac{b}{a}\right)}{\partial x_2}dx_2 + a\,dx_4 + b\,dx_5,$$

so that

$$u = -a\phi\left(x_1, x_2, \frac{b}{a}\right) + ax_4 + bx_5 + c.$$

Now $u=0$ is the intermediate integral: hence

$$ax_4+bx_5+c=a\phi\left(x_1,\, x_2,\, \frac{b}{a}\right),$$

or, in the other notation,

$$p+aq+\beta=\phi(x,\, y,\, a),$$

where a and β are arbitrary constants.

The equations of the characteristic are

$$\frac{dx}{1}=\frac{dy}{a}=\frac{dp}{\dfrac{\partial\phi}{\partial x}}=\frac{dq}{\dfrac{\partial\phi}{\partial y}},$$

and

$$dp=r\,dx+s\,dy, \quad dq=s\,dx+t\,dy:$$

consequently,

$$\left.\begin{aligned} r+sa&=\frac{\partial\phi}{\partial x}\\ s+ta&=\frac{\partial\phi}{\partial y}\end{aligned}\right\}.$$

The required differential equation of the second order is given by eliminating a between these equations.

A general intermediate integral is given by the elimination of a between the equations

$$\left.\begin{aligned} p+aq+\psi(a)&=\phi(x,\, y,\, a)\\ q+\psi'(a)&=\frac{\partial\phi}{\partial a}\end{aligned}\right\},$$

ψ being an arbitrary function. To proceed to the primitive, we take p and q in the forms

$$p=\phi-a\frac{\partial\phi}{\partial a}+a\psi'-\psi,$$

$$q=\frac{\partial\phi}{\partial a}-\psi';$$

and then the relation

$$p\,dx+q\,dy$$

must be an exact differential. That this may be the case, a must be such a function of x and y that

$$\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x},$$

that is,

$$\frac{\partial^2\phi}{\partial x\,\partial a}+\left(\frac{\partial^2\phi}{\partial a^2}-\psi''\right)\frac{\partial a}{\partial x}=\frac{\partial\phi}{\partial y}-a\frac{\partial^2\phi}{\partial y\,\partial a}+a\left(\psi''-\frac{\partial^2\phi}{\partial a^2}\right)\frac{\partial a}{\partial y},$$

or, what is the same thing,

$$\frac{\partial a}{\partial x}+a\frac{\partial a}{\partial y}=\frac{\dfrac{\partial\phi}{\partial y}-a\dfrac{\partial^2\phi}{\partial y\,\partial a}-\dfrac{\partial^2\phi}{\partial x\,\partial a}}{\dfrac{\partial^2\phi}{\partial a^2}-\psi''}=U,$$

say. When the form of ϕ is given, we have an equation for the determination of a: the most general value can be obtained by Lagrange's method of integration. We construct the equations

$$dx=\frac{dy}{a}=\frac{da}{U};$$

if two integrals are

$$\theta(x,\ y,\ a)=\text{constant},$$
$$\chi(x,\ y,\ a)=\text{constant},$$

the most general value of a is given by the elimination of β between the equations

$$\theta(x,\ y,\ a)=\beta,\quad \chi(x,\ y,\ a)=f(\beta),$$

where f is arbitrary. We combine these with

$$p=\phi-a\frac{\partial\phi}{\partial a}+a\psi'-\psi,\quad q=\frac{\partial\phi}{\partial a}-\psi';$$

we substitute in

$$\begin{aligned} dz&=p\,dx+q\,dy\\ &=\left(p\frac{\partial x}{\partial a}+q\frac{\partial y}{\partial a}\right)da+\left(p\frac{\partial x}{\partial\beta}+q\frac{\partial y}{\partial\beta}\right)d\beta, \end{aligned}$$

and effect the quadrature: the result is the general primitive of the differential equation, for it involves two arbitrary functions f and ψ.

Ex. 2. A simple example arises by taking

$$\phi(x,\ y,\ a)=xya;$$

and then the general intermediate integral is given by

$$\left.\begin{aligned} p+aq+\psi(a)&=xya\\ q+\psi'(a)&=xy \end{aligned}\right\}.$$

Hence

$$p=a\psi'-\psi,\quad q=xy-\psi';$$

and now the argument a of ψ must be such as to make

$$p\,dx+q\,dy$$

an exact differential: hence

$$y-\psi''\frac{\partial a}{\partial x}=a\psi''\frac{\partial a}{\partial y}.$$

The most general value of a satisfying this relation is found by Lagrange's rule: it is easily proved to be the result of eliminating β between the two equations

$$y^2=2a\psi'-2\psi+2\beta,$$
$$x=f'(\beta)+\int\frac{\psi''da}{(2a\psi'-2\psi+2\beta)^{\frac{1}{2}}},$$

where β is to be regarded as a constant during the quadrature that occurs in x, and f is an arbitrary function.

Again, we have

$$\begin{aligned} d(px-z)&=x\,dp-q\,dy\\ &=x(dp-y\,dy)+\psi'dy\\ &=-x\,d\beta+\psi'dy: \end{aligned}$$

substituting for x and for dy, and rearranging, we have

$$d\{px - z + f(\beta)\} = A\,da + B\,d\beta,$$

where

$$A = \frac{a\psi'\psi''}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

$$B = \frac{\psi'}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}} - \int \frac{\psi''da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

and, in B, the quantity β during the quadrature is constant, becoming parametric after the quadrature has been effected. We at once verify that

$$\frac{\partial A}{\partial \beta} = \frac{\partial B}{\partial a},$$

so that the right-hand side of the differential relation is a perfect differential: the result of quadrature can be expressed in the form

$$\int \frac{a\psi'\psi''da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},$$

β being kept constant during the integration expressed in the result.

Gathering together the various equations, we have

$$\begin{aligned}
p &= a\psi' - \psi,\\
q &= xy - \psi',\\
y^2 &= 2a\psi' - 2\psi + 2\beta,\\
x &= f'(\beta) + \int \frac{\psi''\,da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},\\
z - px &= f(\beta) - \int \frac{a\psi'\psi''\,da}{(2a\psi' - 2\psi + 2\beta)^{\frac{1}{2}}},
\end{aligned}$$

where the two quadratures (in which β is to be kept unvarying) can be regarded as explicitly possible. The differential equation of the second order, of which this aggregate of equations constitutes the general primitive, is given by eliminating a between the equations

$$\left.\begin{aligned} r + sa &= ya \\ s + ta &= xa \end{aligned}\right\},$$

that is, it is

$$rt - s^2 - rx + sy = 0.$$

Its general intermediate integral is given by

$$p + aq + \psi(a) = xya, \quad q + \psi'(a) = xy,$$

that is,

$$p = f(q - xy).$$

Ex. 3. Obtain a general intermediate integral of the equation

$$st + x(rt - s^2)^2 = 0,$$

in the form

$$\left.\begin{aligned} p &= 2ax^{\frac{1}{2}} - a^2q + 2\psi(a) \\ 0 &= x^{\frac{1}{2}} - aq + \psi'(a) \end{aligned}\right\};$$

and deduce the primitive. (Ampère; Goursat.)

Ex. 4. Integrate the equation

$$(rt-s^2)(sx-yt)+(sy-rx)^2=0.$$

Ex. 5. Another class of equations having intermediate integrals is provided by assuming

$$G=\frac{\partial F(x_1,\ x_2,\ m)}{\partial x_1},$$

$$H=\frac{\partial F(x_1,\ x_2,\ m)}{\partial x_2}+H';$$

when we denote m by x_6, and write

$$\frac{\partial H'}{\partial x_n}=p_n',$$

the equations to determine H' are

$$p_1'+\left(G-x_6\frac{\partial G}{\partial x_6}\right)p_4'+\frac{\partial G}{\partial x_6}p_5'=0,$$

$$p_3'=0.$$

The two equations are a complete Jacobian system: there are six independent variables in the construction of H', viz., x_1, x_2, x_3, x_4, x_5, x_6: and therefore there are four algebraically independent integrals common to the two equations. These are easily found to be

$$x_2,\quad x_4-F+x_6\frac{\partial F}{\partial x_6},\quad x_5-\frac{\partial F}{\partial x_6},\quad x_6;$$

and therefore we can take

$$\begin{aligned}H'&=\phi\left(x_2,\ x_4-F+x_6\frac{\partial F}{\partial x_6},\ x_5-\frac{\partial F}{\partial x_6},\ x_6\right)\\&=\phi\left(y,\ p-F+m\frac{\partial F}{\partial m},\ q-\frac{\partial F}{\partial m},\ m\right),\end{aligned}$$

where ϕ is any arbitrary function of its four possible arguments.

With this value of H', the conditions for the completeness of the system

$$\left.\begin{aligned}F_1&=p_1+p_4G=0\\F_2&=p_2+p_4H=0\\F_3&=p_3\qquad\ \ =0\end{aligned}\right\}$$

are satisfied: it therefore possesses two common integrals, which can be deduced by the usual processes, and the forms of which will be affected by the form of H'.

Let $u=0$ be the most general intermediate integral; its characteristic is

$$\frac{dx}{-p_4}=\frac{dy}{-p_5}=\frac{dp}{p_1}=\frac{dq}{p_2},$$

so that

$$dy=m\,dx,$$

$$dp=G\,dx,$$

$$dq=H\,dx:$$

hence

$$\left.\begin{aligned} r+sm=G&=\frac{\partial F(x,\,y,\,m)}{\partial x} \\ s+tm=H&=\frac{\partial F(x,\,y,\,m)}{\partial y}+H' \end{aligned}\right\}.$$

The elimination of m between these two equations leads to the required equation of the second order.

Third Case.

288. Passing to the alternative case, suppose that B does not vanish; then, as B is homogeneous of the first order in p_4 and p_5, it can be expressed in the form

$$p_4 h\,(x_1,\; x_2,\; x_3,\; x_4,\; x_5,\; m);$$

and so the system of equations is transformable to

$$\left.\begin{aligned} F_1&=p_1+p_4 f\,(x_1,\; x_2,\; x_3,\; x_4,\; x_5,\; m)=0 \\ F_2&=p_2+p_4 g\,(x_1,\; x_2,\; x_3,\; x_4,\; x_5,\; m)=0 \\ F_3&=p_3+p_4 h\,(x_1,\; x_2,\; x_3,\; x_4,\; x_5,\; m)=0 \end{aligned}\right\},$$

where, as before,

$$p_5=mp_4.$$

The system is to be complete, and the necessary conditions must be satisfied. From $(F_1,\,F_2)=0$, we have

$$\frac{\partial g}{\partial x_1}-\frac{\partial f}{\partial x_2}+f\frac{\partial g}{\partial x_4}-g\frac{\partial f}{\partial x_4}-m\frac{\partial\,(f,g)}{\partial\,(m,\,x_4)}+\frac{\partial\,(f,g)}{\partial\,(m,\,x_5)}=0:$$

from $(F_2,\,F_3)=0$, we have

$$\frac{\partial h}{\partial x_2}-\frac{\partial g}{\partial x_3}+g\frac{\partial h}{\partial x_4}-h\frac{\partial g}{\partial x_4}-m\frac{\partial\,(g,\,h)}{\partial\,(m,\,x_4)}+\frac{\partial\,(g,\,h)}{\partial\,(m,\,x_5)}=0:$$

and from $(F_3,\,F_1)=0$, we have

$$\frac{\partial f}{\partial x_3}-\frac{\partial h}{\partial x_1}+h\frac{\partial f}{\partial x_4}-f\frac{\partial h}{\partial x_4}-m\frac{\partial\,(h,f)}{\partial\,(m,\,x_4)}+\frac{\partial\,(h,f)}{\partial\,(m,\,x_5)}=0:$$

and each of these three conditions must be satisfied identically. They may be regarded as three simultaneous equations of the first order, for the determination of f, g, h.

Ex. 1. One simple set of cases is easily obtained. Let $G(x_1, x_2, x_3, m)$ denote any function of x_1, x_2, x_3, m: then the three conditions are satisfied by taking

$$f=\frac{\partial G(x_1, x_2, x_3, m)}{\partial x_1},$$

$$g=\frac{\partial G(x_1, x_2, x_3, m)}{\partial x_2},$$

$$h=\frac{\partial G(x_1, x_2, x_3, m)}{\partial x_3}.$$

Assuming these to be the values of f, g, h, we find

$$(F_r, p_4)=0, \quad (F_r, p_5)=0,$$

for $r=1, 2, 3$: so that p_4 and p_5 are independent integrals of the system. We therefore take

$$F_1=0, \quad F_2=0, \quad F_3=0, \quad p_4=a, \quad p_5=b;$$

and then, as

$$du=\sum_{n=1}^{5} p_n dx_n,$$

we have

$$u=-aF\left(x_1, x_2, x_3, \frac{b}{a}\right)+ap_4+bp_5+c.$$

The intermediate integral is $u=0$; hence, returning to the earlier notation, it is

$$p=F(x, y, z, a)-aq+\beta.$$

The characteristic of this equation of the first order is given by

$$\frac{dx}{1}=\frac{dy}{a}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}},$$

that is, by

$$dy=adx,$$

$$rdx+sdy=\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)dx,$$

$$sdx+tdy=\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)dx;$$

and therefore the differential equation of the second order is obtained by eliminating a between the two equations

$$\left.\begin{aligned} r+sa&=\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\\ s+ta&=\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\end{aligned}\right\},$$

where F denotes $F(x, y, z, m)$.

The general intermediate integral is given by

$$\left.\begin{aligned} p&=F(x, y, z, a)-aq+\theta(a)\\ 0&=\frac{\partial F}{\partial a}-q+\theta'(a)\end{aligned}\right\},$$

where $\theta(a)$ is an arbitrary function. From this equation, we can proceed to the primitive by a process similar to that in § 287, Ex. 1.

Ex. 2. Obtain an intermediate integral of the equation

$$(rt-s^2)^2 z-(rt-s^2)(q^2r-2pqs+p^2t)+q^2rs-pq(rt+s^2)+p^2st=0,$$

in the form

$$abz+ap+bq=1;$$

and find the general primitive.

Ex. 3. Obtain an intermediate integral of the equation

$$q^2rs+pq(rt+s^2)+p^2st+pq=qxr+(px+yq)s+pyt,$$

in the form

$$pq-ax-by+ab=0;$$

and find the general primitive.

Ex. 4. Integrate the equation

$$(rt-s^2)^2=(qr-ps)^2+(qs-pt)^2.$$

Remaining Cases.

289. The remaining case, in which $F_1=0$ and $F_2=0$ can be a complete system only if two other equations are associated with them, is comparatively unimportant. There is only a single integral of that complete system: and so the intermediate integral, which is being sought, contains only a single constant. When resolved with respect to this constant, the intermediate integral has the form

$$v=v(x, y, z, p, q)=a.$$

Let this equation of the first order be integrated so as to obtain its complete primitive, which will contain two additional constants and can have a form

$$f(x, y, z, a, b, c)=0.$$

It may be possible to generalise this by means of Imschenetsky's process and to construct a general integral involving two arbitrary functions.

The significance of the integral and of the various equations is limited by the fact that, in reality, the intermediate integral leads to a couple of equations of the second order, viz.

$$v_x+rv_p+sv_q=0, \quad v_y+sv_p+tv_q=0,$$

neither of which contains any arbitrary element: the integrals cannot be regarded as belonging to either equation alone.

290. We now have to deal with the alternative, left over from § 282, in which the requirement, that the equations

$$f=0, \quad u_x + ru_p + su_q = 0, \quad u_y + su_p + tu_q = 0,$$

shall be indeterminate so far as concerns r, s, t, leads to three algebraically independent relations. These relations are homogeneous in u_x, u_y, u_p, u_q; when resolved, they yield a set (or a number of sets) of relations of the type

$$\left.\begin{aligned} u_x + u_q\rho &= 0 \\ u_y + u_q\sigma &= 0 \\ u_p + u_q\tau &= 0 \end{aligned}\right\},$$

where ρ, σ, τ are functions of x, y, z, p, q alone.

It may be at once noted that $f=0$ is not the only equation of the second order which arises in connection with the intermediate integral: in fact,

$$\left.\begin{aligned} \rho + r\tau - s &= 0 \\ \sigma + s\tau - t &= 0 \end{aligned}\right\}$$

are a couple of equations, with which $f=0$ is not inconsistent and which actually are of the second order. Hence the integral, when it exists, cannot be regarded as belonging to either equation alone; and this alternative case accordingly does not demand further consideration.

Ex. 1. Consider the equation

$$rt - b^2x^2t^2 + py - qs = 0,$$

propounded by Ampère at the end of the first of the memoirs quoted in Chapter XVII.

The necessary conditions for the possession of an intermediate integral are three: viz.

$$\frac{u_x u_y}{u_p u_q} + py - b^2x^2\frac{u_y^2}{u_q^2} = 0,$$

$$\frac{u_y}{u_p} + \frac{u_x}{u_q} - q - 2\frac{b^2x^2u_pu_y}{u_q^2} = 0,$$

$$1 - b^2x^2\frac{u_p^2}{u_q^2} = 0.$$

These resolve themselves into the two sets

$$u_x = \left(q - \frac{py}{q}\right)u_q,$$

$$u_y = -\epsilon\frac{py}{bxq}u_q,$$

$$u_p = \epsilon\frac{1}{bx}u_q:$$

where ϵ is ± 1; and it is easy to prove that, in addition to the given equation, these require

$$py - qs = \epsilon bxqt.$$

The original equation can only have integrals in finite form free from partial quadratures when they satisfy the last equation also (see § 183, Ex. 3).

The discussion of these simultaneous equations of the second order by the method of Vályi (§ 263) is left as an exercise.

Ex. 2. The equations

$$r + f(x, y, z, p, q, s) = 0, \quad t + g(x, y, z, p, q, s) = 0,$$

are a system in involution (§ 263): shew that they possess a common characteristic, the equations of which are

$$dx = \frac{dy}{\frac{\partial f}{\partial s}} = \frac{dz}{p + q\frac{\partial f}{\partial s}} = \frac{dp}{-f + s\frac{\partial f}{\partial s}} = \frac{dq}{s - g\frac{\partial f}{\partial s}} = \frac{ds}{-\frac{df}{dy}}.$$

Indicate a mode of deducing an integral surface from the integrated equations of the characteristic. (Lie; Goursat.)

CHAPTER XXI.

General Transformation of Equations of the Second Order.

As indicated at the beginning of the present chapter, the general theory of the transformation of partial equations of order higher than the first has been only slightly developed. Such developments as have been effected are concerned with equations of the second order in two independent variables: and they are chiefly associated with geometrical properties or interpretations. The discussion in the present chapter relates to matters that had their origin in some investigations by Bäcklund, upon simultaneous partial equations of the first order in two dependent variables*, and upon the theory of the transformation of surfaces in ordinary space†. For the contents of the chapter, reference may be made to these memoirs by Bäcklund, to Goursat's discussions ‡ of the matter, and to a thesis § by Clairin where other references are given.

Processes of Transformation for Equations of the Second Order.

291. Before concluding the discussion of equations of the second order in one dependent variable and two independent variables, it is natural to consider those processes of transformation which can be used in connection with such equations. We have seen how Lie's theory of contact-transformations not merely elucidates the consideration of equations of the first order but actually provides a method of constructing various classes of

* *Math. Ann.*, t. XVII (1880), pp. 285—328.

† *Math. Ann.*, t. XIX (1882), pp. 387—422.

‡ In Chapter IX of his "Lecons sur l'intégration des équations aux dérivées partielles du second ordre," and in a memoir, *Annales de Toulouse*, 2e Sér., t. IV (1902), pp. 299—340.

§ *Ann. de l'École Norm. Sup.*, 3e Sér., t. XIX (1902), Supplément.

integrals of such an equation. Again, Laplace's process of solving a linear equation of the second order (as expounded in Chapter XIII) is really a process of transformation: and other cases have arisen in which transformations, perhaps depending upon the form of particular equations or particular classes of equations, have been used. Such instances raise obvious questions as to what is the most general type of these transformations and as to how far they are, or can be made, effective in throwing fresh light either upon the construction of the integrals or upon the general theory.

It may at once be stated that such results, as have been obtained, do not constitute a theory of nearly the same important range for equations of the second order as does Lie's theory of contact-transformations for equations of the first order: that, in their present development, they are connected with equations in only two independent variables: and that they apply only to equations of the Monge-Ampère form and, even so, not to the most general equations of that form. A comparatively brief sketch of the investigations already achieved will suffice to give an indication of their range and their significance.

Perhaps the simplest mode of initiating the discussion is to propound the question as one concerned with the transformation of surfaces in ordinary space. In accordance with the now familiar notions of Lie's theory, let x, y, z, p, q denote an element of any surface, and x', y', z', p', q' denote an element of any other surface. In order to connect the two elements completely though not uniquely, it is necessary (but not unconditionally sufficient) to have five distinct equations connecting the two sets of variables. But, as each set of five variables is to define an element of a surface, we have the Pfaffian relations

$$dz = p\,dx + q\,dy, \quad dz' = p'dx' + q'dy';$$

and therefore, if we are seeking the equations which are sufficient to determine an element of one surface corresponding to any given element of the other, we need take only four distinct equations, provided regard is paid to the Pfaffian relation. Accordingly, we shall take four equations of a form

$$F_n\,(x, y, z, p, q, x', y', z', p', q') = 0,$$

for $n = 1, 2, 3, 4$; and it will be assumed that the four equations are independent of one another.

A critical Relation.

292. These four equations do undoubtedly make surface-elements in the different spaces correspond with one another: no conditions are required. But if, instead of securing the correspondence of elements, it is desired that the equations should secure that a surface or surfaces in one space should correspond with a surface or surfaces in the other, an inquiry into the circumstances of the correspondence is needed: and it is conceivable that conditions may emerge limiting the correspondence. On the other hand, there should survive a generality in the results which arises from a different cause; for when two surfaces correspond, an unlimited number must correspond arising through the application, to both surfaces, of general contact-transformations. The latter generality will so far be discounted by declaring that, for the present purpose, surfaces transformable into one another by Lie's contact-transformations are equivalent to one another.

As the four equations are independent of one another, we shall consider them resoluble so as to express four out of the five variables x', y', z', p', q', in terms of the remaining quantities that occur. The preceding explanations shew that contact-transformations can be applied to both elements without affecting the immediate issue; we may therefore suppose that the four equations have been resolved so as to express x', y', p', q', in forms

$$\left.\begin{aligned} x' &= X(x, y, z, p, q, z') = X \\ y' &= Y(x, y, z, p, q, z') = Y \\ p' &= P(x, y, z, p, q, z') = P \\ q' &= Q(x, y, z, p, q, z') = Q \end{aligned}\right\}.$$

A simpler form would arise, if X, Y, P, Q were explicitly independent of z'; for the present, however, the more general form will be used.

We have to determine the surface or surfaces in the set x, y, z, p, q which, through the above equations, lead to corresponding surfaces in the set x', y', z', p', q'. In order that the latter set of variables, when varying continuously, may determine a surface, the relation

$$dz' - p'dx' - q'dy' = 0$$

must be satisfied. When the values of x', y', p', q' are substituted and the terms in the same differentials are collected, this relation becomes

$$A\,dz' + (B + Dr + Es)\,dx + (C + Ds + Et)\,dy = 0,$$

where

$$A = P\frac{\partial X}{\partial z'} + Q\frac{\partial Y}{\partial z'} - 1,$$

$$B = P\left(\frac{\partial X}{\partial x} + p\frac{\partial X}{\partial z}\right) + Q\left(\frac{\partial Y}{\partial x} + p\frac{\partial Y}{\partial z}\right),$$

$$C = P\left(\frac{\partial X}{\partial y} + q\frac{\partial X}{\partial z}\right) + Q\left(\frac{\partial Y}{\partial y} + q\frac{\partial Y}{\partial z}\right),$$

$$D = P\frac{\partial X}{\partial p} + Q\frac{\partial Y}{\partial p},$$

$$E = P\frac{\partial X}{\partial q} + Q\frac{\partial Y}{\partial q}.$$

Now the integral equivalent of the differential relation is to consist of a single equation; hence the relation must be integrable. The necessary and sufficient condition of integrability is

$$\begin{aligned} A&\left\{\frac{d}{dy}(B + Dr + Es) - \frac{d}{dx}(C + Ds + Et)\right\} \\ &+ (B + Dr + Es)\left\{\frac{\partial}{\partial z'}(C + Ds + Et) - \frac{dA}{dy}\right\} \\ &+ (C + Ds + Et)\left\{\frac{dA}{dx} - \frac{\partial}{\partial z'}(B + Dr + Es)\right\} = 0, \end{aligned}$$

where

$$\frac{dA}{dx} = \frac{\partial A}{\partial x} + p\frac{\partial A}{\partial z} + r\frac{\partial A}{\partial p} + s\frac{\partial A}{\partial q},$$

$$\frac{dA}{dy} = \frac{\partial A}{\partial y} + q\frac{\partial A}{\partial z} + s\frac{\partial A}{\partial p} + t\frac{\partial A}{\partial q},$$

and similarly for the derivatives of the other quantities. It is easy to see that, in this equation, the derivatives of z of the third order disappear: and the condition becomes

$$U(rt - s^2) + Rr + 2Ss + Tt + V = 0,$$

where

$$U = A\left(\frac{\partial D}{\partial q} - \frac{\partial E}{\partial p}\right) + E\left(\frac{\partial A}{\partial p} - \frac{\partial D}{\partial z'}\right) + D\left(\frac{\partial E}{\partial z'} - \frac{\partial A}{\partial q}\right),$$

and so for other coefficients: the quantities U, R, S, T, V can involve x, y, z, p, q, z', but they do not involve derivatives of z which are of the second order.

293. In the first place, suppose that z' does occur in the conditional equation; then the equation provides a value of z' in terms of the other quantities that occur, viz., x, y, z, p, q, r, s, t. Let this value be substituted in the initial form

$$Adz' + (B + Dr + Es)\,dx + (C + Ds + Et)\,dy = 0$$

of the differential relation, which may be written

$$\alpha dz' + \beta dx + \gamma dy = 0,$$

more briefly; the result of substitution is

$$\left(\alpha \frac{dz'}{dx} + \beta\right) dx + \left(\alpha \frac{dz'}{dy} + \gamma\right) dy = 0,$$

where $\dfrac{dz'}{dx}$ and $\dfrac{dz'}{dy}$ are the complete derivatives of the value of z'. Consequently, z satisfies the equations

$$\alpha \frac{dz'}{dx} + \beta = 0, \quad \alpha \frac{dz'}{dy} + \gamma = 0,$$

which are two equations of the third order.

These two equations, while distinct from one another, are compatible with one another in virtue of the original condition of integrability. For, when complete derivatives with regard to y and to x are taken of the two equations respectively, they give

$$\alpha \frac{d^2z'}{dx\,dy} + \left(\frac{\partial\alpha}{\partial y} + \frac{\partial\alpha}{\partial z'}\frac{dz'}{dy}\right)\frac{dz'}{dx} + \frac{\partial\beta}{\partial y} + \frac{\partial\beta}{\partial z'}\frac{dz'}{dy} = 0,$$

$$\alpha \frac{d^2z'}{dx\,dy} + \left(\frac{\partial\alpha}{\partial x} + \frac{\partial\alpha}{\partial z'}\frac{dz'}{dx}\right)\frac{dz'}{dy} + \frac{\partial\gamma}{\partial x} + \frac{\partial\gamma}{\partial z'}\frac{dz'}{dx} = 0;$$

subtracting, and substituting from the equations for the first derivatives of z' that have remained, we find

$$\alpha\left(\frac{\partial\beta}{\partial y} - \frac{\partial\gamma}{\partial x}\right) + \beta\left(\frac{\partial\gamma}{\partial z'} - \frac{\partial\alpha}{\partial y}\right) + \gamma\left(\frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial z'}\right) = 0,$$

which is the condition of integrability, known to be satisfied.

Hence the two equations have integrals in common. To determine the amount of arbitrary element that occurs in the most general common integral, we can proceed on the basis of the existence-theorems as discussed in Chapter II. Most simply, let initial values $\phi_0(y)$, $\phi_1(y)$, $\phi_2(y)$, when $x = a$, be assigned to z, p, r,.

respectively; then the values of $\dfrac{\partial^3 z}{\partial x \partial y^2}$ and $\dfrac{\partial^3 z}{\partial y^3}$, when $x = a$, can be regarded as known, and the two equations of the third order can then be regarded as determining $\dfrac{\partial^3 z}{\partial x^3}$ and $\dfrac{\partial^3 z}{\partial x^2 \partial y}$, that is, as determining $\dfrac{\partial^3 z}{\partial x^3}$ and $\dfrac{d\phi_2}{dy}$, when $x = a$. Eliminating $\dfrac{\partial^3 z}{\partial x^3}$ between them, we have an ordinary differential equation of the first order for the determination of ϕ_2: the integral of such an equation contains a single arbitrary element. Consequently, when values are assigned to z and p, the two differential equations possess a simple infinitude of common integrals. Taking the initial conditions more generally as in §§ 24, 178, we can say that the two equations of the third order determine a simple infinitude of surfaces, which satisfy them and which touch a given developable along a given curve.

To each such surface of elements x, y, z, p, q, there corresponds a single z'-surface; for z' is known in terms of z and its derivatives, that is, z' is known as a function of x and y.

Hence, on the present assumption, there is a simple infinitude of z-surfaces touching a given developable along a given curve; to each of them there corresponds a single z'-surface.

294. In the second place, suppose that z' does not occur in the conditional equation

$$U(rt - s^2) + Rr + 2Ss + Tt + V = 0;$$

the latter equation is then of the Monge-Ampère form and it serves to determine z. When a value of z satisfying this equation is substituted in the relation

$$A dz' + (B + Dr + Es)\,dx + (C + Ds + Et)\,dy = 0,$$

the latter equation is integrable by a single integral equation which involves an arbitrary constant. We thus obtain a simple infinitude of surfaces in the set x', y', z', p', q', corresponding to a single surface in the set x, y, z, p, q: and the single surface is an integral of an equation of the second order.

This result will always arise when the four equations of the transformation either do not involve z' explicitly, or can be changed, by a contact transformation, so as not to involve z' explicitly: when either condition is satisfied, we shall clearly have the simplest

comprehensive set of transformations. The character of the Monge-Ampère equation, which has been obtained, will be discussed almost immediately.

It is to be remarked that, in both the preceding hypotheses as to the form of the conditional equation, the quantity A is tacitly assumed to be different from zero. The general value of A is

$$P\frac{\partial X}{\partial z'}+Q\frac{\partial Y}{\partial z'}-1:$$

and the tacit assumption will be justified, if neither X nor Y involves z'. If however A should vanish, the conditional relation becomes

$$(B+Dr+Es)\,dx+(C+Ds+Et)\,dy=0,$$

and the condition of integrability can be taken to be

$$\frac{\partial}{\partial z'}\left(\frac{C+Ds+Et}{B+Dr+Es}\right)=0:$$

but the resulting integrated equation will be ineffective, because the arbitrary quantity which arises in the quadrature is a constant, so that the equation will not involve z'.

The relation can, however, be satisfied by taking

$$B+Dr+Es=0,$$
$$C+Ds+Et=0;$$

these two conditions lead, on the elimination of z', to an equation of the second order

$$f(x, y, z, p, q, r, s, t)=0:$$

and then, when z is determined so as to be an integral of this equation, either of the conditions will serve to determine z' in terms of x, y, z, p, q. Usually this determination of z' will be unique, so that then a single surface in the set x', y', z', p', q' corresponds to each surface in the set x, y, z, p, q, arising as an integral of the equation $f=0$.

As the equation $f=0$ arises from the elimination of z' between two equations that are linear in r, s, t, it is not an equation of quite general form: when we regard r, s, t as the non-homogeneous coordinates of a point in a new space, the equation represents a ruled surface having generators parallel to those of the cone $rt-s^2=0$, so that the differential equation $f=0$ possesses a system of characteristics of the first order.

Ex. Shew that, when the four equations

$$F_n=0$$

are not resolved so as to express x', y', p', q' in terms of the other variables, the equation of the second order (which is the condition of integrability for the differential relation $dz'=p'dx'+q'dy'$) can be expressed in the form

$$(12)[34]+(13)[42]+(14)[23]+(34)[12]+(42)[13]+(23)[14]=0,$$

where

$$[ij]=\left(\frac{\partial F_i}{\partial x'}+p'\frac{\partial F_i}{\partial z'}\right)\frac{\partial F_j}{\partial p'}-\left(\frac{\partial F_j}{\partial x'}+p'\frac{\partial F_j}{\partial z'}\right)\frac{\partial F_i}{\partial p'}$$

$$+\left(\frac{\partial F_i}{\partial y'}+q'\frac{\partial F_i}{\partial z'}\right)\frac{\partial F_j}{\partial q'}-\left(\frac{\partial F_j}{\partial y'}+q'\frac{\partial F_j}{\partial z'}\right)\frac{\partial F_i}{\partial q'},$$

$$(ij)=\frac{dF_i}{dx}\frac{dF_j}{dy}-\frac{dF_j}{dx}\frac{dF_i}{dy},$$

and where

$$\frac{d}{dx}=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q},$$

$$\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}.$$

(Bäcklund, Darboux.)

Bäcklund Transformations.

295. It thus appears that, when the transformations

$$F_n(x, y, z, p, q, x', y', z', p', q')=0,$$

for $n=1, 2, 3, 4$, are used so as to construct a surface or surfaces in the set x', y', z', p', q', which correspond with a surface or surfaces in the set x, y, z, p, q, there are certain cases when the variable z is an integral of a partial differential equation of the second order having the form of the equations considered by Monge and by Ampère. Conversely, if we seek to construct a surface or surfaces in the set x, y, z, p, q, which, under the same transformations, shall correspond with a surface or surfaces in the set x', y', z', p', q', it may happen that the variable z' is also an integral of a similar partial differential equation of the second order.

When the variables z and z' thus separately satisfy partial equations of the second order, these equations can be regarded as transformable into one another by the four relations

$$F_n(x, y, z, p, q, x', y', z', p', q')=0,$$

for $n = 1, 2, 3, 4$. This set of four relations is usually called a *Bäcklund transformation*, because the properties were first discussed in some of Bäcklund's investigations*.

As the transformations in question change, into one another, surfaces which are integrals of equations of the second order or (what is the same thing, when the geometrical link is dropped) transform equations of the second order into one another, it is important to obtain the limitations and the restrictions (if any) upon the equations, other than that of belonging to the Monge-Ampère type. Thus two questions arise for discussion at the outset. One is to determine whether a given Monge-Ampère type admits a Bäcklund transformation: the other question is the construction of Bäcklund transformations in general.

296. It is easy to see that there are several kinds of Bäcklund transformations†, discriminated according to the correspondence of the surfaces.

One kind of transformation arises when a single surface in one set of elements corresponds to only a single surface in the other set, and conversely. As there is only a single surface in x', y', z', p', q', the quadrature of the relation

$$dz' = p'dx' + q'dy',$$

after it has been duly modified, does not occur for operation: and the condition

$$A = P\frac{\partial X}{\partial z'} + Q\frac{\partial Y}{\partial z'} - 1 = 0$$

is therefore satisfied. The equivalent condition, associated with the transformations in the form

$$F_n = 0,$$

for $n = 1, 2, 3, 4$, is that the Jacobian

$$\left(\frac{F_1, F_2, F_3, F_4}{x', y', p', q'}\right)$$

should vanish, the x'-derivatives being $\frac{\partial F}{\partial x'} + p'\frac{\partial F}{\partial z'}$; and similarly for the y'-derivatives.

* These are contained in the two memoirs in volumes XVII and XIX of the *Mathematische Annalen*, which have already (p. 425 of this volume) been quoted.

† The following classification is due to Clairin, *Ann. de l'Éc. Norm.*, 3e Sér., t. XIX (1902), Supplément, p. 15.

Another kind of transformation arises, when a single surface in one set of elements corresponds to a single surface in the other set, while to the single surface in the latter set there corresponds a simple infinitude of surfaces in the former.

A third kind of transformation arises when, to each surface in either set of elements, there corresponds a simple infinitude of surfaces in the other.

Ex. 1. Prove that if two surfaces, arising as integrals of equations of the second order, correspond to one another under a Bäcklund transformation, their characteristics also correspond to one another. (Goursat.)

Ex. 2. Prove that the equations

$$z=\frac{\partial w}{\partial x'},\quad z^2+(x-y)^2pq=2\frac{\partial w}{\partial y'},\quad \frac{2z}{x-y}+p-q=q',\quad \frac{2}{x-y}=p',$$

where w is a function of x' and y' only, constitute a Bäcklund transformation; and obtain the equations of the second order satisfied by z and z' respectively. (Cosserat, Goursat.)

Ex. 3. Discuss the transformation constituted by the equations

$$z=\frac{\partial w}{\partial x'},\quad z^2+(x-y)^2pq=2\frac{\partial w}{\partial y'},\quad \frac{x+y}{x-y}=p',\quad px-qy+\frac{x+y}{x-y}z=q',$$

where w is a function of x' and y' only. (Goursat.)

297. A question next arises as to the degree of generality possessed by the equations of the Monge-Ampère form, which arise in connection with a Bäcklund transformation: does a perfectly general equation of that form necessarily admit such a transformation? It will be seen that the answer is in the negative.

It has appeared that the equations of the transformation may be taken in the form

$$x'=X,\quad y'=Y,\quad p'=P,\quad q'=Q,$$

where X, Y, P, Q are functions of x, y, z, p, q, and of z'. All the analysis is much more complicated when z' occurs explicitly in these functions: we shall therefore be content to deal with those transformations of the equations which are such that z' does not occur explicitly in the functions X, Y, P, Q.

Proceeding to consider in detail the simplest form of the original equations when they do not explicitly involve z', so that the equation

$$U(rt-s^2)+Rr+2Ss+Tt+V=0$$

does not explicitly involve z', we see at once that

$$A = -1;$$

and we easily find the coefficients U, R, S, T, V, in the preceding equations, to be

$$\left.\begin{aligned}
U &= \left(\frac{X, P}{p, q}\right) + \left(\frac{Y, Q}{p, q}\right) = \frac{\partial D}{\partial q} - \frac{\partial E}{\partial p} \\
R &= \left(\frac{X, P}{p, y}\right) + \left(\frac{Y, Q}{p, y}\right) = \frac{dD}{dy} - \frac{\partial C}{\partial p} \\
T &= \left(\frac{P, X}{q, x}\right) + \left(\frac{Q, Y}{q, x}\right) = \frac{\partial B}{\partial q} - \frac{dE}{dx} \\
V &= \left(\frac{P, X}{y, x}\right) + \left(\frac{Q, Y}{y, x}\right) = \frac{dB}{dy} - \frac{dC}{dx} \\
2S &= \left(\frac{X, P}{q, y}\right) + \left(\frac{P, X}{p, x}\right) + \left(\frac{Y, Q}{q, y}\right) + \left(\frac{Q, Y}{p, x}\right) \\
&= \frac{\partial B}{\partial p} - \frac{\partial C}{\partial q} - \frac{dD}{dx} + \frac{dE}{dy}
\end{aligned}\right\},$$

where, as usual,

$$\left(\frac{X, P}{p, q}\right) = \frac{\partial X}{\partial p}\frac{\partial P}{\partial q} - \frac{\partial X}{\partial q}\frac{\partial P}{\partial p},$$

$$\left(\frac{X, P}{p, y}\right) = \frac{\partial X}{\partial p}\frac{dP}{dy} - \frac{dX}{dy}\frac{\partial P}{\partial p},$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z},$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z},$$

and so for the other combinations. The quantities B, C, D, E are the forms of the coefficients of the equation of § 292, simplified by the non-occurrence of z', so that they are

$$\left.\begin{aligned}
B &= P\frac{dX}{dx} + Q\frac{dY}{dx} \\
C &= P\frac{dX}{dy} + Q\frac{dY}{dy} \\
D &= P\frac{\partial X}{\partial p} + Q\frac{\partial Y}{\partial p} \\
E &= P\frac{\partial X}{\partial q} + Q\frac{\partial Y}{\partial q}
\end{aligned}\right\},$$

the equations of transformation themselves being

$$\left.\begin{aligned} x' &= X(x, y, z, p, q) = X \\ y' &= Y(x, y, z, p, q) = Y \\ p' &= P(x, y, z, p, q) = P \\ q' &= Q(x, y, z, p, q) = Q \end{aligned}\right\}.$$

The new form of the relation, by hypothesis, does not involve z': therefore either it is identically satisfied, or it is an equation of the second order for the determination of z.

SIGNIFICANCE OF THE CRITICAL RELATION.

298. In the first place, suppose that the conditional relation is identically satisfied: in that case, each coefficient in the relation must vanish. As U then vanishes, we have

$$\frac{\partial D}{\partial q} - \frac{\partial E}{\partial p} = 0;$$

and therefore we may take

$$D = \frac{\partial \theta}{\partial p}, \quad E = \frac{\partial \theta}{\partial q},$$

where, so far as D and E are concerned, θ can be any function of x, y, z, p, q. Next, because R vanishes, we have

$$\begin{aligned} \frac{\partial C}{\partial p} = \frac{dD}{dy} &= \left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z}\right)\frac{\partial \theta}{\partial p} \\ &= \frac{\partial}{\partial p}\left(\frac{\partial \theta}{\partial y} + q\frac{\partial \theta}{\partial z}\right); \end{aligned}$$

and therefore

$$\begin{aligned} C &= \frac{\partial \theta}{\partial y} + q\frac{\partial \theta}{\partial z} + \phi(x, y, z, q) \\ &= \frac{d\theta}{dy} + \phi(x, y, z, q), \end{aligned}$$

where, so far as C is concerned, ϕ is an arbitrary function of the arguments indicated. Similarly, from the vanishing of T, we find

$$B = \frac{d\theta}{dx} + \psi(x, y, z, p),$$

where, so far as B is concerned, ψ is an arbitrary function. When these values are substituted in the expression for S, and when account is taken of the fact that S vanishes, we find

$$\frac{\partial \psi}{\partial p} = \frac{\partial \phi}{\partial q};$$

as q cannot occur on the left-hand side nor p on the right, we clearly have

$$\phi = qH + K, \quad \psi = pH + L,$$

where, so far as concerns S, θ, ϕ, the quantities H, K, L, are functions of the variables x, y, z only. Lastly, because V vanishes, we have

$$\frac{d\psi}{dy} = \frac{d\phi}{dx},$$

that is,

$$p\frac{\partial H}{\partial y} + pq\frac{\partial H}{\partial z} + \frac{\partial L}{\partial y} + q\frac{\partial L}{\partial z}$$
$$= q\frac{\partial H}{\partial x} + pq\frac{\partial H}{\partial z} + \frac{\partial K}{\partial x} + p\frac{\partial K}{\partial z}.$$

Hence

$$\frac{\partial H}{\partial y} = \frac{\partial K}{\partial z}, \quad \frac{\partial L}{\partial z} = \frac{\partial H}{\partial x}, \quad \frac{\partial K}{\partial x} = \frac{\partial L}{\partial y};$$

and therefore some function I of x, y, z exists, such that

$$H = \frac{\partial I}{\partial z}, \quad K = \frac{\partial I}{\partial y}, \quad L = \frac{\partial I}{\partial x}.$$

Consequently,

$$\phi = qH + K = \frac{dI}{dy}, \quad \psi = pH + L = \frac{dI}{dx};$$

and therefore, if

$$\Theta = \theta + I,$$

we have

$$B = \frac{d\Theta}{dx}, \quad C = \frac{d\Theta}{dy}, \quad D = \frac{\partial\Theta}{\partial p}, \quad E = \frac{\partial\Theta}{\partial q},$$

so that

$$Bdx + Cdy + Ddp + Edq$$
$$= d\Theta - \frac{\partial\Theta}{\partial z}(dz - pdx - qdy).$$

Again,

$$p'dx' + q'dy'$$
$$= Bdx + Cdy + Ddp + Edq + \left(P\frac{\partial X}{\partial z} + Q\frac{\partial Y}{\partial z}\right)(dz - pdx - qdy),$$

and therefore

$$d\Theta - p'dx' - q'dy' = \left(\frac{\partial\Theta}{\partial z} - P\frac{\partial X}{\partial z} - Q\frac{\partial Y}{\partial z}\right)(dz - pdx - qdy).$$

Accordingly, we take

$$z' = \Theta;$$

and this equation, together with the four initial equations

$$x' = X, \quad y' = Y, \quad p' = P, \quad q' = Q,$$

defines a contact-transformation which, as is known, changes every surface into some other surface.

It has already been declared that surfaces, which are transformable into one another by contact-transformations, are to be regarded as equivalent to one another for the present purpose. Consequently, the case, when the equation

$$U(rt - s^2) + Rr + 2Ss + Tt + V = 0$$

is identically satisfied, provides no new or independent transformations.

299. In the next place, suppose that the relation is satisfied, though not identically; then it is an equation of the Monge-Ampère form which serves to determine z. We proceed to shew that this equation is not perfectly general: that is, we cannot assume that any postulated equation of the form can be associated with a Bäcklund transformation.

To justify this statement, two forms of equation will be considered, according as the term in $rt - s^2$ is present or is absent. First, suppose the term in $rt - s^2$ to be present: then the equation may be taken

$$rt - s^2 + ar + bs + ct + e = 0,$$

where a, b, c, e are functions of x, y, z, p, q. If this equation is effectively the same as the equation which arises in the discussion of the Bäcklund transformations, a quantity μ must exist, such that

$$U = \mu, \quad R = \mu a, \quad 2S = \mu b, \quad T = \mu c, \quad V = \mu e.$$

The equations for the determination of μ, B, C, D, E are

$$\mu = \frac{\partial D}{\partial q} - \frac{\partial E}{\partial p},$$

$$\mu a = \frac{dD}{dy} - \frac{\partial C}{\partial p},$$

$$\mu b = \frac{\partial B}{\partial p} - \frac{\partial C}{\partial q} - \frac{dD}{dx} + \frac{dE}{dy},$$

$$\mu c = \frac{\partial B}{\partial q} - \frac{dE}{dx},$$

$$\mu e = \frac{dB}{dy} - \frac{dC}{dx},$$

five equations for five quantities. Let a new variable θ, an unknown function of x, y, z, p, q, be introduced by the relation

$$E = \frac{\partial \theta}{\partial q};$$

and let other unknown quantities be introduced by the relations

$$B = \frac{d\theta}{dx} + B',$$

$$C = \frac{d\theta}{dy} + C',$$

$$D = \frac{\partial \theta}{\partial p} + D'.$$

The foregoing equations become

$$\mu = \frac{\partial D'}{\partial q},$$

$$\mu a = \frac{dD'}{dy} - \frac{\partial C'}{\partial p},$$

$$\mu b = \frac{\partial B'}{\partial p} - \frac{\partial C'}{\partial q} - \frac{dD'}{dx},$$

$$\mu c = \frac{\partial B'}{\partial q},$$

$$\mu e = \frac{dB'}{dy} - \frac{dC'}{dx}.$$

We still have five equations, but they now involve explicitly only four unknown quantities μ, B', C', D': and θ remains undetermined. Manifestly, all the equations cannot be satisfied simultaneously

unless certain conditions are fulfilled: and therefore an arbitrarily postulated Monge-Ampère equation of the assumed form does not arise (and cannot be made to arise) through a Bäcklund transformation.

Next, suppose that the postulated Monge-Ampère equation is devoid of the term in $rt - s^2$, so that it has the form

$$ar + bs + ct + e = 0.$$

When we proceed as in the preceding case, the quantities B, C, D, E, and an unknown multiplier λ, must satisfy the equations

$$\begin{aligned}
0 &= \frac{\partial D}{\partial q} - \frac{\partial E}{\partial p}, \\
\lambda a &= \frac{dD}{dy} - \frac{\partial C}{\partial p}, \\
\lambda b &= \frac{\partial B}{\partial p} - \frac{\partial C}{\partial q} - \frac{dD}{dx} + \frac{dE}{dy}, \\
\lambda c &= \frac{\partial B}{\partial q} - \frac{dE}{dx}, \\
\lambda e &= \frac{dB}{dy} - \frac{dC}{dx}.
\end{aligned}$$

The first of these is satisfied by taking

$$D = \frac{\partial \theta}{\partial p}, \qquad E = \frac{\partial \theta}{\partial q},$$

where θ can be any function of x, y, z, p, q; and then, when we write

$$\begin{aligned}
B &= \frac{d\theta}{dx} + B', \\
C &= \frac{d\theta}{dy} + C',
\end{aligned}$$

the remaining equations are

$$\begin{aligned}
\lambda a &= -\frac{\partial C'}{\partial p}, \\
\lambda b &= \frac{\partial B'}{\partial p} - \frac{\partial C'}{\partial q}, \\
\lambda c &= \frac{\partial B'}{\partial q}, \\
\lambda e &= \frac{dB'}{dy} - \frac{dC'}{dx}.
\end{aligned}$$

We thus have four equations for the determination of the three quantities λ, B', C', the variable θ being left arbitrary: the equations cannot be satisfied unconditionally.

Hence any arbitrarily assumed Monge-Ampère equation cannot be associated with a Bäcklund transformation.

Ex. 1. Implicit functions P and Q, of p and of q respectively, are defined by the equations

$$P-1=e^{p-P}, \quad Q-1=e^{q-Q};$$

prove that the relations

$$\left.\begin{aligned} p&=e^{-(x+y)p'+\frac{z}{x+y}}-(x+y)p'+\frac{z}{x+y}+1 \\ q&=e^{z'+\frac{z}{x+y}}+z'+\frac{z}{x+y}+1 \end{aligned}\right\}$$

constitute a Bäcklund transformation connecting the equations

$$(x+y)s=PQ,$$

$$s'+\frac{p'}{x+y}=\frac{z'}{(x+y)^2}.$$

Discuss the relation of the integrals of the equations. (Clairin.)

Ex. 2. Prove that the equations

$$s'=0, \quad s+qe^{-z}=0,$$

are transformed into one another by the Bäcklund transformation

$$p=p'+e^{-z}, \quad q=e^{z'-z}. \qquad \text{(Clairin.)}$$

Ex. 3. Prove that the equation

$$p-z\frac{s}{q}=f\left(x, \frac{s}{q}\right)$$

can be changed to the equation $s'=0$, by a Bäcklund transformation: and obtain the transformation. (Clairin.)

Application to the Linear Equation.

300. Consider the linear equation*

$$s+\alpha p+\beta q+\gamma z=0,$$

where α, β, γ are functions of x and y only, so as to construct the Bäcklund transformations (if any) of the form

$$x'=x, \quad y'=y, \quad p'=P, \quad q'=Q,$$

which can be associated with the equation.

* The equation, in this connection, is discussed by Goursat, in the memoir quoted on p. 425.

With these assumptions as to the form of the transformation, we at once have (from § 292) the values of B, C, D, E in the form

$$B = P, \quad C = Q, \quad D = 0, \quad E = 0.$$

Thus the quantity θ in the preceding investigation is a function of x, y, and z only; also

$$\frac{\partial C'}{\partial p} = 0, \quad \frac{\partial B'}{\partial q} = 0,$$

so that

$$B' = \text{function of } x, y, z, p = \phi(x, y, z, p),$$

$$C' = \text{function of } x, y, z, q = \psi(x, y, z, q).$$

We then have

$$P = \frac{d\theta}{dx} + \phi(x, y, z, p),$$

$$Q = \frac{d\theta}{dy} + \psi(x, y, z, q):$$

and the equations, to be satisfied by the functions ϕ and ψ, are

$$\left.\begin{aligned} \lambda &= \frac{\partial\phi}{\partial p} - \frac{\partial\psi}{\partial q} \\ \lambda(\alpha p + \beta q + \gamma z) &= \frac{d\phi}{dy} - \frac{d\psi}{dx} \end{aligned}\right\},$$

where λ is the (unknown) multiplier: these equations being satisfied consistently with the limitations imposed on the forms of ϕ and ψ.

Writing

$$\mu = \lambda(\alpha p + \beta q + \gamma z),$$

we can take the equations as a set for the determination of ϕ, in the form

$$\frac{\partial\phi}{\partial p} = \frac{\partial\psi}{\partial q} + \lambda,$$

$$\frac{d\phi}{dy} = \frac{d\psi}{dx} + \mu,$$

$$\frac{\partial\phi}{\partial q} = 0.$$

The Jacobian conditions of coexistence of these three equations are

$$0=\frac{d}{dy}\left(\frac{\partial\psi}{\partial q}\right)+\frac{d\lambda}{dy}-\frac{\partial\mu}{\partial p}-\frac{\partial\psi}{\partial z},$$

$$0=\frac{\partial^2\psi}{\partial q^2}+\frac{\partial\lambda}{\partial q},$$

$$\frac{\partial\phi}{\partial z}=\frac{\partial}{\partial q}\left(\frac{d\psi}{dx}\right)+\frac{\partial\mu}{\partial q};$$

in the first of these, regard has been paid to the fact that ψ is independent of p. In order that the last of them may coexist with the three equations which involve derivatives of ϕ, we similarly have

$$0=\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial q}\right)+\frac{\partial\lambda}{\partial z}-\frac{\partial^2}{\partial p\partial q}\left(\frac{d\psi}{dx}\right)-\frac{\partial^2\mu}{\partial p\partial q},$$

$$0=\frac{\partial}{\partial z}\left(\frac{d\psi}{dx}\right)+\frac{\partial\mu}{\partial z}-\frac{d}{dy}\left\{\frac{\partial}{\partial q}\left(\frac{d\psi}{dx}\right)\right\}-\frac{d}{dy}\left(\frac{\partial\mu}{\partial q}\right),$$

$$0=\frac{\partial^2}{\partial q^2}\left(\frac{d\psi}{dx}\right)+\frac{\partial^2\mu}{\partial q^2}.$$

We thus obtain five equations involving derivatives of ψ only: combining them with

$$\frac{\partial\psi}{\partial p}=0,$$

which is satisfied in virtue of the form of ψ, we see that the necessary and sufficient conditions for the coexistence of the whole set are

$$\frac{\partial^2\lambda}{\partial p\partial q}=0,$$

$$\frac{d}{dx}\left(\frac{\partial\lambda}{\partial q}\right)=\frac{\partial^2\mu}{\partial q^2},$$

$$\frac{d}{dy}\left(\frac{\partial\lambda}{\partial p}\right)=\frac{\partial^2\mu}{\partial p^2},$$

$$\frac{\partial\lambda}{\partial z}=\frac{\partial^2\mu}{\partial p\partial q},$$

$$\frac{d^2\lambda}{dxdy}=\frac{d}{dx}\left(\frac{\partial\mu}{\partial p}\right)+\frac{d}{dy}\left(\frac{\partial\mu}{\partial q}\right)-\frac{\partial\mu}{\partial z}.$$

The form of λ is already known, being

$$\lambda=\frac{\partial\phi}{\partial p}-\frac{\partial\psi}{\partial q},$$

where ϕ is explicitly independent of q, and ψ of p: this form satisfies the first of these equations. Also

$$\mu = \lambda(\alpha p + \beta q + \gamma z),$$

where α, β, γ are functions of x and y only.

When these values are substituted in the fourth of these equations, we have

$$\begin{aligned}\frac{\partial^2\phi}{\partial p\partial z} - \beta\frac{\partial^2\phi}{\partial p^2} &= \frac{\partial^2\psi}{\partial q\partial z} - \alpha\frac{\partial^2\psi}{\partial q^2}\\ &= \frac{\partial^2 u}{\partial z^2},\end{aligned}$$

where u is any function of x, y, z alone, because ϕ is independent of q and ψ of p. Hence

$$\phi = p\frac{\partial u}{\partial z} + \beta u + v + f(x, y, \xi),$$

$$\psi = q\frac{\partial u}{\partial z} + \alpha u + w + g(x, y, \eta),$$

where

$$\xi = p + \beta z, \quad \eta = q + \alpha z,$$

v and w are any functions of x, y, z, and f and g are any functions of their arguments. With these values, the value of λ is given by

$$\lambda = \frac{\partial f}{\partial \xi} - \frac{\partial g}{\partial \eta}.$$

When this value of λ and the corresponding value of μ are substituted in the third of the equations, we find

$$\left\{\alpha\xi + \left(\gamma - \alpha\beta - \frac{\partial\beta}{\partial y}\right)z\right\}\frac{\partial^3 f}{\partial\xi^3} + 2\alpha\frac{\partial^2 f}{\partial\xi^2} = \frac{\partial^3 f}{\partial y\partial\xi^2}.$$

We assume that the coefficient of z, which is one of the invariants of the linear equation, does not vanish: and then the preceding equation for f leads to the two equations

$$\frac{\partial^3 f}{\partial\xi^3} = 0,$$

$$\frac{\partial^3 f}{\partial y\partial\xi^2} = 2\alpha\frac{\partial^2 f}{\partial\xi^2}.$$

Hence

$$\frac{\partial f}{\partial\xi} = l + m\xi,$$

where

$$\frac{\partial m}{\partial y} = 2\alpha m,$$

and l, m are functions of x and y only.

Proceeding similarly with the second of the equations, and assuming that $\frac{\partial \alpha}{\partial x} + \alpha\beta - \gamma$ (which is another invariant of the linear equation) does not vanish, we find

$$\frac{\partial g}{\partial \eta} = l' - m'\eta,$$

where

$$\frac{\partial m'}{\partial x} = 2\beta m',$$

and l', m' are functions of x and y only.

The value of λ now is given by

$$\begin{aligned}\lambda &= l + m\xi - l' + m'\eta \\ &= mp + m'q + (m\beta + m'\alpha) z + n,\end{aligned}$$

say, where n is a function of x and y only, and

$$\frac{\partial m}{\partial y} = 2\alpha m, \quad \frac{\partial m'}{\partial x} = 2\beta m'.$$

When this value of λ is substituted in the remaining equation, viz. in

$$\frac{d^2\lambda}{dx dy} = \frac{d}{dx}\left(\frac{\partial \mu}{\partial p}\right) + \frac{d}{dy}\left(\frac{\partial \mu}{\partial q}\right) - \frac{\partial \mu}{\partial z},$$

we find

$$\begin{aligned}&\frac{\partial^2}{\partial x \partial y}(m\beta + m'\alpha) \\ &= \frac{\partial}{\partial x}(m\gamma + m\alpha\beta + m'\alpha^2) + \frac{\partial}{\partial y}(m'\gamma + m'\alpha\beta + m\beta^2) - 2\gamma(m\beta + m'\alpha),\end{aligned}$$

which must be satisfied in order that the term in z may vanish; and

$$\frac{\partial^2 n}{\partial x \partial y} = \frac{\partial (n\alpha)}{\partial x} + \frac{\partial (n\beta)}{\partial y} - n\gamma,$$

in order that the terms independent of z, p, q may vanish: the terms in p and in q respectively are unconditionally evanescent.

Denoting the two invariants of the original equation by h and k, where

$$h = \frac{\partial \alpha}{\partial x} + \alpha\beta - \gamma, \quad k = \frac{\partial \beta}{\partial y} + \alpha\beta - \gamma,$$

and taking account of the equations satisfied by m and m', the first of the two preceding equations can be changed to the form

$$\frac{\partial (mk)}{\partial x} + \frac{\partial (m'h)}{\partial y} = 2\alpha m'h + 2\beta mk.$$

It is to be noticed that the equation satisfied by n is the adjoint of the original equation.

There are thus three equations in the two unknown quantities m and m': consequently, some condition must be satisfied. Let

$$\alpha = \frac{\partial A}{\partial y}, \quad \beta = \frac{\partial B}{\partial x},$$

where A can have as an arbitrary additive term any function of x, and B can have as an arbitrary additive term any function of y: then

$$m = Xe^{2A}, \quad m' = Ye^{2B},$$

where X and Y are arbitrary functions of x and of y respectively, which may vanish but will not vanish in the most general case. Assuming that X and Y do not vanish, and absorbing the X into e^{2A} and the Y into e^{2B}, we can take

$$m = e^{2A}, \quad m' = e^{2B}.$$

Substituting these values in the third equation which involves m and m', we have

$$e^{2A}\left(\frac{\partial k}{\partial x} + 2k\frac{\partial A}{\partial x} - 2\beta k\right) + e^{2B}\left(\frac{\partial h}{\partial y} + 2h\frac{\partial B}{\partial y} - 2\alpha h\right) = 0.$$

This relation involving the coefficients in the equation

$$s + \alpha p + \beta q + \gamma z = 0$$

must be satisfied, if the equation is to be capable of a Bäcklund transformation.

It is to be noted that the three equations can be simultaneously satisfied by taking X and Y both zero, so that m and m' vanish. In that case, we have

$$\lambda = n,$$

where n, a function of x and y only, is an integral of the adjoint equation; and the equations for ϕ and ψ become

$$\frac{\partial \phi}{\partial p} - \frac{\partial \psi}{\partial q} = n,$$

$$\frac{d\phi}{dy} - \frac{d\psi}{dx} = n\,(\alpha p + \beta q + \gamma z).$$

It is easy to verify that possible values of ϕ and ψ are given by

$$\left.\begin{aligned} \phi &= up + z\left(\frac{\partial u'}{\partial x} + nb\right) + \frac{d\theta}{dx} \\ \psi &= u'q + z\left(\frac{\partial u}{\partial y} - na\right) + \frac{d\theta}{dy} \end{aligned}\right\},$$

where

$$u - u' = n,$$

and θ is any function of x, y, z; and that then the transformation is given by

$$x' = x, \quad y' = y, \quad p' = \phi, \quad q' = \psi,$$

$$\theta + zu - z' = \int\left\{\left(z\frac{\partial n}{\partial x} - znb\right)dx + (nq + zna)\,dy\right\}.$$

The explicit value of z' is known as soon as the values of z and n are known.

Ex. Verify that a Bäcklund transformation of the preceding equation is given by

$$p' = \left(nb - \frac{\partial n}{\partial x}\right)z, \quad q' = -nq - naz;$$

and that another is given by

$$p' = -np - nbz, \quad q' = \left(na - \frac{\partial n}{\partial y}\right)z.$$

(Goursat.)

301. The process adopted by Goursat for a determination of quantities ϕ and ψ, which are to satisfy the equations

$$\lambda = \frac{\partial \phi}{\partial p} - \frac{\partial \psi}{\partial q},$$

$$\mu = \lambda\,(\alpha p + \beta q + \gamma z) = \frac{d\phi}{dy} - \frac{d\psi}{dx},$$

is as follows, on the assumption that λ is known. Take two particular functions $\phi_1(x, y, z, p)$ and $\psi_1(x, y, z, q)$, such that

$$\frac{\partial \phi_1}{\partial p} - \frac{\partial \psi_1}{\partial q} = \lambda;$$

then

$$\frac{\partial(\phi-\phi_1)}{\partial p}=\frac{\partial(\psi-\psi_1)}{\partial q}=U,$$

where U is any function only of x, y, z, so that

$$\phi=\phi_1+Up+V,\quad \psi=\psi_1+Uq+W,$$

where V and W are functions only of x, y, z. When these values are substituted in the second equation, we have

$$\frac{\partial V}{\partial y}-\frac{\partial W}{\partial x}+p\left(\frac{\partial U}{\partial y}-\frac{\partial W}{\partial z}\right)+q\left(\frac{\partial V}{\partial z}-\frac{\partial U}{\partial x}\right)=\rho,$$

where

$$\rho=\mu-\frac{d\phi_1}{dy}+\frac{d\psi_1}{dx}.$$

Evidently ρ is a linear function of p and q; and

$$\frac{\partial U}{\partial y}-\frac{\partial W}{\partial z}=\frac{\partial\rho}{\partial p},$$

$$\frac{\partial V}{\partial z}-\frac{\partial U}{\partial x}=\frac{\partial\rho}{\partial q},$$

$$\frac{\partial V}{\partial y}-\frac{\partial W}{\partial x}=\rho-p\frac{\partial\rho}{\partial p}-q\frac{\partial\rho}{\partial q},$$

ρ being determined by the equation

$$\frac{\partial^2\rho}{\partial x\partial p}+\frac{\partial^2\rho}{\partial y\partial q}=\frac{\partial\rho}{\partial z}-p\frac{\partial^2\rho}{\partial p\partial z}-q\frac{\partial^2\rho}{\partial q\partial z},$$

or, what is the same thing, by the equation

$$\frac{d}{dx}\left(\frac{\partial\rho}{\partial p}\right)+\frac{d}{dy}\left(\frac{\partial\rho}{\partial q}\right)-\frac{\partial\rho}{\partial z}=0.$$

When the value of ρ is substituted, we have

$$\frac{d}{dx}\left(\frac{\partial\mu}{\partial p}\right)+\frac{d}{dy}\left(\frac{\partial\mu}{\partial q}\right)-\frac{\partial\mu}{\partial z}=\frac{d^2}{dxdy}\left(\frac{\partial\phi_1}{\partial p}-\frac{\partial\psi_1}{\partial q}\right)$$

$$=\frac{d^2\lambda}{dxdy},$$

one of the equations already (§ 300) known as one to be satisfied. Assuming the value of μ to be such that this equation is satisfied, we see that the three equations in U, V, W are equivalent to a couple only, so that one of the three quantities U, V, W can be taken arbitrarily. Goursat takes

$$U=0;$$

and then

$$V = \int_{z_0}^{z} \frac{\partial\rho}{\partial q}\, dz + f(x, y),$$

$$W = -\int_{z_0}^{z} \frac{\partial\rho}{\partial p}\, dz + g(x, y),$$

provided

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = \rho - p\frac{\partial\rho}{\partial p} - q\frac{\partial\rho}{\partial q} - \int_{z_0}^{z}\left(\frac{\partial^2\rho}{\partial x\partial p} + \frac{\partial^2\rho}{\partial y\partial q}\right) dz$$

$$= \left[\rho - p\frac{\partial\rho}{\partial p} - q\frac{\partial\rho}{\partial q}\right]_{z=z_0}.$$

The values of V and W (and therefore of ϕ and ψ) will be known, if f and g are any two functions satisfying this last relation.

When the particular form

$$\mu = \lambda(\alpha p + \beta q + \gamma z)$$

is taken, so that λ is an integral of the equation adjoint to

$$s + \alpha p + \beta q + \gamma z = 0,$$

and therefore can be regarded as a function of x and y only, we have first to obtain particular functions ϕ_1 and ψ_1, such that

$$\frac{\partial\phi_1}{\partial p} - \frac{\partial\psi_1}{\partial q} = \lambda.$$

Obviously we can take

$$\phi_1 = \tfrac{1}{2}\lambda p, \quad \psi_1 = -\tfrac{1}{2}\lambda q;$$

and then

$$\rho = \lambda(\alpha p + \beta q + \gamma z) - \tfrac{1}{2}p\frac{\partial\lambda}{\partial y} - \tfrac{1}{2}q\frac{\partial\lambda}{\partial x}.$$

Making $z_0 = 0$, we have

$$V = \int_0^z \frac{\partial\rho}{\partial q}\, dz + f(x, y) = \left(\lambda\beta - \tfrac{1}{2}\frac{\partial\lambda}{\partial x}\right) z + f,$$

$$W = -\int_0^z \frac{\partial\rho}{\partial p}\, dz + g(x, y) = -\left(\lambda\alpha - \tfrac{1}{2}\frac{\partial\lambda}{\partial y}\right) z + g,$$

and

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = \left[\rho - p\frac{\partial\rho}{\partial p} - q\frac{\partial\rho}{\partial q}\right]_{z=0}$$

$$= 0,$$

so that we can take

$$f = \frac{\partial H}{\partial x}, \quad g = \frac{\partial H}{\partial y},$$

where H is an arbitrary function of x and y. The values of ϕ and ψ are

$$\left.\begin{aligned} p' = \phi &= \tfrac{1}{2}\lambda p + \left(\lambda\beta - \tfrac{1}{2}\frac{\partial\lambda}{\partial x}\right)z + \frac{\partial H}{\partial x} \\ q' = \psi &= -\tfrac{1}{2}\lambda q - \left(\lambda\alpha - \tfrac{1}{2}\frac{\partial\lambda}{\partial y}\right)z + \frac{\partial H}{\partial y} \end{aligned}\right\};$$

earlier investigations shew that we can add $\dfrac{d\theta}{dx}$ to p, and $\dfrac{d\theta}{dy}$ to q, where θ is any function of x, y, z.

Simultaneous Equations of the First Order.

302. In connection with these investigations upon the transformation of equations of the second order, it is worth while considering another set of investigations on lines initiated by Bäcklund. Reference was made to them in the introductory note to Chapter XI; they are concerned with the discussion of simultaneous equations of the first order, the number of dependent variables being equal to the number of equations. The simplest case arises when there are two simultaneous equations involving two dependent variables: denoting the latter by z and z', and their derivatives by p, q, p', q', respectively, we may take the equations in the form

$$\left.\begin{aligned} f(x, y, z, z', p, q, p', q') = 0 \\ g(x, y, z, z', p, q, p', q') = 0 \end{aligned}\right\}.$$

When these two equations $f=0$ and $g=0$, can be resolved algebraically for (say) p and q, the condition being that the Jacobian of f and g with regard to p and q does not vanish identically, the resolution leads to equations

$$\left.\begin{aligned} p &= f_1(x, y, z, z', p', q') \\ q &= f_2(x, y, z, z', p', q') \end{aligned}\right\}.$$

Now we must have

$$\frac{dp}{dy} = \frac{dq}{dx},$$

where the respective derivatives of p and q are complete: hence

$$\begin{aligned} &\frac{\partial f_1}{\partial y} + f_2\frac{\partial f_1}{\partial z} + q'\frac{\partial f_1}{\partial z'} + s'\frac{\partial f_1}{\partial p'} + t'\frac{\partial f_1}{\partial q'} \\ &\quad = \frac{\partial f_2}{\partial x} + f_1\frac{\partial f_2}{\partial z} + p'\frac{\partial f_2}{\partial z'} + r'\frac{\partial f_2}{\partial p'} + s'\frac{\partial f_2}{\partial q'}, \end{aligned}$$

being a relation which is linear in r', s', t', and which, in general, involves x, y, z, z', p', q'. When z occurs, we can conceive this relation resolved so as to express z, in terms of the independent variables, of z', and of the derivatives of z' up to the second order inclusive; when the value of z so given is substituted in the given equations (or in their resolved equivalents), they become two equations of the third order for the determination of z'. From the general theory we know that, unless the original equations cannot be resolved with respect to p and p', or with respect to q and q', they possess common integrals; consequently, the two equations of the third order which are satisfied by z' must be compatible with one another, and they must therefore lead to a value or values of z' which involve (or may involve) arbitrary elements. To each such value of z', there corresponds the resolved value of z as given above.

If, however, it should happen that the relation, which expresses the condition

$$\frac{dp}{dy} = \frac{dq}{dx},$$

should be explicitly free from z, then it becomes a single equation for z', of the second order and linear. When a value of z' has been obtained satisfying this equation of the second order, and when it is substituted in the equations expressing p and q, then a process of quadrature leads to a value of z which involves an arbitrary constant. We may therefore infer that an infinitude of integrals z will correspond to a single integral z'.

Note. Exceptional cases arise when the relation, which expresses the condition

$$\frac{dp}{dy} = \frac{dq}{dx},$$

does not involve r', s', t'; that this may happen, we must have

$$\frac{\partial f_2}{\partial p'} = 0, \quad \frac{\partial f_1}{\partial q'} = 0, \quad \frac{\partial f_1}{\partial p'} = \frac{\partial f_2}{\partial q'}.$$

If the relation in this special event should involve z, then z' satisfies two equations of the second order. If it should not involve z, then z' satisfies a single equation of the first order; and the relations between the integrals z and z' are the same as in the more general event.

303. When the two equations $f=0$, $g=0$, cannot be resolved algebraically for p and q, because the Jacobian of f and g with regard to p and q vanishes, either identically or in virtue of the two equations, then the elimination of p between the two equations compels the elimination of q also: the result of the elimination is an equation

$$g(x, y, z, z', p', q')=0.$$

If g involves z, we can imagine $g=0$ resolved with regard to z; and then the original equations can be replaced by a set (or by a number of sets) of equations of the form

$$f(x, y, z, z', p, q, p', q')=0,$$
$$z-g_1(x, y, z', p', q')=0.$$

Substituting for z, we usually have an equation of the second order for the determination of z'; to every integral of that equation, there corresponds a single value of z.

These are the results which arise from resolution of the two original equations with regard to p and q. The equations can equally be resolved with regard to p' and q', or the resolution can equally fail: and there will be corresponding relations between the integrals z' and z. We thus have the same three kinds of pairs of equations as arise in the Bäcklund transformations; they are as follows:—

(i) the equations may be such that an integral z' corresponds to a single integral, and conversely:

(ii) the equations may be such that a single integral of one equation corresponds to a single integral of the other, while a simple infinitude of integrals of that other corresponds to a single integral of the first:

(iii) the equations may be such that to each integral of either equation there corresponds a simple infinitude of integrals of the other.

In each case, the simple infinitude of integrals arises through the presence of an arbitrary parameter: and the equations are equations of the second order.

Ex. 1. Of the first case, the general type is such that the two initial equations can be expressed in the form

$$z'=F(x, y, z, p, q),$$
$$z=G(x, y, z', p', q'),$$

where F and G are explicit functions: the elimination of either of the dependent variables leads to an equation of the second order for the determination of the other.

The best known instance is associated with Laplace's linear equation, which is

$$s+ap+bq+cz=0,$$

a, b, c being functions of x and y only. When we take

$$z'=q+az, \quad z''=p+bz,$$

we find, with the notation of Chapter XIII,

$$hz=p'+bz', \quad kz=q''+az'':$$

then z' and z'' satisfy the respective equations

$$s'+a'p'+b'q'+c'z'=0,$$
$$s''+a''p''+b''q''+c''z''=0,$$

with definite values of a', b', c', a'', b'', c''. A single value of z and a single value of z' correspond uniquely to one another: likewise for a single value of z and a single value of z''. Hence a single value of z' and a single value of z'' correspond uniquely to one another through the medium of the unique corresponding z: but the analytical expression of the correspondence between z' and z'' is of a different character, for it involves derivatives of the second order.

Ex. 2. In the equation

$$as+bp+\psi(x, y, z, q, t)=0,$$

the coefficients a and b are functions of x, y, z, q; shew that, if

$$z'=\phi(x, y, z, q),$$

where ϕ is any integral of

$$b\frac{\partial\phi}{\partial q}-a\frac{\partial\phi}{\partial z}=0,$$

z' satisfies an equation of the second order. Shew also that each integral of either equation corresponds to only a single integral of the other.

(Teixeira.)

Ex. 3. Of the second kind of correspondence in the text, a type is represented by a couple of equations

$$z=f(x, y, z', p', q'),$$
$$0=g(x, y, p, q, z', p', q'),$$

these equations being supposed resoluble with regard to p' and q', in a form

$$p'=F(x, y, z, p, q, z'),$$
$$q'=G(x, y, z, p, q, z').$$

It is clear that z' satisfies an equation of the second order: if the relation

$$\frac{dF}{dy}=\frac{dG}{dx}$$

does not contain z' (and this will be assumed to be the fact), then z also satisfies an equation of the second order. To each integral z' there corresponds one, and only one, integral z: to each integral z, there corresponds a simple infinitude of integrals z'.

The simplest set of cases is given by the equations

$$z=f(x, y, p', q'),$$
$$0=g(x, y, p, q, p', q'),$$

the Jacobian of f and g with respect to p' and q' being supposed not to vanish.

Ex. 4. Of the third kind of correspondence in the text, a type is represented by a couple of equations

$$f(x, y, p, q, p', q')=0,$$
$$g(x, y, p, q, p', q')=0,$$

when they can be resolved with respect to p and q, and also with respect to p' and q': the sufficient condition is that neither of the Jacobians

$$J\left(\frac{f,\ g}{p,\ q}\right),\quad J\left(\frac{f,\ g}{p',\ q'}\right),$$

should vanish.

It is easy to see that z and z' both satisfy equations of the second order. To each integral z', there corresponds a simple infinitude of integrals z; and to each integral z, there corresponds a simple infinitude of integrals z'.

Ex. 5. Discuss the character of the correspondence of the integrals of the equations

$$p+p'=\sin(z-z'),$$
$$q-q'=\sin(z+z'),$$

being equations connected with surfaces of constant curvature. Obtain integrals z and z', which are functions of x^2+y^2 only; and interpret the results. (Bianchi; Darboux.)

Ex. 6. Shew that every equation $f(x, y, z, p, q, r, s, t)=0$ of the second order which, under the transformation

$$z'=q,$$

leads to an equation of the second order for the determination of z', can be brought to the form

$$X_1r+X_2p+X_3z+g(x, y, q, s, t)=0,$$

where X_1, X_2, X_3 are any functions of x alone; and discuss the correspondence of the integrals of the two equations. (Goursat.)

Ex. 7. Shew that, if z satisfies a linear equation

$$Ar+2Bs+Ct+Dp+Eq+Fz=0,$$

and if a new quantity u be defined by the relation

$$u=\alpha p+\beta q+\gamma z,$$

where A, B, C, D, E, F, a, β, γ are functions of x and y only, then, when u satisfies an equation of the second order, $u=0$ must be satisfied by two distinct integrals of the original equation, provided $A\beta^2-2Ba\beta+Ca^2$ does not vanish.

Obtain the condition, necessary to secure that u satisfies an equation of the second order, when $A\beta^2-2Ba\beta+Ca^2$ does vanish. (Goursat.)

Ex. 8. Given two equations

$$p'=az'+\beta z+\gamma p+\delta q+\eta,$$
$$q'=a'z'+\beta' z+\gamma' p+\delta' q+\eta',$$

where all the coefficients are functions of x and y only; prove that, if z is to satisfy an equation of the second order resulting from the elimination of z', it is necessary and sufficient that

$$\frac{\partial a}{\partial y}=\frac{\partial a_1}{\partial x}.$$

When this relation is satisfied, and when $\gamma\delta'-\gamma'\delta=0$, shew that the elimination of z leads to an equation of the second order for z': and obtain the conditions for this result, when $\gamma\delta'-\gamma'\delta$ does not vanish.

Discuss the correspondence of the integrals. (Goursat.)

CHAPTER XXII.

EQUATIONS OF THE THIRD AND HIGHER ORDERS, IN TWO INDEPENDENT VARIABLES.

THE present chapter is only a brief outline of the application of the preceding theory and the various preceding methods (where they can be applied) to equations of order higher than the second; and, with the fewest exceptions, these applications are made to equations of the third order only.

The earlier sections are devoted to the discussion of equations of the third order having an intermediate integral of the second order. On this topic, a paper by Tanner* may be consulted: the method of exposition adopted is different from Tanner's, and is the extension of the method expounded in Chapter XVI. Some results relating to equations of the third order had previously been given by Falk†, who also gave some results, mostly formal, relating to equations of order n. Reference also may be made to the treatise by Natani‡, and to the investigations by Hamburger§ on equations of order higher than the second.

304. The methods, that have been expounded, were devised in connection with equations of the second order. They can be applied, with the appropriate modifications, to equations of order higher than the second: and some instances will now be given. As however the development does not seem to lead to any new kinds of properties, but is concerned with details of a kind already familiar in the equations of the second order, and as no new method of constructing integrals has been devised for equations of higher orders, we shall give only a brief discussion of this part of the subject.

* *Proc. Lond. Math. Soc.*, vol. VIII (1877), pp. 229—261.

† *Acta Ups.*, t. VIII (1872), pp. 1—40.

‡ *Die höhere Analysis*, quoted at the beginning of Chap. XI.

§ *Crelle*, t. XCIII (1882), pp. 201—214.

With the notation already adopted, the derivatives of z with regard to x and y of the third order will be denoted by α, β, γ, δ. The differential equation of the third order can then be taken in the form

$$f(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0;$$

and, for the purposes of discussion, the equation will be supposed polynomial in the derivatives of highest order.

We naturally begin with those equations which possess an intermediate integral or which admit an equation of lower order compatible with themselves. The simplest form in which such an integral will occur is

$$\theta(u, v) = 0,$$

where θ is an arbitrary functional form, while u and v are definite functions of x, y, z, p, q, r, s, t: the corresponding differential equation is

$$\begin{aligned} E(\alpha\gamma - \beta^2) + F(\alpha\delta - \beta\gamma) + G(\beta\delta - \gamma^2) \\ + A\alpha + B\beta + C\gamma + D\delta + H = 0, \end{aligned}$$

where A, B, C, D, E, F, G, H do not involve derivatives of the third order. This form is a first condition that an equation of the third order should have an intermediate integral of the specified type; yet the necessary form is only one among other conditions. Another simple form of intermediate integral is

$$g(x, y, z, p, q, r, s, t, a, b) = 0,$$

where a and b are arbitrary constants: the elimination of a and b between the equations

$$g = 0, \quad \frac{dg}{dx} = 0, \quad \frac{dg}{dy} = 0,$$

leads to an equation of the third order having $g = 0$ for an intermediate integral. And other cases can be constructed when other types of intermediate integral are postulated.

To construct the classes of equations that have intermediate integrals of a form

$$u(x, y, z, p, q, r, s, t) = 0,$$

without any specification of the element or elements of generality in u, we use the property that the three equations

$$\begin{aligned} f &= 0, \\ u_x + \alpha u_r + \beta u_s + \gamma u_t &= 0, \\ u_y + \beta u_r + \gamma u_s + \delta u_t &= 0, \end{aligned}$$

where

$$\left.\begin{aligned} u_x &= \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} \\ u_y &= \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} \end{aligned}\right\},$$

are not independent of one another; so that, regarded as three equations from which two of the four quantities α, β, γ, δ can be eliminated, they will provide an evanescent eliminant. Suppose that we use the second and the third of the equations to express α and δ in terms of β and γ, the coefficients in the expressions being homogeneous of order zero in the derivatives of u. Let these expressions be substituted in $f=0$, where f is a polynomial in the third derivatives: after substitution and collection of terms, there would be three terms at least in an arbitrary equation, viz. a term in β, a term in γ, and a term free from β and γ. The result of the substitution must, on the hypothesis adopted, lead to an evanescent form: hence the coefficient of each term in β and γ is to be evanescent, and the vanishing of each such coefficient gives a condition to be satisfied by u. Consequently, there will be three conditions at least. Each of the conditions must involve u_x, u_y, u_r, u_s, u_t homogeneously, and therefore there cannot be more than four independent conditions. We therefore have two cases to consider:—

(i) when there are three independent conditions for u;

(ii) when there are four independent conditions for u.

When u is known, or when we infer that u can be determined, then the differential equation possesses an intermediate integral.

Note. It has been assumed that the number of independent conditions is not two. If there were only two, we could consider them resolved for u_x and u_y in the form

$$u_x + g(u_r, u_s, u_t) = 0, \quad u_y + h(u_r, u_s, u_t) = 0:$$

the differential equation should then result from the elimination of the ratios $u_r : u_s : u_t$ from

$$\alpha u_r + \beta u_s + \gamma u_t = g, \quad \beta u_r + \gamma u_s + \delta u_t = h,$$

and there are, in general, too few equations for the performance of the elimination.

A general Class of Equations.

305. To illustrate the working by a particular case, consider the equation

$$E(\alpha\gamma - \beta^2) + F(\alpha\delta - \beta\gamma) + G(\beta\delta - \gamma^2) \\ + A\alpha + B\beta + C\gamma + D\delta + H = 0,$$

which may have an intermediate integral.

According to the preceding argument, we take

$$\alpha = -\beta\frac{u_s}{u_r} - \gamma\frac{u_t}{u_r} - \frac{u_x}{u_r},$$

$$\delta = -\beta\frac{u_r}{u_t} - \gamma\frac{u_s}{u_t} - \frac{u_y}{u_t};$$

we substitute these values of α and δ in the equation, and we then make the resulting equation evanescent so far as regards the determination of β and γ.

The terms in β^2, $\beta\gamma$, γ^2 disappear in virtue of a single relation

$$Eu_t - Fu_s + Gu_r = 0.$$

The term in β disappears in virtue of the relation

$$E\frac{u_y}{u_r} + F\frac{u_x}{u_t} - A\frac{u_s}{u_r} - D\frac{u_r}{u_t} + B = 0,$$

the preceding relation being used to simplify the form. Similarly, from the disappearance of the term in γ, we have

$$F\frac{u_y}{u_r} + G\frac{u_x}{u_t} - A\frac{u_t}{u_r} - D\frac{u_s}{u_t} + C = 0;$$

and lastly, the aggregate of terms independent of β and γ gives the relation

$$F\frac{u_x u_y}{u_r u_t} - A\frac{u_x}{u_r} - D\frac{u_y}{u_t} + H = 0.$$

There are apparently four equations.

From the second and the third of these relations, we find (also using the first relation)

$$(EG - F^2)(Fu_x - Du_r) = (-AFG + BF^2 - CFE + DE^2)\,u_t,$$

$$(EG - F^2)(Fu_y - Au_t) = (AG^2 - BFG + CF^2 - DEF)\,u_r;$$

and the fourth relation can be written

$$(Fu_x - Du_r)(Fu_y - Au_t) = (AD - FH)\,u_r u_t.$$

Consequently

$$(AFG - BF^2 + CFE - DE^2)(AG^2 - BFG + CF^2 - DEF)$$
$$= (EG - F^2)^2 (FH - AD),$$

a relation between the coefficients of the original equation which must be satisfied. When satisfied, it renders the fourth relation for u a mere identity in the presence of the other relations; also, it can be regarded as determining H, on the removal of an obviously superfluous factor F.

Following Tanner*, we select one class of equations determined by the property that $EG - F^2$ does not vanish: and this class will be composed of two sub-classes, according as F does not or does vanish. The subsidiary equations are

$$\left.\begin{aligned} Fu_x - Du_r - \theta u_t &= 0 \\ Fu_y - \theta' u_r - Au_t &= 0 \\ Fu_s - Gu_r - Eu_t &= 0 \end{aligned}\right\},$$

when F and $EG - F^2$ do not vanish, and the quantities θ and θ' are

$$\left.\begin{aligned} (EG - F^2)\,\theta &= -AFG + BF^2 - CFE + DE^2 \\ (EG - F^2)\,\theta' &= AG^2 - BFG + CF^2 - DEF \end{aligned}\right\};$$

obviously

$$\theta\theta' = AD - FH.$$

But, when F does vanish while neither E nor G vanishes, the subsidiary equations are

$$\left.\begin{aligned} &Eu_t + Gu_r = 0 \\ &\qquad Eu_y = -\left(D\frac{E}{G} + B\right)u_r + Au_s \\ &\qquad Eu_x = \ \ \left(A\frac{G}{E} + C\right)u_r + D\frac{E}{G}u_s \end{aligned}\right\},$$

while the value of H is easily found to be

$$H = \frac{A^2G + ACE}{E^2} + \frac{D^2E + BDG}{G^2};$$

the fourth relation then is an identity.

* *Proc. L. M. S.*, vol. VIII (1877), p. 237.

For either of the sub-classes of equations, we have a system of homogeneous linear equations of the first order in one dependent variable. This system can be treated in the ordinary way, by being made a complete Jacobian system. The most useful case arises when the completed system contains six equations so that, as there are eight variables, the system will have two algebraically independent integrals: if these be u_1 and u_2, the intermediate integral is

$$f(u_1, u_2) = 0,$$

where f is an arbitrary function. If the completed system contains seven equations, the only intermediate integral is of the form

$$u = a;$$

and that integral leads to two equations of the third order, not to one only. If the completed system contains eight equations, there is no intermediate integral.

Ex. 1. Consider the equation

$$a\gamma - \beta^2 + a\delta - \beta\gamma - (\beta\delta - \gamma^2) = 0.$$

Here $E=1$, $F=1$, $G=-1$, and all the other coefficients vanish. The necessary relation between the set of coefficients is satisfied: and so, formally, we have an instance of the first sub-class. Also

$$\theta=0, \quad \theta'=0;$$

thus the subsidiary equations are

$$u_x=0, \quad u_y=0, \quad u_s - u_r + u_t = 0.$$

Taken in full, these are

$$\Delta_1 = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} = 0,$$

$$\Delta_2 = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} = 0,$$

$$\Delta_3 = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0.$$

We have

$$(\Delta_1, \Delta_2) = 0,$$

$$(\Delta_1, \Delta_3) = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} = 0,$$

$$(\Delta_2, \Delta_3) = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} = 0.$$

by the Jacobian conditions of coexistence: hence

$$\frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0.$$

With these equations, we have now

$$\Delta_1' = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} = 0,$$

$$\Delta_2' = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} = 0:$$

also

$$\left(\Delta_1', \frac{\partial u}{\partial p}\right) = \frac{\partial u}{\partial z} = 0,$$

$$\left(\Delta_2', \frac{\partial u}{\partial q}\right) = \frac{\partial u}{\partial z} = 0.$$

Hence our equations are the complete Jacobian system

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial p} = 0, \quad \frac{\partial u}{\partial q} = 0,$$

$$\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0.$$

Two independent integrals are $r - s$, $s + t$: hence an intermediate integral of the original equation is

$$f(r - s, \ s + t) = 0.$$

Ex. 2. Obtain intermediate integrals of the equations :—

(i) $a\delta - \beta\gamma = 0$;

(ii) $t(a\gamma - \beta^2) - s(a\delta - \beta\gamma) + r(\beta\delta - \gamma^2) = 0$;

(iii) $t(a\gamma - \beta^2) - r(\beta\delta - \gamma^2) + sta - t(r+1)\beta + s(r+1)\gamma - r^2\delta = rt - s^2$.

(Tanner.)

306. Another class of these equations is the class determined by the condition

$$EG - F^2 = 0,$$

while E, F, G do not vanish together. Obviously both E and G cannot vanish, for then F also would vanish: and the case $E = 0$, $F = 0$, and $G \gtrless 0$, is obtainable from the case $G = 0$, $F = 0$, and $E \gtrless 0$, by interchange of variables. Hence there are two sub-classes in the present class:

(i) E, F, G all different from zero:

(ii) $E = 0$, $F = 0$, and G different from zero.

First, take the case when no one of the quantities E, F, G vanishes: then we may take

$$F = mE, \quad G = m^2E,$$

as fulfilling the general condition, m being not zero. The equations for θ and θ' then give

$$Am^3 - Bm^2 + Cm - D = 0,$$

as the necessary relation among the coefficients. In order to obtain the equations for u, we revert to the original relations of § 305, the first of which now is

$$u_t - mu_s + m^2 u_r = 0.$$

The second is

$$\frac{u_y}{u_r} + m\frac{u_x}{u_t} = \frac{A}{E}\frac{u_s}{u_r} + \frac{D}{E}\frac{u_r}{u_t} - \frac{B}{E},$$

while the third is

$$m\frac{u_y}{u_r} + m^2\frac{u_x}{u_t} = \frac{A}{E}\frac{u_t}{u_r} + \frac{D}{E}\frac{u_s}{u_t} - \frac{C}{E};$$

and these two are equivalent to one another, in virtue of the first relation and of the condition satisfied among the coefficients. The third relation is

$$(Fu_x - Du_r)(Fu_y - Au_t) = (AD - FH)\, u_r u_t.$$

Resolving the modified relations so as to obtain u_x, u_y, u_s in terms of u_r and u_t, we have

$$\left.\begin{aligned} u_x &= \frac{D}{mE}\, u_r + \frac{\theta}{m^2 E}\, u_t \\ u_y &= \frac{\beta}{mE}\, u_r + \frac{A}{mE}\, u_t \\ u_s &= mu_r \quad + \frac{1}{m}\, u_t \end{aligned}\right\},$$

where θ is either root of the quadratic equation

$$\theta^2 - (m^2 A - mB)\,\theta + m\,(AD - FH) = 0,$$

and where

$$\beta = \frac{m}{\theta}(AD - FH),$$

β being the other root of the quadratic equation.

In general, the quadratic has distinct roots: and thus there can be two distinct systems. In the most favourable combination, each of the systems would lead to an intermediate integral; and we should then have two intermediate integrals.

If, however, the condition

$$m\,(mA - B)^2 = 4\,(AD - FH)$$

is satisfied, the quadratic has equal roots: there is only a single system, and there cannot then be more than a single intermediate integral.

In either event, the system of simultaneous equations is treated in the usual manner.

Ex. 1. Consider the equation

$$\alpha\gamma-\beta^2+\alpha\delta-\beta\gamma+\beta\delta-\gamma^2=0.$$

We have $E=1$, $F=1$, $G=1$: it thus is an example of the preceding case.

Also, there is only a single subsidiary system for u, because the quadratic equation is

$$\theta^2=0.$$

This subsidiary system is easily found to be

$$u_x=0, \quad u_y=0,$$

$$u_r-u_s+u_t=0:$$

two independent integrals are

$$r+s, \quad s+t;$$

and therefore an intermediate integral of the original equation is

$$f(r+s,\ s+t)=0.$$

A primitive of the equation is easily constructed.

Ex. 2. Integrate the equations :—

(i) $\alpha\gamma-\beta^2+\alpha\delta-\beta\gamma+\beta\delta-\gamma^2$

$$+\frac{1}{y}(s+t)(\alpha+\beta)+\frac{1}{x}(r+s)(\gamma+\delta)+\frac{1}{xy}(r+s)(s+t)=0;$$

(ii) $\alpha\gamma-\beta^2+\alpha\delta-\beta\gamma+\beta\delta-\gamma^2+a(\alpha+\beta)+b(\beta+\gamma)=0,$

where, in the latter, a and b are constants. (Tanner.)

Ex. 3. Obtain an intermediate integral of the equation

$$r^2(\beta\delta-\gamma^2)+rs(\beta\gamma-\alpha\delta)+s^2(\alpha\gamma-\beta^2)=0.$$

307. Next, take the case when $E=0$, $F=0$, and G is different from zero. The first of the relations in § 305 becomes

$$u_r=0,$$

so that u cannot involve r; and we therefore must reinvestigate from the beginning. Our differential equation is

$$G(\beta\delta-\gamma^2)+A\alpha+B\beta+C\gamma+D\delta+H=0;$$

and it is presumed to have an intermediate integral

$$u=u(x,\ y,\ z,\ p,\ q,\ s,\ t)=0.$$

Consequently, when we proceed from the relations

$$u_x + \beta u_s + \gamma u_t = 0,$$
$$u_y + \gamma u_s + \delta u_t = 0,$$

to eliminate the derivatives of the third order from the given equation, the result must be evanescent: hence

$$G\frac{u_x u_y}{u_s u_t} - B\frac{u_x}{u_s} - D\frac{u_y}{u_t} + H = 0,$$
$$G\left(\frac{u_x}{u_t} + \frac{u_y}{u_s}\right) - B\frac{u_t}{u_s} + C - D\frac{u_s}{u_t} = 0,$$
$$A = 0,$$

together, of course, with

$$u_r = 0.$$

We then resolve these equations: and we easily find that they are equivalent to the set

$$\left.\begin{aligned} u_r &= 0 \\ Gu_x - Du_s - \theta u_t &= 0 \\ Gu_y - \theta' u_s - Bu_t &= 0 \end{aligned}\right\},$$

where θ and θ' are the roots of the quadratic

$$\mu^2 + C\mu + BD - GH = 0.$$

Ex. 1. In the case of the equation

$$\beta\delta - \gamma^2 + t\beta - 2s\gamma + r\delta + rt - s^2 = 0,$$

we have

$$G = 1, \quad B = t, \quad C = -2s, \quad D = r, \quad H = rt - s^2;$$

hence

$$\theta = \theta' = s,$$

and the subsidiary equations for u are

$$\left.\begin{aligned} u_x - ru_s - su_t &= 0 \\ u_y - su_s - tu_t &= 0 \\ u_r &= 0 \end{aligned}\right\}.$$

When the system is rendered complete, it is found to possess two independent integrals

$$s + p, \quad t + q;$$

and therefore an intermediate integral exists in the form

$$f(s+p, \ t+q) = 0,$$

where f is an arbitrary function.

Ex. 2. Obtain an intermediate integral of the equation

$$pq(\beta\delta - \gamma^2) + qst\beta - (pst + qrt)\gamma + s^2p\delta - st(rt - s^2) = 0.$$

308. Next, we consider equations, for which $E=0$, $F=0$, $G=0$, and which therefore are linear of the form

$$A\alpha + B\beta + C\gamma + D\delta + H = 0.$$

Proceeding as usual to eliminate α and δ from the equation by means of the derivatives

$$u_x + \alpha u_r + \beta u_s + \gamma u_t = 0, \quad u_y + \beta u_r + \gamma u_s + \delta u_t = 0,$$

of the supposed intermediate integral $u=0$, and making the resulting equation evanescent as a relation in β and γ, we find

$$A\frac{u_s}{u_r} + D\frac{u_r}{u_t} - B = 0,$$

$$A\frac{u_t}{u_r} + D\frac{u_s}{u_t} - C = 0,$$

$$A\frac{u_x}{u_r} + D\frac{u_y}{u_t} - H = 0,$$

as the necessary and sufficient conditions. Taking

$$D\frac{u_r}{u_t} = A\mu,$$

the first of these relations gives

$$\frac{u_s}{u_r} = \frac{B}{A} - \mu;$$

and the second then becomes

$$\frac{D}{\mu} + A\mu\left(\frac{B}{A} - \mu\right) - C = 0,$$

that is,

$$A\mu^3 - B\mu^2 + C\mu - D = 0.$$

Let l, m, n be the roots of this cubic: then the three relations can be replaced by the set

$$\left.\begin{aligned} u_t &= mnu_r \\ u_s &= (m+n)\,u_r \\ A\,(u_x + lu_y) &= Hu_r \end{aligned}\right\};$$

and when the roots of the cubic are unequal, there are three such systems.

If $D=0$ and A is not zero, the inference merely is that one of the roots of the cubic is zero; and the corresponding subsidiary

systems are simplified in form. If $A=0$ and D is not zero, we interchange the variables: and we then have the preceding case.

If $A=0$ and $D=0$, it is simplest to reinvestigate the subsidiary equations from the beginning. The coefficients B and C cannot vanish simultaneously: we shall assume that B does not vanish. Then the equation

$$B\beta + C\gamma + H = 0$$

is to be an inference from

$$u_x + \alpha u_r + \beta u_s + \gamma u_t = 0, \quad u_y + \beta u_r + \gamma u_s + \delta u_t = 0,$$

and therefore quantities λ and μ must exist such that

$$\begin{aligned} H &= \lambda u_x + \mu u_y, \\ B &= \lambda u_s + \mu u_r, \\ C &= \lambda u_t + \mu u_s, \\ 0 &= \lambda u_r, \\ 0 &= \mu u_t : \end{aligned}$$

consequently, the subsidiary systems are

$$\left.\begin{aligned} u_r = 0, \quad u_t &= 0 \\ Hu_s = Bu_x &+ Cu_y \end{aligned}\right\},$$

$$\left.\begin{aligned} Cu_r &= Bu_s \\ Cu_y &= Hu_s \\ u_t &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} Bu_t &= Cu_s \\ Bu_x &= Hu_s \\ u_r &= 0 \end{aligned}\right\}.$$

In every case, we have a system or systems of subsidiary equations for the determination of u. Each of the equations is homogeneous and linear in the derivatives of u; and they can be treated by the customary Jacobian process of integration.

Ex. 1. Consider the equation

$$xy(\beta - \gamma) + xr - 2(x-y)s - yt + 2p - 2q = 0.$$

Here

$$A = 0 = D, \quad B = xy, \quad C = -xy,$$

$$H = xr - 2(x-y)s - yt + 2p - 2q.$$

The first of the subsidiary systems is

$$\frac{\partial u}{\partial r}=0,$$

$$\frac{\partial u}{\partial t}=0,$$

$$H\frac{\partial u}{\partial s}-xy\,(u_x-u_y)=0.$$

The Poisson-Jacobi conditions of coexistence of the first and third, and of the second and third, give

$$-xy\frac{\partial u}{\partial p}+x\frac{\partial u}{\partial s}=0,\quad xy\frac{\partial u}{\partial q}-y\frac{\partial u}{\partial s}=0:$$

so that we have

$$\frac{\partial u}{\partial r}=0,\quad \frac{\partial u}{\partial t}=0,\quad \frac{\partial u}{\partial s}-y\frac{\partial u}{\partial p}=0,\quad \frac{\partial u}{\partial s}-x\frac{\partial u}{\partial q}=0\,;$$

and, using these, the other equation is

$$(sy-sx+2p-2q)\frac{\partial u}{\partial s}-xy\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+xy\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0.$$

The Poisson-Jacobi conditions of coexistence of the last equation with the third and with the fourth of the modified system are satisfied in virtue of

$$\frac{\partial u}{\partial s}-xy\frac{\partial u}{\partial z}=0,$$

regard being had to all the equations. The system can be replaced by

$$\frac{\partial u}{\partial r}=0,\quad \frac{\partial u}{\partial t}=0,$$

$$y\frac{\partial u}{\partial p}=\frac{\partial u}{\partial s},\quad x\frac{\partial u}{\partial q}=\frac{\partial u}{\partial s},\quad xy\frac{\partial u}{\partial z}=\frac{\partial u}{\partial s},$$

$$xy\left(\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial s}(sy-sx+p-q)=0\,;$$

and, in this form, it is easily seen to be a complete Jacobian system. Consequently, it possesses two algebraically independent integrals: two such are

$$xys+yq+xp+z,\quad x+y\,;$$

and therefore an intermediate integral of the original equation is

$$xys+yq+xp+z=g''(x+y),$$

where g is an arbitrary function.

Proceeding in the same way with the second of the subsidiary systems, we find an intermediate integral in the form

$$xy\,(r-s)+(2y-x)\,p-yq-z=h''(x),$$

where h is an arbitrary function.

And similarly proceeding from the third of the subsidiary systems, we find an intermediate integral in the form

$$xy(t-s)-xp+(2x-y)q-z=k''(y),$$

where k is an arbitrary function.

Each one of these three intermediate integrals admits of integration. Further, they can be treated as existing simultaneously: the proof is simple. Either by integration of one of the integrals, or by quadrature that is based upon all three of them, we find a primitive in the form

$$xyz=h(x)+k(y)+g(x+y).$$

Ex. 2. Integrate the equations:—

(i) $x^3\alpha+3x^2y\beta+3xy^2\gamma+y^3\delta+2(x^2r+2xys+y^2t)=0$;

(ii) $\alpha+3u\beta+3u^2\gamma+u^3\delta=0$,

where u is given by the equation

$$r+2su+tu^2=0. \qquad \text{(Falk.)}$$

Ex. 3. Obtain an intermediate integral, involving an arbitrary function, and obtain further (as far as possible) the primitive, of the following equations:—

(i) $t(\alpha\gamma-\beta^2)-s(\alpha\delta-\beta\gamma)+r(\beta\delta-\gamma^2)=0$;

(ii)
$$q^2(\alpha\gamma-\beta^2)+pq(\alpha\delta-\beta\gamma)+p^2(\beta\delta-\gamma^2)$$
$$+2qst\alpha+2(pst-qs^2-qrt)\beta$$
$$+2prs\delta+2(qrs-ps^2-prt)\gamma=(rt-s^2)^2;$$

(iii) $(s-t)^2(\alpha\gamma-\beta^2)+(s-r)(s-t)(\alpha\delta-\beta\gamma)+(s-r)^2(\beta\gamma-\delta^2)=0$;

(iv)
$$\alpha\gamma-\beta^2+\frac{r}{a-s}(\alpha\delta-\beta\gamma)+\left(\frac{r}{a-s}\right)^2(\beta\gamma-\delta^2)$$
$$+\frac{\lambda r}{a-s}\alpha+\left\{\frac{\lambda r^2}{(a-s)^2}-\frac{\mu r}{a-s}\right\}\beta-\frac{\mu r^2}{(a-s)^2}\gamma=0,$$

where a, λ, μ are constants;

(v) $\alpha-3c\beta+3c^2\gamma-c^3\delta=0$,

where c is a constant;

(vi) $(xs+yt)\beta-(xr+ys)\gamma=\left(\frac{x}{q}-\frac{y}{p}\right)s(rt-s^2)$;

(vii) $(qs-pt)\beta-(qr-ps)\gamma=s(rt-s^2)$;

(viii) $$\beta=s\gamma. \qquad \text{(Tanner.)}$$

Ex. 4. Denoting the derivatives of z of the fourth order with respect to x and y by ι, κ, λ, μ, ν, prove that, if the equation

$$A(\iota\lambda-\kappa^2)+B(\iota\mu-\kappa\lambda)+C(\iota\nu-\kappa\mu)+D(\kappa\mu-\lambda^2)+E(\kappa\nu-\lambda\mu)+F(\lambda\nu-\mu^2)$$
$$=I\iota+K\kappa+L\lambda+M\mu+N\nu+H,$$

where $A, \ldots, F, I, \ldots, N, H$ are functions of $x, y, z, p, q, r, s, t, a, \beta, \gamma, \delta$, possesses an intermediate integral

$$\phi(u, v) = 0,$$

where ϕ is arbitrary, and where u and v are definite functions of z and its derivatives up to the third order inclusive, and of x and y, then the relation

$$AF - BE + CD = 0$$

must be satisfied, as well as two other relations which involve only these coefficients $A, \ldots, H$ in the differential equation.

Assuming the conditions satisfied, prove that u and v are independent integrals of a system

$$\left.\begin{aligned} Cw_x - Nw_a + \theta w_\delta &= 0 \\ Cw_y + \phi w_a - Iw_\delta &= 0 \\ Cw_\beta - Ew_a - Aw_\delta &= 0 \\ Cw_\gamma - Fw_a - Bw_\delta &= 0 \end{aligned}\right\},$$

where

$$(AF - C^2)\,\theta = -KC^2 + MAC - NAB + IEC,$$
$$(AF - C^2)\,\phi = KCF - MC^2 + NBC - IEF.$$

Equations having Intermediate Integrals.

309. Proceeding now with the equation of general form and dealing first with the case when there are three equations for the determination of u, where $u = 0$ is the supposed intermediate integral, we may suppose that these equations (which are homogeneous of order zero in u_x, u_y, u_r, u_s, u_t) are resolved so as to express three of these quantities, say u_x, u_y, u_s, in terms of the other two. Let

$$x, y, z, p, q, r, s, t = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8,$$

respectively; and let

$$\frac{\partial u}{\partial x_i} = p_i,$$

for $i = 1, \ldots, 8$; then the three equations can be taken in the form

$$\Delta = p_1 + x_4 p_3 + x_6 p_4 + x_7 p_5 + h(x_1, \ldots, x_8, p_6, p_8) = 0,$$
$$\Delta' = p_2 + x_5 p_3 + x_7 p_4 + x_8 p_5 + k(x_1, \ldots, x_8, p_6, p_8) = 0,$$
$$\Delta'' = p_7 \qquad\qquad\qquad\qquad + l(x_1, \ldots, x_8, p_6, p_8) = 0,$$

where each of the functions h, k, l is homogeneous of the first order in p_6 and p_8.

As $\Delta = 0$, $\Delta' = 0$, $\Delta'' = 0$, are simultaneous equations of the first order in one dependent variable, they must satisfy the Poisson-Jacobi conditions of coexistence

$$(\Delta, \Delta') = 0, \quad (\Delta, \Delta'') = 0, \quad (\Delta', \Delta'') = 0.$$

The condition $(\Delta, \Delta'') = 0$, when expressed in full, is

$$\frac{\partial l}{\partial x_1} + x_4 \frac{\partial l}{\partial x_3} + x_6 \frac{\partial l}{\partial x_4} + x_7 \frac{\partial l}{\partial x_5} - \left(p_5 + \frac{\partial h}{\partial x_7}\right) + \frac{\partial (h, l)}{\partial (p_6, x_6)} + \frac{\partial (h, l)}{\partial (p_8, x_8)} = 0;$$

and the condition $(\Delta', \Delta'') = 0$, when expressed in full, is

$$\frac{\partial l}{\partial x_2} + x_5 \frac{\partial l}{\partial x_3} + x_7 \frac{\partial l}{\partial x_4} + x_8 \frac{\partial l}{\partial x_5} - \left(p_4 + \frac{\partial k}{\partial x_7}\right) + \frac{\partial (k, l)}{\partial (p_6, x_6)} + \frac{\partial (k, l)}{\partial (p_8, x_8)} = 0.$$

Before using the other condition, it is convenient to use these two relations: they clearly are independent of the three equations already obtained, and so they are new equations expressing p_5 and p_4 respectively in terms of the other quantities. Let

$$\Delta_4 = p_4 + f_4 (x_1, \ldots, x_8, p_6, p_8) = 0,$$
$$\Delta_5 = p_5 + f_5 (x_1, \ldots, x_8, p_6, p_8) = 0:$$

substituting these values, let Δ become ∇, where

$$\nabla = p_1 + x_4 p_3 + h' (x_1, \ldots, x_8, p_6, p_8) = 0.$$

Now we must have

$$(\nabla, \Delta_4) = 0:$$

expressed in full, the condition is

$$\frac{\partial f_4}{\partial x_1} + x_4 \frac{\partial f_4}{\partial x_3} - \left(p_3 + \frac{\partial h'}{\partial x_4}\right) + \frac{\partial (h', f_4)}{\partial (p_6, x_6)} + \frac{\partial (h', f_4)}{\partial (p_8, x_8)} = 0.$$

This again is obviously a new equation: and it expresses p_3 in terms of the other quantities. Inserting this value, and gathering together the various equations, we have

$$\left.\begin{aligned}
0 = \Delta_1 &= p_1 + f_1 (x_1, \ldots, x_8, p_6, p_8) = p_1 + p_8 \phi_1 (x_1, \ldots, x_8, m)\\
0 = \Delta_2 &= p_2 + f_2 (x_1, \ldots, x_8, p_6, p_8) = p_2 + p_8 \phi_2 (x_1, \ldots, x_8, m)\\
0 = \Delta_3 &= p_3 + f_3 (x_1, \ldots, x_8, p_6, p_8) = p_3 + p_8 \phi_3 (x_1, \ldots, x_8, m)\\
0 = \Delta_4 &= p_4 + f_4 (x_1, \ldots, x_8, p_6, p_8) = p_4 + p_8 \phi_4 (x_1, \ldots, x_8, m)\\
0 = \Delta_5 &= p_5 + f_5 (x_1, \ldots, x_8, p_6, p_8) = p_5 + p_8 \phi_5 (x_1, \ldots, x_8, m)\\
0 = \Delta_7 &= p_7 + f_7 (x_1, \ldots, x_8, p_6, p_8) = p_7 + p_8 \phi_7 (x_1, \ldots, x_8, m)
\end{aligned}\right\},$$

where

$$m = \frac{p_6}{p_8},$$

all the functions $f_1, f_2, f_3, f_4, f_5, f_7$ being homogeneous of the first order in p_6 and p_8. Thus

$$\frac{\partial f_\alpha}{\partial p_6} = \frac{\partial \phi_\alpha}{\partial m}, \quad \frac{\partial f_\alpha}{\partial p_8} = \phi_\alpha - m\frac{\partial \phi_\alpha}{\partial m};$$

and so the Poisson-Jacobi conditions of coexistence, being

$$(\Delta_i, \Delta_j) = 0,$$

take the forms

$$\frac{\partial \phi_j}{\partial x_i} - \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_i}{\partial m}\frac{\partial \phi_j}{\partial x_6} - \frac{\partial \phi_j}{\partial m}\frac{\partial \phi_i}{\partial x_6}$$

$$+ \left(\phi_i - m\frac{\partial \phi_i}{\partial m}\right)\frac{\partial \phi_j}{\partial x_8} - \left(\phi_j - m\frac{\partial \phi_j}{\partial m}\right)\frac{\partial \phi_i}{\partial x_8} = 0,$$

on the removal of a superfluous factor p_8.

The above system of equations will be regarded as complete, in order that there may be an intermediate integral; hence these relations, for all the combinations $i, j = 1, 2, 3, 4, 5, 7$, must be satisfied. It is then obvious that they can only be satisfied identically.

Suppose that functions $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_7$ are known, satisfying the foregoing conditions: then the Jacobian system is complete, and it possesses integrals satisfying all the equations, that is, there is an intermediate integral. As, however, the equations are no longer necessarily linear and homogeneous in the derivatives of u, we cannot declare that the intermediate integral necessarily involves an arbitrary functional form, though it will involve some arbitrary element. Let it be denoted by

$$u = 0;$$

then, as

$$mp_8 = p_6,$$

we have

$$m\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}.$$

The equation of the third order is given by the elimination of the arbitrary element and of m between the equations

and

$$\left.\begin{aligned} m\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial r} \\ \alpha m + \beta\phi_7 + \gamma + \phi_1 + p\phi_3 + r\phi_4 + s\phi_5 &= 0 \\ \beta m + \gamma\phi_7 + \delta + \phi_2 + q\phi_3 + s\phi_4 + t\phi_5 &= 0 \end{aligned}\right\}.$$

The development of the analysis follows a course similar to that adopted (in Chapter XX) for the corresponding questions relating to equations of the second order in two independent variables.

Ex. 1. Prove that the equation

$$\alpha^2\delta^2 + \alpha\gamma^3 + \beta^3\delta = 3\alpha\beta\gamma\delta$$

possesses an intermediate integral, in the form of an equation of the second order involving two independent arbitrary constants. Obtain this integral: and deduce a primitive.

Ex. 2. Obtain intermediate integrals of the equations :—

(i) $\alpha\delta - \beta\gamma = 0$;

(ii) $(\beta\delta - \gamma^2)^2 r + (\beta\gamma - \alpha\delta)(\beta\delta - \gamma^2) s + (\alpha\gamma - \beta^2)(\beta\delta - \gamma^2) t$
$$= (\beta\gamma - \alpha\delta)(\alpha\gamma - \beta^2);$$

(iii) $(\alpha\delta - \beta\gamma)\{(\alpha\delta - \beta\gamma) r + 2(\beta^2 - \alpha\gamma) s\}$
$$+ \{(\alpha\delta - \beta\gamma) t + 2(\gamma^2 - \beta\delta) s\}^2 = 0.$$

310. Next dealing with the case where four equations arise in the process of § 304 for the determination of u, we may suppose these equations resolved for the four ratios $u_x : u_y : u_r : u_s : u_t$, in a form

$$\left.\begin{aligned} u_x + u_s\theta_1 &= 0 \\ u_y + u_s\theta_2 &= 0 \\ u_r + u_s\theta_3 &= 0 \\ u_t + u_s\theta_4 &= 0 \end{aligned}\right\},$$

where θ_1, θ_2, θ_3, θ_4 can be functions of all the eight variables, subject of course to the necessary conditions of coexistence of the four equations.

It is, however, comparatively unnecessary to discuss the detailed development of this case: for even when the conditions are satisfied, so that an intermediate integral exists, that integral

leads to two equations of the third order and not to only a single equation. In fact, these equations are

$$\left.\begin{aligned}\theta_1 + \alpha\theta_3 + \beta + \gamma\theta_4 = 0\\ \theta_2 + \beta\theta_3 + \gamma + \delta\theta_4 = 0\end{aligned}\right\};$$

we shall not further consider the case.

Ex. Obtain intermediate integrals of the equations :—

(i) $$\frac{a\gamma - \beta^2}{r} = \frac{a\delta - \beta\gamma}{2s} = \frac{\beta\delta - \gamma^2}{t};$$

(ii) $$\left.\begin{aligned}(ta + r\gamma)(t\beta + r\delta) = 4rt\beta\gamma\\ t(a\gamma - \beta^2) - r(\beta\delta - \gamma^2) = 0\end{aligned}\right\}.$$

Ampère's Method applied to Equations of the Third Order.

311. When a given equation of the third order does not possess an intermediate integral, in the form of an equation of the second order involving an arbitrary element, so that the preceding method does not apply, we still may be able to proceed to a primitive by applying Ampère's method, as used for equations of the second order, or Darboux's method, for the construction of compatible equations of the third or of higher order.

Using an extension of Ampère's method, we denote by u the argument of any one of the three arbitrary functions that occur in the integral equivalent, supposed to be free from partial quadratures: and we change the independent variables from x and y to x and u, on the assumption that u is not a function of x alone. Adopting the notation of Chapter XVII and denoting derivatives with regard to x and u by $\frac{\delta}{\delta x}$ and $\frac{\delta}{\delta u}$, we have the former relations, viz.

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}, \quad \frac{\delta z}{\delta u} = q\frac{\delta y}{\delta u},$$

$$\frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x}, \quad \frac{\delta p}{\delta u} = s\frac{\delta y}{\delta u},$$

$$\frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x}, \quad \frac{\delta q}{\delta u} = t\frac{\delta y}{\delta u},$$

as well as the further relations

$$\frac{\delta r}{\delta x} = \alpha + \beta \frac{\delta y}{\delta x}, \quad \frac{\delta r}{\delta u} = \beta \frac{\delta y}{\delta u},$$

$$\frac{\delta s}{\delta x} = \beta + \gamma \frac{\delta y}{\delta x}, \quad \frac{\delta s}{\delta u} = \gamma \frac{\delta y}{\delta u},$$

$$\frac{\delta t}{\delta x} = \gamma + \delta \frac{\delta y}{\delta x}, \quad \frac{\delta t}{\delta u} = \delta \frac{\delta y}{\delta u}.$$

Keeping the value of δ as given by the last equation, viz.,

$$\delta = \frac{\delta t}{\delta u} \div \frac{\delta y}{\delta u},$$

we have

$$\gamma = \frac{\delta t}{\delta x} - \delta \frac{\delta y}{\delta x},$$

$$\beta = \frac{\delta s}{\delta x} - \frac{\delta t}{\delta x}\frac{\delta y}{\delta x} + \delta \left(\frac{\delta y}{\delta x}\right)^2,$$

$$\alpha = \frac{\delta r}{\delta x} - \frac{\delta s}{\delta x}\frac{\delta y}{\delta x} + \frac{\delta t}{\delta x}\left(\frac{\delta y}{\delta x}\right)^2 - \delta \left(\frac{\delta y}{\delta x}\right)^3.$$

Let these values of α, β, γ be substituted in the given equation

$$f(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0,$$

which will be supposed to be a polynomial in the derivatives of the third order. After the substitution, f will become a polynomial in δ alone; thus the equation, arranged in powers of δ, acquires a form

$$Q_0 + Q_1\delta + \ldots + Q_m\delta^m = 0,$$

where the original degree of f as a polynomial in α, β, γ, δ is m at least.

Now the equation is to be satisfied identically when the proper value of z is substituted. In that value, there occur an arbitrary function of u and its derivatives up to finite order; and these derivatives occur in p, q, r, s, t. Now, in the derivative $\frac{\delta t}{\delta u}$, the order of the highest derivative of the arbitrary function is greater than the order of the derivatives of the arbitrary function which occur in any of the quantities $Q_0, \ldots, Q_m$: that is, in the transformed equation, the quantity δ contains higher derivatives of the arbitrary function than occur elsewhere. The transformed equation must be satisfied identically in connection with the integral system:

when account is taken of the successive powers of δ, it is easy to see that the requirement as to the equation can be fulfilled, only if

$$Q_m = 0, \quad Q_{m-1} = 0, \ldots, Q_1 = 0, \quad Q_0 = 0.$$

Further, having regard to the preceding values of α, β, γ substituted in $f=0$, we see that the equation

$$\frac{\partial f}{\partial \delta} - \frac{\partial f}{\partial \gamma}\frac{\delta y}{\delta x} + \frac{\partial f}{\partial \beta}\left(\frac{\delta y}{\delta x}\right)^2 - \frac{\partial f}{\partial \alpha}\left(\frac{\delta y}{\delta x}\right)^3 = 0$$

must be satisfied: but it is not additional to the other equations, being satisfied in virtue of them and of the subsidiary equations. Now $\frac{\delta y}{\delta x}$ is given by the relation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\delta y}{\delta x} = 0;$$

and therefore

$$\frac{\partial f}{\partial \delta}\left(\frac{\partial u}{\partial y}\right)^3 + \frac{\partial f}{\partial \gamma}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial u}{\partial x} + \frac{\partial f}{\partial \beta}\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial f}{\partial \alpha}\left(\frac{\partial u}{\partial x}\right)^3 = 0,$$

which accordingly is an equation satisfied by the argument of any arbitrary function that occurs in the integral equivalent of the given equation, on the hypothesis adopted as to the character of that equivalent.

Reverting to the earlier relations, we see that they give a number of simultaneous equations. If these equations are consistent with one another, and are also consistent with $f=0$, regard being paid to the relations between the derivatives relative to the old independent variables and the new, then the original equation can possess an integral system of the specified type. The quantities $Q_0, Q_1, \ldots, Q_m$ contain z, p, q, r, s, t, and also the derivatives of these with regard to x: and we also have

$$\frac{\delta z}{\delta x} = p + q\frac{\delta y}{\delta x}, \quad \frac{\delta p}{\delta x} = r + s\frac{\delta y}{\delta x}, \quad \frac{\delta q}{\delta x} = s + t\frac{\delta y}{\delta x}.$$

Thus the system of equations contains no derivatives with regard to u: it can be regarded as a system of simultaneous ordinary equations.

Ex. 1. Consider the equation

$$\beta - \gamma = \frac{r-t}{x+y}.$$

When we substitute the values of β and γ in terms of δ, we have two equations after the application of the preceding process. One of these equations is

$$\left(\frac{\delta y}{\delta x}\right)^2+\frac{\delta y}{\delta x}=0:$$

as this is a degenerate form of the cubic, the arguments of the three arbitrary functions that occur in the integral equivalent are

$$y, \quad x, \quad x+y.$$

The remaining equation is

$$\frac{\delta s}{\delta x}-\frac{\delta t}{\delta x}\frac{\delta y}{\delta x}-\frac{\delta t}{\delta x}=\frac{r-t}{x+y}.$$

Taking the argument u, where

$$u=x+y,$$

we have this equation in the form

$$u\frac{\delta s}{\delta x}=r-t,$$

where u is constant in derivation with respect to x. Also,

$$\frac{\delta p}{\delta x}=r+s\frac{\delta y}{\delta x}=r-s,$$

$$\frac{\delta q}{\delta x}=s+t\frac{\delta y}{\delta x}=s-t,$$

in the present case; thus

$$\frac{\delta p}{\delta x}+\frac{\delta q}{\delta x}=r-t,$$

and therefore

$$u\frac{\delta s}{\delta x}=\frac{\delta p}{\delta x}+\frac{\delta q}{\delta x},$$

so that, as u is parametric, we have

$$us=p+q+\text{constant}.$$

The constant on the right-hand side is subject to the constancy of u: let it be

$$u\theta''(u)-2\theta'(u),$$

where θ is any arbitrary function. Thus

$$us=p+q+u\theta''(u)-2\theta'(u),$$

and therefore

$$s=\frac{p+q}{x+y}+\theta''(x+y)-\frac{2}{x+y}\theta'(x+y),$$

which is an intermediate integral.

The primitive is

$$z=X+Y-\tfrac{1}{2}(x+y)(X'+Y')+\theta(x+y),$$

where X and Y are arbitrary functions of x and of y respectively.

Ex. 2. Integrate the equation

$$\beta+\gamma=\frac{r+2s+t}{x+y};$$

and obtain a primitive of

$$(x-2cy)^2(c\beta+\gamma)=(x-2cy)\,t+2cq$$

in the form

$$z=X+2c\{Y+\theta(x-cy)\}-(x-2cy)\{Y'-c\theta'(x-cy)\},$$

where X, Y, $\theta(x-cy)$ are arbitrary functions of x, of y, and of $x-cy$ respectively.

Darboux's Method applied to Equations of the Third Order.

312. It is natural to consider the extension of Darboux's method, as explained in Chapter XVIII, to equations of order higher than the second. When it appears that an equation

$$f=f(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta)=0$$

has no intermediate integral in the form of an equation of the second order, the method seeks to obtain a new equation of the third order, say

$$u=u(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta)=0,$$

which may coexist with $f=0$, though it is not resoluble into $f=0$. Let the derivatives of z of the fourth order be denoted by ι, κ, λ, μ, ν: and write

$$\frac{d}{dx}=\frac{\partial}{\partial x}+p\frac{\partial}{\partial z}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q}+\alpha\frac{\partial}{\partial r}+\beta\frac{\partial}{\partial s}+\gamma\frac{\partial}{\partial t},$$

$$\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}+\beta\frac{\partial}{\partial r}+\gamma\frac{\partial}{\partial s}+\delta\frac{\partial}{\partial t}.$$

Then, in accordance with the earlier explanations, we assign the conditions that the equations

$$\left.\begin{aligned}
0&=\frac{df}{dx}+\iota\frac{\partial f}{\partial \alpha}+\kappa\frac{\partial f}{\partial \beta}+\lambda\frac{\partial f}{\partial \gamma}+\mu\frac{\partial f}{\partial \delta}\\
0&=\frac{df}{dy}\qquad+\kappa\frac{\partial f}{\partial \alpha}+\lambda\frac{\partial f}{\partial \beta}+\mu\frac{\partial f}{\partial \gamma}+\nu\frac{\partial f}{\partial \delta}\\
0&=\frac{du}{dx}+\iota\frac{\partial u}{\partial \alpha}+\kappa\frac{\partial u}{\partial \beta}+\lambda\frac{\partial u}{\partial \gamma}+\mu\frac{\partial u}{\partial \delta}\\
0&=\frac{du}{dy}\qquad+\kappa\frac{\partial u}{\partial \alpha}+\lambda\frac{\partial u}{\partial \beta}+\mu\frac{\partial u}{\partial \gamma}+\nu\frac{\partial u}{\partial \delta}
\end{aligned}\right\}$$

are not linearly independent of one another. That this may be the case, quantities l, m, n must exist such that

$$\begin{aligned}
&\frac{du}{dx} + l\,\frac{du}{dy} - m\,\frac{df}{dx} - n\,\frac{df}{dy} = 0,\\
&\frac{\partial u}{\partial \alpha} \qquad\quad - m\,\frac{\partial f}{\partial \alpha} \qquad\quad = 0,\\
&\frac{\partial u}{\partial \beta} + l\,\frac{\partial u}{\partial \alpha} - m\,\frac{\partial f}{\partial \beta} - n\,\frac{\partial f}{\partial \alpha} = 0,\\
&\frac{\partial u}{\partial \gamma} + l\,\frac{\partial u}{\partial \beta} - m\,\frac{\partial f}{\partial \gamma} - n\,\frac{\partial f}{\partial \beta} = 0,\\
&\frac{\partial u}{\partial \delta} + l\,\frac{\partial u}{\partial \gamma} - m\,\frac{\partial f}{\partial \delta} - n\,\frac{\partial f}{\partial \gamma} = 0,\\
&\qquad\; l\,\frac{\partial u}{\partial \delta} \qquad\quad - n\,\frac{\partial f}{\partial \delta} = 0.
\end{aligned}$$

Now the quantity $n - lm$ cannot be zero: for if it were, we should have

$$\frac{\partial u}{\partial \alpha} = m\,\frac{\partial f}{\partial \alpha}, \quad \frac{\partial u}{\partial \beta} = m\,\frac{\partial f}{\partial \beta}, \quad \frac{\partial u}{\partial \gamma} = m\,\frac{\partial f}{\partial \gamma}, \quad \frac{\partial u}{\partial \delta} = m\,\frac{\partial f}{\partial \delta},$$

and u would not be functionally independent of f, so far as concerns α, β, γ, δ. Multiply the second equation by l^4, the third by $-l^3$, the fourth by l^2, the fifth by $-l$, and add all these to the sixth: then we have

$$(n - ml)\left(l^3\,\frac{\partial f}{\partial \alpha} - l^2\,\frac{\partial f}{\partial \beta} + l\,\frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \delta}\right) = 0,$$

so that l is a root of the equation

$$\theta^3\,\frac{\partial f}{\partial \alpha} - \theta^2\,\frac{\partial f}{\partial \beta} + \theta\,\frac{\partial f}{\partial \gamma} - \frac{\partial f}{\partial \delta} = 0.$$

Let ρ, σ, τ be the three roots of this cubic, and let

$$l = \tau;$$

we shall assume that ρ, σ, τ are unequal. Again, multiply those equations in order by ρ^3, $-\rho^2$, ρ, -1, $\frac{1}{\rho}$ respectively, and add: we have

$$\left(1 - \frac{l}{\rho}\right)\left(\rho^3\frac{\partial u}{\partial \alpha} - \rho^2\,\frac{\partial u}{\partial \beta} + \rho\,\frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta}\right) = 0,$$

and l is not equal to ρ; so that

$$\rho^3\frac{\partial u}{\partial \alpha}-\rho^2\frac{\partial u}{\partial \beta}+\rho\frac{\partial u}{\partial \gamma}-\frac{\partial u}{\partial \delta}=0.$$

Similarly,

$$\sigma^3\frac{\partial u}{\partial \alpha}-\sigma^2\frac{\partial u}{\partial \beta}+\sigma\frac{\partial u}{\partial \gamma}-\frac{\partial u}{\partial \delta}=0.$$

Also

$$m=\frac{\dfrac{\partial u}{\partial \alpha}}{\dfrac{\partial f}{\partial \alpha}}$$

$$n=l\frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial \delta}}=\frac{1}{\rho\sigma}\frac{\dfrac{\partial u}{\partial \delta}}{\dfrac{\partial f}{\partial \alpha}};$$

and therefore the first equation is

$$\left(\frac{du}{dx}+\tau\frac{du}{dy}\right)\frac{\partial f}{\partial \alpha}=\frac{df}{dx}\frac{\partial u}{\partial \alpha}+\frac{1}{\rho\sigma}\frac{df}{dy}\frac{\partial u}{\partial \delta}.$$

With each arrangement of the roots of

$$\theta^3\frac{\partial f}{\partial \alpha}-\theta^2\frac{\partial f}{\partial \beta}+\theta\frac{\partial f}{\partial \gamma}-\frac{\partial f}{\partial \delta}=0,$$

we have three equations satisfied by u; one integral obviously is $u=f$. If they possess an integral, which is distinct from f and involves some of the derivatives α, β, γ, δ, then

$$u=0$$

is a new equation independent of, and compatible with, $f=0$. The test, as to whether they do or do not possess such an integral, is obtained as usual: the set of partial equations of the first order in u is made a complete Jacobian system. If when thus completed, the system contains n equations, it possesses $11-n$ new integrals: for there are twelve variables that can occur, and f is certainly an integral.

It may happen that one distribution of the roots of the cubic may provide a system which possesses new integrals, and that another distribution does not. The most favourable case occurs when three integrals are provided: the least favourable case occurs when no integrals are obtained. Moreover, when $n=9$ for any

system, being the value of n which often occurs when the method proves effective, there are two integrals, say u_1 and u_2; the most general integral is then $\phi(u_1, u_2)$, where ϕ is arbitrary: and the equation compatible with $f=0$ is

$$\phi(u_1, u_2) = 0.$$

When the process leads to no such integral, then we attempt to find an equation of the fourth order compatible with, but not composed of, the equations

$$\frac{df}{dx} = 0, \qquad \frac{df}{dy} = 0,$$

the complete derivatives of $f=0$.

In all cases, the subsidiary equations in this extension of Darboux's method are homogeneous and linear of the first order.

313. Instead of subsidiary equations which are linear and homogeneous partial equations of the first order, we can obtain a subsidiary system in differential elements as follows. Let

$$X = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q} + \alpha\frac{\partial f}{\partial r} + \beta\frac{\partial f}{\partial s} + \gamma\frac{\partial f}{\partial t},$$

$$Y = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q} + \beta\frac{\partial f}{\partial r} + \gamma\frac{\partial f}{\partial s} + \delta\frac{\partial f}{\partial t},$$

$$A = \frac{\partial f}{\partial \alpha}, \quad B = \frac{\partial f}{\partial \beta}, \quad C = \frac{\partial f}{\partial \gamma}, \quad D = \frac{\partial f}{\partial \delta};$$

then, because of the equation

$$f = 0,$$

we have

$$A\frac{\partial \alpha}{\partial x} + B\frac{\partial \beta}{\partial x} + C\frac{\partial \gamma}{\partial x} + D\frac{\partial \delta}{\partial x} = -X,$$

$$A\frac{\partial \alpha}{\partial y} + B\frac{\partial \beta}{\partial y} + C\frac{\partial \gamma}{\partial y} + D\frac{\partial \delta}{\partial y} = -Y.$$

Now the system

$$\begin{aligned} dr &= \alpha dx + \beta dy, & dp &= r dx + s dy, \\ ds &= \beta dx + \gamma dy, & dq &= s dx + t dy, \\ dt &= \gamma dx + \delta dy, & dz &= p dx + q dy, \end{aligned}$$

is to be perfectly integrable; hence, among other relations, we must have

$$\frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y}, \quad \frac{\partial \gamma}{\partial x} = \frac{\partial \beta}{\partial y}, \quad \frac{\partial \delta}{\partial x} = \frac{\partial \gamma}{\partial y},$$

the derivatives with regard to x and y being complete. Hence

$$\begin{aligned}
d\alpha &= \frac{\partial\alpha}{\partial x}\,dx + \frac{\partial\beta}{\partial x}\,dy,\\
d\beta &= \frac{\partial\beta}{\partial x}\,dx + \frac{\partial\gamma}{\partial x}\,dy = \frac{\partial\alpha}{\partial y}\,dx + \frac{\partial\beta}{\partial y}\,dy,\\
d\gamma &= \frac{\partial\gamma}{\partial x}\,dx + \frac{\partial\delta}{\partial x}\,dy = \frac{\partial\beta}{\partial y}\,dx + \frac{\partial\gamma}{\partial y}\,dy,\\
d\delta &\qquad\qquad\qquad\qquad\quad = \frac{\partial\gamma}{\partial y}\,dx + \frac{\partial\delta}{\partial y}\,dy.
\end{aligned}$$

In accordance with Hamburger's method of procedure (§§ 167, 173), we form the combinations

$$\begin{aligned}
\lambda_1 d\alpha + \lambda_2 d\beta + \lambda_3 d\gamma &= \frac{\partial\alpha}{\partial x}\,\lambda_1 dx + \frac{\partial\beta}{\partial x}(\lambda_2 dx + \lambda_1 dy)\\
&\quad + \frac{\partial\gamma}{\partial x}(\lambda_3 dx + \lambda_2 dy) + \frac{\partial\delta}{\partial x}\,\lambda_3 dy,\\
\lambda_1 d\beta + \lambda_2 d\gamma + \lambda_3 d\delta &= \frac{\partial\alpha}{\partial y}\,\lambda_1 dx + \frac{\partial\beta}{\partial y}(\lambda_2 dx + \lambda_1 dy)\\
&\quad + \frac{\partial\gamma}{\partial y}(\lambda_3 dx + \lambda_2 dy) + \frac{\partial\delta}{\partial y}\,\lambda_3 dy;
\end{aligned}$$

and then, comparing these with the two first derivatives of $f = 0$, we construct the linear equations

$$\begin{aligned}
&\frac{\lambda_1 dx}{A} = \frac{\lambda_2 dx + \lambda_1 dy}{B} = \frac{\lambda_3 dx + \lambda_2 dy}{C} = \frac{\lambda_3 dy}{D}\\
&= \frac{\lambda_1 d\alpha + \lambda_2 d\beta + \lambda_3 d\gamma}{-X} = \frac{\lambda_1 d\beta + \lambda_2 d\gamma + \lambda_3 d\delta}{-Y}.
\end{aligned}$$

The equality of the first four fractions determines τ and the ratios $\lambda_1 : \lambda_2 : \lambda_3$, where

$$dy = \tau\, dx.$$

Writing each of the fractions as equal to Jdx, we have

$$\begin{aligned}
\lambda_1 &= JA,\\
\lambda_2 &= J(B - \tau A),\\
\lambda_3 &= J(C - \tau B + \tau^2 A),\\
\tau\lambda_3 &= JD:
\end{aligned}$$

hence τ is determined by the cubic

$$A\tau^3 - B\tau^2 + C\tau - D = 0,$$

the significance of which will appear later. Using these values of $\lambda_1, \lambda_2, \lambda_3$, we further have

$$A\,d\alpha + (B - \tau A)\,d\beta + \frac{1}{\tau}D\,d\gamma = -X\,dx,$$

$$A\,d\beta + (B - \tau A)\,d\gamma + \frac{1}{\tau}D\,d\delta = -Y\,dx,$$

together with

$$\begin{aligned} dz &= (p + \tau q)\,dx, & dr &= (\alpha + \tau\beta)\,dx, \\ dp &= (r + \tau s)\,dx, & ds &= (\beta + \tau\gamma)\,dx, \\ dq &= (s + \tau t)\,dx, & dt &= (\gamma + \tau\delta)\,dx. \end{aligned}$$

The first two of these equations can be modified. Let τ denote any root of the cubic equation, and let the other two roots be denoted by ρ and σ: then

$$B - \tau A = (\rho + \sigma)\,A,$$

$$\frac{1}{\tau}D = \rho\sigma A,$$

and so the first two equations become

$$d\alpha + (\rho + \sigma)\,d\beta + \rho\sigma\,d\gamma = -\frac{X}{A}\,dx,$$

$$d\beta + (\rho + \sigma)\,d\gamma + \rho\sigma\,d\delta = -\frac{Y}{A}\,dx.$$

Whichever form be adopted, we have a system of equations linear in the differential elements; and permutation of the roots of the cubic, when these are unequal, gives three such systems.

What is desired, in every case, is an integrable combination of the equations. The following process leads to the subsidiary equations in Darboux's method.

314. Let du be a linear combination of the equations of a system which is an exact differential: then multipliers $\lambda_1, \ldots, \lambda_9$ must exist such that the relation

$$\begin{aligned} du = {} & \lambda_1\,(dy - \tau\,dx) + \lambda_2\,\{dz - (p + \tau q)\,dx\} \\ & + \lambda_3\,\{dp - (r + \tau s)\,dx\} + \lambda_4\,\{dq - (s + \tau t)\,dx\} \\ & + \lambda_5\,\{dr - (\alpha + \tau\beta)\,dx\} + \lambda_6\,\{ds - (\beta + \tau\gamma)\,dx\} \\ & + \lambda_7\,\{dt - (\gamma + \tau\delta)\,dx\} + \lambda_8\left\{d\alpha + (\rho + \sigma)\,d\beta + \rho\sigma\,d\gamma + \frac{X}{A}\,dx\right\} \\ & + \lambda_9\left\{d\beta + (\rho + \sigma)\,d\gamma + \rho\sigma\,d\delta + \frac{Y}{A}\,dx\right\} \end{aligned}$$

holds identically, these multipliers being free from differential elements. We at once have

$$\lambda_1 = \frac{\partial u}{\partial y}, \quad \lambda_2 = \frac{\partial u}{\partial z},$$

$$\lambda_3 = \frac{\partial u}{\partial p}, \quad \lambda_4 = \frac{\partial u}{\partial q},$$

$$\lambda_5 = \frac{\partial u}{\partial r}, \quad \lambda_6 = \frac{\partial u}{\partial s}, \quad \lambda_7 = \frac{\partial u}{\partial t};$$

also

$$\frac{\partial u}{\partial \alpha} = \lambda_8,$$

$$\frac{\partial u}{\partial \beta} = \lambda_8 (\rho + \sigma) + \lambda_9,$$

$$\frac{\partial u}{\partial \gamma} = \lambda_8 \rho\sigma + \lambda_9 (\rho + \sigma),$$

$$\frac{\partial u}{\partial \delta} = \lambda_9 \rho\sigma;$$

and

$$\frac{\partial u}{\partial x} = \lambda_9 \frac{Y}{A} + \lambda_8 \frac{X}{A} - \lambda_5 (\alpha + \tau\beta) - \lambda_6 (\beta + \tau\gamma) - \lambda_7 (\gamma + \tau\delta)$$
$$- \lambda_3 (r + \tau s) - \lambda_4 (s + \tau t) - \lambda_2 (p + \tau q) - \lambda_1 \tau.$$

Hence

$$\left.\begin{aligned} \frac{du}{dx} + \tau \frac{du}{dy} - \frac{X}{A}\frac{\partial u}{\partial \alpha} - \frac{Y}{A\rho\sigma}\frac{\partial u}{\partial \delta} &= 0 \\ \rho^3 \frac{\partial u}{\partial \alpha} - \rho^2 \frac{\partial u}{\partial \beta} + \rho \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} &= 0 \\ \sigma^3 \frac{\partial u}{\partial \alpha} - \sigma^2 \frac{\partial u}{\partial \beta} + \sigma \frac{\partial u}{\partial \gamma} - \frac{\partial u}{\partial \delta} &= 0 \end{aligned}\right\}.$$

These are the partial differential equations in Darboux's method; they can be used to determine integrable combinations (if any) of the subsidiary system in the differential elements. One such combination is f: it is ineffective for our purpose, because $f = 0$ is the original equation: and so what is required is some other combination.

Moreover, by permuting the roots of the cubic

$$A\tau^3 - B\tau^2 + C\tau - D = 0,$$

we shall have three systems; and we proceed to obtain integrable combinations (if any) other than f belonging to each of the systems. The most favourable case occurs *when each of the systems gives an integrable combination involving* α, β, γ, δ: *if these be*

$$u=0, \quad v=0, \quad w=0,$$

they can be combined with $f=0$, *so as to express* α, β, γ, δ *in terms of the other variables: and the construction of the primitive is then merely a matter of quadratures.* The proof is the same as for former similar propositions.

Ex. 1. Consider the equation

$$\alpha+a\beta-\gamma-a\delta=\frac{2}{x}(r+as),$$

where a is a constant unequal to 1 or -1. The critical cubic is

$$\theta^3-a\theta^2-\theta+a=0;$$

the roots of this cubic are -1, $+1$, a: and therefore the arguments of the arbitrary functions in the primitive are

$$y+x, \quad y-x, \quad y-ax.$$

Taking l, m, $n=-1$, 1, a in some order, we have the equations subsidiary to an intermediate integral of the second order (if it exists) in the form

$$\begin{aligned}[u_x]+l\,[u_y]-\frac{2}{x}(r+as)\,u_r&=0,\\ u_s-(m+n)\,u_r&=0,\\ u_t-mn&=0,\end{aligned}$$

where

$$\begin{aligned}[u_x]&=u_x+pu_z+ru_p+su_q,\\ [u_y]&=u_y+qu_z+su_p+tu_q.\end{aligned}$$

It is not difficult to prove that these equations do not possess a common integral; hence there is no intermediate integral.

We must therefore seek to construct some equation or equations in α, β, γ, δ, which are compatible with, but not resoluble into, the given equation. Taking the preceding method, we shall have three subsidiary systems, corresponding respectively to the three arrangements

$$\begin{array}{llll}\text{(i)} & \tau=\ \ a; & \rho,\ \sigma=\ \ 1, & -1;\\ \text{(ii)} & \tau=\ \ 1; & \rho,\ \sigma=-1, & a;\\ \text{(iii)} & \tau=-1; & \rho,\ \sigma=\ \ 1, & a.\end{array}$$

The subsidiary system for the distribution (i), when made a complete Jacobian system, is found to contain ten equations; it thus possesses two integrals. One of these must be

$$\alpha+a\beta-\gamma-a\delta-\frac{2}{x}(r+as),$$

which vanishes owing to the original equation: the other is

$$y - ax,$$

which is not useful for our purpose. The distribution (i) therefore leads to no new equation.

The subsidiary system for the distribution (ii), when made a complete Jacobian system, is found to contain nine equations: it thus possesses two integrals, in addition to the vanishing integral

$$\alpha + a\beta - \gamma - a\delta - \frac{2}{x}(r + as).$$

These two integrals are obtainable in the forms

$$\frac{1}{x}\{\alpha - 2\beta + \gamma + a(\beta - 2\gamma + \delta)\}, \quad y - x.$$

Accordingly, the distribution (ii) provides a new equation which can be taken in the form

$$\alpha - 2\beta + \gamma + a(\beta - 2\gamma + \delta) = 4xf'''(y - x),$$

where f is an arbitrary function.

Similarly, the distribution (iii) provides a new equation which can be taken in the form

$$\alpha + 2\beta + \gamma + a(\beta + 2\gamma + \delta) = 4xg'''(y + x),$$

where g is an arbitrary function.

We thus have two new equations compatible with, but not resoluble into, the original equation. When they are treated as simultaneous equations, they give

$$\begin{aligned} \alpha + a\beta &= \frac{1}{x}(r + as) + xf''' + xg''', \\ \beta + a\gamma &= -xf''' + xg''', \\ \gamma + a\delta &= -\frac{1}{x}(r + as) + xf''' + xg'''. \end{aligned}$$

The construction of the primitive depends upon quadratures in the first instance. We have

$$\begin{aligned} dr + a\,ds &= (\alpha + a\beta)\,dx + (\beta + a\gamma)\,dy \\ &= (r + as)\frac{dx}{x} + x(dg'' - df''); \end{aligned}$$

and therefore

$$\frac{r + as}{x} = g'' - f''.$$

Next,

$$\begin{aligned} ds + a\,dt &= (\beta + a\gamma)\,dx + (\gamma + a\delta)\,dy \\ &= x(dg'' + df'') - \frac{r + as}{x}\,dy \\ &= x\,dg'' - g''dy + x\,df'' + f''dy; \end{aligned}$$

and therefore

$$s + at = xg'' - g' + xf'' + f'.$$

Consequently, we have

$$dp + a\,dq = (r + as)\,dx + (s + at)\,dy$$
$$= x\,dg'' - g'dy + x\,df'' + f'dy,$$

and therefore

$$p + aq = xg' - g + xf' + f.$$

This is an equation of the first order: integrating it by the usual process, we have

$$z = \psi(y - ax)$$
$$+ x\theta'(y + x) - \frac{2+a}{1+a}\theta(y + x),$$
$$+ x\phi'(y - x) + \frac{2-a}{1-a}\phi(y - x),$$

where

$$(1 + a)\,\theta' = g, \quad (a - 1)\,\phi' = f;$$

and, in this last form, θ, ϕ, ψ are the three arbitrary functions in the primitive.

Ex. 2. Obtain the primitives of the equations :—

$$\text{(i)} \quad \alpha + \beta - \gamma - \delta = \frac{2}{x}(r + s);$$

$$\text{(ii)} \quad \alpha - \beta - \gamma + \delta = \frac{2}{x}(r - s).$$

EQUATIONS OF THE nTH ORDER.

315. After the preceding investigations dealing with equations of the third order in two independent variables, it will be sufficient to state the results of a similar type which appertain to equations of order n also in two independent variables. We write

$$p_{l,m} = \frac{\partial^{l+m} z}{\partial x^l \partial y^m},$$

and we assume the equation to be

$$f(x, y, z, p_{10}, p_{01}, \ldots, p_{n,0}, p_{n-1,1}, \ldots, p_{1,n-1}, p_{0,n}) = 0.$$

The general primitive of such an equation, in whatever form it occurs, involves n arbitrary functions, each of a single argument: if α denote any of these arguments, supposed to be a quantity depending upon both x and y, then the derivative of y with regard to x on the supposition that α is constant, being given by the relation

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{\delta y}{\delta x} = 0,$$

satisfies the equation

$$\frac{\partial f}{\partial p_{n,0}}\left(\frac{\delta y}{\delta x}\right)^n - \frac{\partial f}{\partial p_{n-1,1}}\left(\frac{\delta y}{\delta x}\right)^{n-1} + \ldots + (-1)^{n-1}\frac{\partial f}{\partial p_{1,n-1}}\frac{\delta y}{\delta x} + (-1)^n\frac{\partial f}{\partial p_{0,n}} = 0,$$

while α satisfies the equivalent equation

$$\frac{\partial f}{\partial p_{n,0}}\left(\frac{\partial \alpha}{\partial x}\right)^n + \frac{\partial f}{\partial p_{n-1,1}}\left(\frac{\partial \alpha}{\partial x}\right)^{n-1}\frac{\partial \alpha}{\partial y} + \ldots + \frac{\partial f}{\partial p_{0,n}}\left(\frac{\partial \alpha}{\partial y}\right)^n = 0.$$

Ex. 1. Shew that, if the equation

$$f(x, y, z, p_{1,0}, p_{0,1}, \ldots, p_{n,0}, \ldots, p_{0,n}) = 0$$

possesses an intermediate integral in the form of an equation

$$\phi(u, v) = 0,$$

where ϕ is an arbitrary function, and u and v are definite functions of x, y, z and of all the derivatives of z up to order $n-1$ inclusive, then $f=0$ is of the form

$$\Sigma A_{l,l'}(p_{n-l,n-m}\,p_{n-l',n-m'} - p_{n-l+1,n-m-1}\,p_{n-l'-1,n-m'+1}) + \Sigma B_l\, p_{n-l,n-m} + C = 0,$$

where

$$l+m=n, \quad l'+m'=n,$$

and the coefficients $A_{l,l'}$, B_l, C do not involve any derivatives of order n.

When the particular equation of order n possesses an intermediate integral in the form of an equation

$$\phi(u, v) = 0$$

of order $n-1$, where ϕ is an arbitrary function, shew that the coefficients $A_{l,l'}$ identically satisfy relations

$$A_{l,l'}\,A_{k,k'} - A_{l,k}\,A_{l',k'} + A_{l,k'}\,A_{l',k} = 0,$$

for all values of l and l' different from k and k'. Obtain other identical relations which must be satisfied by the coefficients $A_{l,l'}$, when the given equation has an intermediate integral.

Ex. 2. Shew that, if the equation in the preceding example does not necessarily possess an intermediate integral in the form of an equation of lower order, while it does admit the existence of a compatible equation

$$g(x, y, z, p_{10}, p_{01}, \ldots, p_{n,0}, \ldots, p_{0,n}) = 0,$$

which is of order n and is not resoluble into $f=0$, then g satisfies one of the sets of equations

$$\frac{dg}{dx} + \tau_i\frac{dg}{dy} = \frac{X}{P_n}\frac{\partial g}{\partial p_{n,0}} + \tau_i\frac{Y}{P_0}\frac{\partial g}{\partial p_{0,n}},$$

$$\sum_{\lambda=0}^{n}(-1)^{n-\lambda}\tau_j^{\,n-\lambda}\frac{\partial g}{\partial p_{n-\lambda,\lambda}} = 0,$$

where j has all the values in $1, \ldots, n$ other than i, and the sets of equations are varied by giving the values $1, \ldots, n$ to i in succession: and where

$$P_n = \frac{\partial f}{\partial p_{n,0}}, \quad P_0 = \frac{\partial f}{\partial p_{0,n}}, \quad X = \frac{df}{dx}, \quad Y = \frac{df}{dy},$$

while also $\dfrac{d}{dx}$ and $\dfrac{d}{dy}$, as applied to f and to g, imply complete derivation with regard to x and to y respectively through all derivatives of z up to those of order $n-1$ inclusive: and where, lastly, τ_i is any one of the roots of the equation

$$\sum_{\lambda=0}^{n} (-1)^{n-\lambda} \tau_j{}^{n-\lambda} \frac{\partial f}{\partial p_{n-\lambda,\lambda}} = 0.$$

CHAPTER XXIII.

EQUATIONS OF THE SECOND ORDER IN MORE THAN TWO INDEPENDENT VARIABLES, HAVING AN INTERMEDIATE INTEGRAL.

As indicated in the opening sentences, the aim of this chapter is the extension, to equations involving a number of independent variables greater than two, of the methods of Monge, Boole, and Goursat, which are applicable to equations that involve only two independent variables and possess an intermediate integral. The results are given, and even the notation is specially devised, for the case when the number of independent variables is three: but many of the results can obviously be generalised to the case when the number of independent variables is n, though it has seemed unnecessary to state them explicitly.

Much of the material of the chapter is derived from a memoir by the author*; reference may also be made to memoirs by Tanner†, Sersawy‡, von Weber§, Vivanti‖, and Coulon¶.

316. The preceding discussions have shewn that the theory of partial equations of the first order in one dependent variable and any number of independent variables can be regarded as complete. It is in a much slighter degree that the same claim can be made as regards equations of order higher than the first when there are

* *Camb. Phil. Trans.*, t. XVI (1898), pp. 191—218: other references are there given.

† *Proc. Lond. Math. Soc.*, t. VII (1876), pp. 43—60, 75—90, *ib.*, t. IX (1878), pp. 41—61, 76—90.

‡ *Wien. Denkschr.*, t. XLIX (1885), pp. 1—104; many results are stated for n variables.

§ *Math. Ann.*, t. XLVII (1896), pp. 230—262.

‖ *Math. Ann.*, t. XLVIII (1897), pp. 474—513.

¶ "Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode des caractéristiques," (*These*, Hermann, Paris, 1902), where other references also will be found.

two independent variables. Still, methods have been given which suffice for the integration of large classes of equations; most of them depend upon some subsidiary equations, in which all the magnitudes involved are temporarily held to be functions of one of the independent variables. Among the methods thus devised for equations of the second order, those associated with the names of Monge and of Boole presuppose (if they are to be effective) that the equation is of a special form and that some of its elements satisfy certain appropriate conditions: the most general methods are those due to Ampère and to Darboux respectively, being effective when the integral is expressible in finite terms without partial quadratures. There is also (§ 238) a method, intermediate in generality between these two classes of processes: it deals with equations of the second order (or of higher orders) in two independent variables which possess an intermediate integral, not of the particular type considered by Monge and by Boole.

When we proceed to equations of more extensive type, the natural generalisation is to be found in an increase in the number of independent variables: and such equations occur in various branches of mathematical physics, involving three independent variables (as in the theory of space-potential) or four independent variables (as in the theory of the conduction of heat in a solid body, and in the theory of propagation of vibrations in three dimensions). These equations are of a very special form, and very special analysis is needed for the full development of the particular solutions: but their occurrence challenges a consideration of equations of general form, to which the individually special methods are quite inapplicable.

Accordingly, in this chapter, we shall discuss general equations of the second order which involve three independent variables: the restriction of the number of independent variables to three is made for the sake of brevity: and, in spite of the notation adopted for special service in the equations considered, it is not difficult to see that many of the properties can be extended to equations in any number of independent variables.

The three independent variables are denoted by x, y, z: the dependent variable is denoted by v, and its first and second derivatives are denoted according to the scheme:—

$$\frac{\partial v}{\partial x} = l, \quad \frac{\partial v}{\partial y} = m, \quad \frac{\partial v}{\partial z} = n,$$

$$\frac{\partial^2 v}{\partial x^2} = a, \quad \frac{\partial^2 v}{\partial y^2} = b, \quad \frac{\partial^2 v}{\partial z^2} = c,$$

$$\frac{\partial^2 v}{\partial y \partial z} = f, \quad \frac{\partial^2 v}{\partial z \partial x} = g, \quad \frac{\partial^2 v}{\partial x \partial y} = h;$$

and then the general differential equation of the second order can be represented by

$$F(x, y, z, v, l, m, n, a, b, c, f, g, h) = 0.$$

317. It will be convenient, in the first place, to consider (so as to set on one side as being definite) those equations which have an intermediate integral; and the discussion will be limited to those equations of the second order which are the sole consequence, in that order, of the intermediate integral. Let this integral, supposed to exist, have the form

$$u(x, y, z, v, l, m, n) = 0;$$

and, in accordance with earlier notations, write

$$u_x = \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v}, \qquad u_l = \frac{\partial u}{\partial l},$$

$$u_y = \frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v}, \qquad u_m = \frac{\partial u}{\partial m},$$

$$u_z = \frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v}, \qquad u_n = \frac{\partial u}{\partial n}.$$

Then we have

$$u_x + au_l + hu_m + gu_n = 0,$$

$$u_y + hu_l + bu_m + fu_n = 0,$$

$$u_z + gu_l + fu_m + cu_n = 0.$$

Owing to the hypothesis of an intermediate integral, the equation $F = 0$ is to be satisfied in virtue of these three equations, that is, when we resolve the three equations for any three of the second derivatives (say for a, b, c) and substitute the deduced values in $F = 0$, the latter must become evanescent: and therefore the coefficients of the various combinations of f, g, h must vanish. This requirement provides a number of relations that are homogeneous in the quantities u_x, u_y, u_z, u_l, u_m, u_n, so that there cannot be more than five such relations; the actual number less than five

will depend upon the equation itself. Each relation is a partial equation of the first order: and u is provided by the common integral (if any) of the system of simultaneous equations.

As regards the actual number of relations, it is easy to see that, for such equations as are amenable to the method, there are generally three relations at least. After substitution for a, b, c, has taken place, the modified equation is of the form

$$T + Pf + Qg + Rh + \dots = 0;$$

in order that it may be evanescent, we must have

$$\dots, \quad R = 0, \quad Q = 0, \quad P = 0, \quad T = 0.$$

If these were equivalent to only one relation, the equivalence would arise through the occurrence of a vanishing factor common to all the expressions $\dots, R, Q, P, T$: let this factor be

$$u_x - \theta(u_y, u_z, u_l, u_m, u_n),$$

where θ is homogeneous of the first order in u_y, u_z, u_l, u_m, u_n. We then have three equations

$$\begin{aligned} \theta + au_l + hu_m + gu_n &= 0, \\ u_y + hu_l + bu_m + fu_n &= 0, \\ u_z + gu_l + fu_m + cu_n &= 0, \end{aligned}$$

involving five quantities homogeneously, and the elimination of these quantities is to lead to the equation of the second order: such elimination is not possible in general.

If the equations were equivalent to only two relations, they may be taken in the form

$$\begin{aligned} \phi(u_x, u_y, u_z, u_l, u_m, u_n) &= 0, \\ \psi(u_x, u_y, u_z, u_l, u_m, u_n) &= 0, \end{aligned}$$

where ϕ and ψ are homogeneous: the original equation is to be derivable by the elimination of the six derivatives of u between these two equations and the other three

$$\left.\begin{aligned} u_x + au_l + hu_m + gu_n &= 0 \\ u_y + hu_l + bu_m + fu_n &= 0 \\ u_z + gu_l + fu_m + cu_n &= 0 \end{aligned}\right\};$$

and this elimination is not possible in general.

The respective eliminations might be possible for very particular cases: we shall put them on one side as being too special. Accordingly, we conclude in general that, when an equation of the second order possesses an intermediate integral, the number of algebraically independent relations determining the quantity u in this method of proceeding is either three, or four, or five.

Note 1. In testing whether a given equation possesses an intermediate integral, it might be convenient to eliminate f, g, h, rather than a, b, c, by means of the equations derived from $u=0$: all that is necessary, in order to obtain the appropriate relations, is to use the three derived equations in order to eliminate three of the second derivatives of v from the given equation.

Note 2. As our object is rather to indicate the method which is of general effectiveness than to apply it in an exhaustive discussion of all inclusible cases, we shall make an initial limitation.

If there were four algebraically independent relations determining the quantity u, these could have a form

$$\alpha_i(u_x, u_y, u_z, u_l, u_m, u_n) = 0,$$

for $i = 1, 2, 3, 4$. Each of these is homogeneous of order unity in the six quantities $u_x, u_y, u_z, u_l, u_m, u_n$. Hence, when we proceed to the equation of the second order, we have to eliminate the two ratios $u_l : u_m : u_n$ between the four equations

$$\alpha_i(-au_l - hu_m - gu_n, -hu_l - bu_m - fu_n, \\ -gu_l - fu_m - cu_n, u_l, u_m, u_n) = 0,$$

for $i = 1, 2, 3, 4$: that is, two equations will appear in the eliminant, which accordingly will consist of two differential equations of the second order.

Similarly, if there were five algebraically independent relations determining the quantity u, there would result three simultaneous differential equations of the second order.

Both these cases will be left on one side: the method will be expounded sufficiently for the most important case, when the relations lead to only a single differential equation of the second order.

Case when the Three Relations are a Complete Jacobian System.

318. Dealing therefore with the case when the number of algebraically independent relations is three, we imagine them resolved with regard to three of the quantities u_x, u_y, u_z, u_l, u_m, u_n, choosing the first three by preference. The relations then have the form

$$L = u_x + \lambda(u_l, u_m, u_n) = 0,$$
$$M = u_y + \mu(u_l, u_m, u_n) = 0,$$
$$N = u_z + \nu(u_l, u_m, u_n) = 0,$$

where λ, μ, ν are homogeneous of the first order in u_l, u_m, u_n, and otherwise may involve the variables v, x, y, z, l, m, n. When we write

$$v, x, y, z, l, m, n = x_1, x_2, x_3, x_4, x_5, x_6, x_7,$$

and

$$\frac{\partial u}{\partial x_i} = p_i,$$

for $i = 1, \ldots, 7$, the equations for u are

$$\left.\begin{aligned} L &= x_5 p_1 + p_2 + \lambda(x_1, \ldots, x_7, p_5, p_6, p_7) = 0 \\ M &= x_6 p_1 + p_3 + \mu(x_1, \ldots, x_7, p_5, p_6, p_7) = 0 \\ N &= x_7 p_1 + p_4 + \nu(x_1, \ldots, x_7, p_5, p_6, p_7) = 0 \end{aligned}\right\}.$$

These equations must satisfy the necessary Poisson-Jacobi conditions for coexistence: that is, the relations

$$(L, M) = 0, \quad (M, N) = 0, \quad (N, L) = 0,$$

must be satisfied, either identically, or in virtue of the equations $L = 0$, $M = 0$, $N = 0$, or as new equations in the system. Now the relation

$$(L, M) = p_1\left(\frac{\partial \mu}{\partial p_5} - \frac{\partial \lambda}{\partial p_6}\right) + x_6\frac{\partial \lambda}{\partial x_1} - x_5\frac{\partial \mu}{\partial x_1} + \frac{\partial \lambda}{\partial x_3} - \frac{\partial \mu}{\partial x_2} + \sum_{i=5}^{7}\frac{\partial(\lambda, \mu)}{\partial(x_i, p_i)} = 0$$

manifestly cannot be satisfied in virtue of $L = 0$, $M = 0$, $N = 0$, for it does not involve either p_2, p_3, or p_4; hence it is either an identity or a new equation.

In order that the relation may be an identity, the term in p_1 must vanish by itself, for p_1 does not occur elsewhere; hence, as a first condition, we have

$$\frac{\partial \mu}{\partial p_5} = \frac{\partial \lambda}{\partial p_6}.$$

Similarly, if $(M, N) = 0$, and $(N, L) = 0$ are identities, we have

$$\frac{\partial \nu}{\partial p_6} = \frac{\partial \mu}{\partial p_7}, \quad \frac{\partial \lambda}{\partial p_7} = \frac{\partial \nu}{\partial p_5}.$$

Other relations, connected with the remaining terms in the thre identities, will have to be satisfied: assuming them satisfied, w see that (on the hypothesis adopted) the three equations constitut a complete Jacobian system. The relations, connecting the deriva tives of λ, μ, ν with respect to p_5, p_6, p_7, shew that a function Θ exists, such that

$$\lambda = \frac{\partial \Theta}{\partial p_5}, \quad \mu = \frac{\partial \Theta}{\partial p_6}, \quad \nu = \frac{\partial \Theta}{\partial p_7}.$$

This function Θ consists of two parts: the first is a quantity homogeneous of the second order in p_5, p_6, p_7, and involving th variables $x_1, \ldots, x_7$: the second is a quantity independent o p_5, p_6, p_7. Let

$$p_5 = \rho p_7, \quad p_6 = \sigma p_7;$$

then Θ may be taken in the form

$$\Theta = p_7^2\, \theta\,(x_1, \ldots, x_7, \rho, \sigma) + X,$$

where X is the additive part of Θ independent of p_5, p_6, p_7, an(where now there is no restriction upon the form of θ so far a regards homogeneity. Clearly

$$\lambda = p_7 \frac{\partial \theta}{\partial \rho},$$

$$\mu = p_7 \frac{\partial \theta}{\partial \sigma},$$

$$\nu = p_7 \left(2\theta - \rho \frac{\partial \theta}{\partial \rho} - \sigma \frac{\partial \theta}{\partial \sigma}\right),$$

the derivatives of X not appearing in λ, μ, ν.

Substituting these values of λ and μ in the remaining term of $(L, M) = 0$ and removing a factor p_7, which is common to all th terms after the substitution has been effected, we find

$$x_6 \frac{\partial^2 \theta}{\partial x_1 \partial \rho} - x_5 \frac{\partial^2 \theta}{\partial x_1 \partial \sigma} + \frac{\partial^2 \theta}{\partial x_3 \partial \rho} - \frac{\partial^2 \theta}{\partial x_2 \partial \sigma}$$
$$+ \frac{\partial^2 \theta}{\partial x_5 \partial \rho} \frac{\partial^2 \theta}{\partial \rho \partial \sigma} - \frac{\partial^2 \theta}{\partial x_5 \partial \sigma} \frac{\partial^2 \theta}{\partial \rho^2}$$
$$+ \frac{\partial^2 \theta}{\partial x_6 \partial \rho} \frac{\partial^2 \theta}{\partial \sigma^2} - \frac{\partial^2 \theta}{\partial x_6 \partial \sigma} \frac{\partial^2 \theta}{\partial \rho \partial \sigma}$$
$$+ \frac{\partial^2 \theta}{\partial x_7 \partial \rho} \left(\frac{\partial \theta}{\partial \sigma} - \rho \frac{\partial^2 \theta}{\partial \rho \partial \sigma} - \sigma \frac{\partial^2 \theta}{\partial \sigma^2}\right) - \frac{\partial^2 \theta}{\partial x_7 \partial \sigma} \left(\frac{\partial \theta}{\partial \rho} - \rho \frac{\partial^2 \theta}{\partial \rho^2} - \sigma \frac{\partial^2 \theta}{\partial \rho \partial \sigma}\right) = 0.$$

When substitution of the values of μ and ν takes place in the remaining terms of $(M, N)=0$, a similar equation of the second order arises: and another equation of the second order is provided by the remaining terms of $(N, L)=0$.

These three equations must be satisfied by θ: when any integral common to all three is known, we have the means of constructing the corresponding equation of the second order possessing an intermediate integral. For

$$u_x + au_l + hu_m + gu_n = 0,$$

that is,

$$\lambda + ap_5 + hp_6 + gp_7 = 0,$$

and therefore

$$-\frac{\partial\theta}{\partial\rho} + a\rho + h\sigma + g = 0.$$

Similarly,

$$-\frac{\partial\theta}{\partial\sigma} + h\rho + b\sigma + f = 0,$$

and

$$-2\theta + \rho\frac{\partial\theta}{\partial\rho} + \sigma\frac{\partial\theta}{\partial\sigma} + g\rho + f\sigma + c = 0;$$

the latter, in connection with the other two, can be replaced by

$$-2\theta + a\rho^2 + 2h\rho\sigma + b\sigma^2 + 2g\rho + 2f\sigma + c = 0.$$

Eliminating ρ and σ between the equations or (what is the same thing) equating to zero the discriminant of the quantity on the left-hand side of the last equation, we have an equation of the second order possessing an intermediate integral as required.

As regards the intermediate integral u itself, it is determined by the three equations which form a complete Jacobian system. This system involves seven independent variables, and therefore it possesses four algebraically independent integrals; let these be u_1, u_2, u_3, u_4. We proceed from the equations

$$L=0, \quad M=0, \quad N=0,$$

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

and resolve these for $p_1, \ldots, p_7$; substituting in

$$du = \sum_{i=1}^{7} p_i dx_i,$$

effecting the quadrature, and dividing out by one of the constants a_1, a_2, a_3, a_4, we have

$$\frac{1}{a_4} u = \phi(x_1, \ldots, x_7, a, b, c) + a',$$

where a, b, c are three arbitrary constants, and a' is an additive constant. As $u = 0$ is the intermediate integral, we can take the latter in the form

$$\phi(x_1, \ldots, x_7, a, b, c) + a' = 0,$$

that is,

$$\phi(v, x, y, z, l, m, n, a, b, c) + a' = 0,$$

which is an equation of the first order.

319. To obtain the primitive of the differential equation constructed with $\phi + a' = 0$ as an intermediate integral, we might proceed to construct the primitive of the equation of the first order: but the theory of § 284 can be generalised, so as to allow the primitive to be constructed merely by operations of elimination. When we substitute

$$u = \phi + a'$$

in the differential equations $L = 0$, $M = 0$, $N = 0$, these are satisfied identically: hence

$$x_5 \frac{\partial p_1}{\partial a} + \frac{\partial p_2}{\partial a} + \frac{\partial \lambda}{\partial p_5}\frac{\partial p_5}{\partial a} + \frac{\partial \lambda}{\partial p_6}\frac{\partial p_6}{\partial a} + \frac{\partial \lambda}{\partial p_7}\frac{\partial p_7}{\partial a} = 0,$$

$$x_6 \frac{\partial p_1}{\partial a} + \frac{\partial p_3}{\partial a} + \frac{\partial \mu}{\partial p_5}\frac{\partial p_5}{\partial a} + \frac{\partial \mu}{\partial p_6}\frac{\partial p_6}{\partial a} + \frac{\partial \mu}{\partial p_7}\frac{\partial p_7}{\partial a} = 0,$$

$$x_7 \frac{\partial p_1}{\partial a} + \frac{\partial p_4}{\partial a} + \frac{\partial \nu}{\partial p_5}\frac{\partial p_5}{\partial a} + \frac{\partial \nu}{\partial p_6}\frac{\partial p_6}{\partial a} + \frac{\partial \nu}{\partial p_7}\frac{\partial p_7}{\partial a} = 0.$$

Now let the Poisson-Jacobi combination for u and $\dfrac{\partial u}{\partial a}$ be constructed: it is

$$\begin{aligned}\left[u, \frac{\partial u}{\partial a}\right] = \; & (p_2 + x_5 p_1)\frac{\partial p_5}{\partial a} - p_5\left(\frac{\partial p_2}{\partial a} + x_5 \frac{\partial p_1}{\partial a}\right) \\ & + (p_3 + x_6 p_1)\frac{\partial p_6}{\partial a} - p_6\left(\frac{\partial p_3}{\partial a} + x_6 \frac{\partial p_1}{\partial a}\right) \\ & + (p_4 + x_7 p_1)\frac{\partial p_7}{\partial a} - p_7\left(\frac{\partial p_4}{\partial a} + x_7 \frac{\partial p_1}{\partial a}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\lambda \frac{\partial p_5}{\partial a} + p_5 \left(\frac{\partial \lambda}{\partial p_5} \frac{\partial p_5}{\partial a} + \frac{\partial \lambda}{\partial p_6} \frac{\partial p_6}{\partial a} + \frac{\partial \lambda}{\partial p_7} \frac{\partial p_7}{\partial a} \right) \\
&\quad - \mu \frac{\partial p_6}{\partial a} + p_6 \left(\frac{\partial \mu}{\partial p_5} \frac{\partial p_5}{\partial a} + \frac{\partial \mu}{\partial p_6} \frac{\partial p_6}{\partial a} + \frac{\partial \mu}{\partial p_7} \frac{\partial p_7}{\partial a} \right) \\
&\quad - \nu \frac{\partial p_7}{\partial a} + p_7 \left(\frac{\partial \nu}{\partial p_5} \frac{\partial p_5}{\partial a} + \frac{\partial \nu}{\partial p_6} \frac{\partial p_6}{\partial a} + \frac{\partial \nu}{\partial p_7} \frac{\partial p_7}{\partial a} \right) \\
&= \frac{\partial p_5}{\partial a} \left(-\lambda + p_5 \frac{\partial \lambda}{\partial p_5} + p_6 \frac{\partial \mu}{\partial p_5} + p_7 \frac{\partial \nu}{\partial p_5} \right) \\
&\quad + \frac{\partial p_6}{\partial a} \left(-\mu + p_5 \frac{\partial \lambda}{\partial p_6} + p_6 \frac{\partial \mu}{\partial p_6} + p_7 \frac{\partial \nu}{\partial p_6} \right) \\
&\quad + \frac{\partial p_7}{\partial a} \left(-\nu + p_5 \frac{\partial \lambda}{\partial p_7} + p_6 \frac{\partial \mu}{\partial p_7} + p_7 \frac{\partial \nu}{\partial p_7} \right) \\
&= \frac{\partial p_5}{\partial a} \left(-\lambda + p_5 \frac{\partial \lambda}{\partial p_5} + p_6 \frac{\partial \lambda}{\partial p_6} + p_7 \frac{\partial \lambda}{\partial p_7} \right) \\
&\quad + \frac{\partial p_6}{\partial a} \left(-\mu + p_5 \frac{\partial \mu}{\partial p_5} + p_6 \frac{\partial \mu}{\partial p_6} + p_7 \frac{\partial \mu}{\partial p_7} \right) \\
&\quad + \frac{\partial p_7}{\partial a} \left(-\nu + p_5 \frac{\partial \nu}{\partial p_5} + p_6 \frac{\partial \nu}{\partial p_6} + p_7 \frac{\partial \nu}{\partial p_7} \right),
\end{aligned}
$$

on account of the relations between the derivatives of λ, μ, ν with regard to p_5, p_6, p_7. As λ, μ, ν are homogeneous of the first order in p_5, p_6, p_7, the coefficients of $\frac{\partial p_5}{\partial a}$, $\frac{\partial p_6}{\partial a}$, $\frac{\partial p_7}{\partial a}$ vanish separately; hence

$$\left[u, \frac{\partial u}{\partial a}\right] = 0.$$

Similarly, we can prove that

$$\left[u, \frac{\partial u}{\partial b}\right] = 0, \quad \left[u, \frac{\partial u}{\partial c}\right] = 0.$$

Corresponding analysis leads to relations

$$\left[\frac{\partial u}{\partial a}, \frac{\partial u}{\partial b}\right] = 0, \quad \left[\frac{\partial u}{\partial b}, \frac{\partial u}{\partial c}\right] = 0, \quad \left[\frac{\partial u}{\partial c}, \frac{\partial u}{\partial a}\right] = 0.$$

It therefore follows that the conditions of coexistence of the equations

$$u = 0, \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta, \quad \frac{\partial u}{\partial c} = \gamma,$$

are satisfied. The elimination of l, m, n among these four equations leads to a relation between $v, x, y, z, a', a, b, c, \alpha, \beta, \gamma$, which is consistent with all of them: it is a complete primitive.

The last proposition shews how to deduce the complete primitive of the equations under consideration from the intermediate integral when the latter is known. The general primitive can also be deduced from that integral; the result can be established by analysis precisely analogous to that in § 284 used for the establishment of the corresponding result in the case of two variables. The mode of deduction is as follows:

Let a, b, c be the three non-additive constants in u; and let the result of eliminating l, m, n between

$$u = 0, \quad \frac{\partial u}{\partial a} = \alpha, \quad \frac{\partial u}{\partial b} = \beta, \quad \frac{\partial u}{\partial c} = \gamma,$$

be denoted by

$$g(x, y, z, v, a, b, c, \alpha, \beta, \gamma) = 0.$$

Then the general primitive is given by the elimination of a and b between the three equations

$$g\left\{x, y, z, v, a, b, \phi(a, b), -\frac{\partial \phi}{\partial a}\chi(a, b), -\frac{\partial \phi}{\partial b}\chi(a, b), \chi(a, b)\right\} = 0,$$

$$\frac{dg}{da} = 0, \quad \frac{dg}{db} = 0,$$

ϕ and χ being arbitrary functions.

In the present case, when the subsidiary equations for u possess four algebraically independent integrals, we can construct the complete primitive and the general primitive by direct operations effected upon the intermediate integral.

The three equations which θ must satisfy are complicated in form; and they involve a larger number of variables than the equations under our consideration, while at the same time they are of the second order. Consequently, we can hardly expect, at the present stage, to obtain the most general function θ which satisfies the equations: one or two examples will suffice to illustrate the theory.

Ex. 1. It is not difficult to verify that the three equations are satisfied, when θ is any function of ρ and σ involving no other variables: let such a value of θ be chosen. In that case, which merely is the generalisation of

the case considered by Goursat (§ 284, Ex. 1), the equation of the second order is obtained by equating the discriminant of the equation

$$-2\theta+a\rho^2+2h\rho\sigma+b\sigma^2+2g\rho+2f\sigma+c=0$$

to zero, so that the equation will be of the form

$$F(a,\ b,\ c,\ f,\ g,\ h)=0,$$

involving derivatives of the second order only.

The intermediate integral $u=0$ depends upon the three equations

$$F_1=0=p_2+x_5p_1+p_7\frac{\partial\theta}{\partial\rho},$$

$$F_2=0=p_3+x_6p_1+p_7\frac{\partial\theta}{\partial\sigma},$$

$$F_3=0=p_4+x_7p_1+p_7\left(2\theta-\rho\frac{\partial\theta}{\partial\rho}-\sigma\frac{\partial\theta}{\partial\sigma}\right).$$

This is a complete Jacobian system, and therefore it possesses four algebraically independent integrals. The simpler these are taken, the better: for we know how to obtain the complete primitive and the general primitive from the intermediate integral, if only the last should contain the proper number of arbitrary constants. Now it is clear that

$$(F_1,\ p_i)=0,\quad (F_2,\ p_i)=0,\quad (F_3,\ p_i)=0,$$

for $i=1,\ 2,\ 3,\ 4$: and therefore we may take $p_1,\ p_2,\ p_3,\ p_4$ as the four common integrals.

To determine u, we proceed from the equations

$$F_1=0,\quad F_2=0,\quad F_3=0,$$

$$p_1=a_1,\quad p_2=a_2,\quad p_3=a_3,\quad p_4=a_4;$$

we resolve them with regard to $p_1,\ \dots,\ p_7$, and we substitute in

$$du=\sum_{i=1}^{7} p_i dx_i.$$

Hence

$$d(u-a_1x_1-a_2x_2-a_3x_3-a_4x_4)=p_7(\rho\,dx_5+\sigma\,dx_6+dx_7),$$

where ρ, σ, and p_7 are given by

$$\left.\begin{aligned} p_7\frac{\partial\theta}{\partial\rho}&=-a_2-a_1x_5\\ p_7\frac{\partial\theta}{\partial\sigma}&=-a_3-a_1x_6\\ p_7\left(2\theta-\rho\frac{\partial\theta}{\partial\rho}-\sigma\frac{\partial\theta}{\partial\sigma}\right)&=-a_4-a_1x_7\end{aligned}\right\}.$$

The right-hand side of the equation giving du must be an exact differential: let it be denoted by dU, and (for the evaluation of U) let the independent variables be changed from x_5, x_6, x_7, to ρ, σ, x_7. Taking

$$\frac{\partial\theta}{\partial\rho}=\theta_1,\quad \frac{\partial\theta}{\partial\sigma}=\theta_2,\quad 2\theta-\rho\theta_1-\sigma\theta_2=\Delta,$$

we have

$$p_7 = -\frac{1}{\Delta}(a_4 + a_1 x_7),$$

$$\frac{\theta_1}{\Delta} = \frac{a_2 + a_1 x_5}{a_4 + a_1 x_7},$$

$$\frac{\theta_2}{\Delta} = \frac{a_3 + a_1 x_6}{a_4 + a_1 x_7},$$

so that

$$a_1 dx_5 = a_1 \frac{\theta_1}{\Delta} dx_7 + (a_4 + a_1 x_7)\, d\frac{\theta_1}{\Delta},$$

$$a_1 dx_6 = a_1 \frac{\theta_2}{\Delta} dx_7 + (a_4 + a_1 x_7)\, d\frac{\theta_2}{\Delta}:$$

consequently,

$$\begin{aligned} -dU &= \frac{a_4 + a_1 x_7}{\Delta^2}\, 2\theta\, dx_7 + \frac{(a_4 + a_1 x_7)^2}{a_1 \Delta}\left\{\rho d\frac{\theta_1}{\Delta} + \sigma d\frac{\theta_2}{\Delta}\right\} \\ &= \frac{\theta}{\Delta^2}\, 2\,(a_4 + a_1 x_7)\, dx_7 + \frac{(a_4 + a_1 x_7)^2}{a_1}\, d\left(\frac{\theta}{\Delta^2}\right), \end{aligned}$$

on reduction; hence

$$-dU = d\left\{\frac{\theta\,(a_4 + a_1 x_7)^2}{a_1 \Delta^2}\right\},$$

and consequently

$$u = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 - \frac{(a_4 + a_1 x_7)^2}{a_1}\frac{\theta}{\Delta^2} + a_5.$$

Now $u=0$ is the intermediate integral: hence, dividing throughout by a_1, and changing the constants and the variables, we can take it in the form

$$v + ax + by + cz - (c+n)^2 \frac{\theta}{\Delta^2} = a',$$

where θ and Δ are known functions of ρ and σ, and where l and m are given by the equations

$$\frac{\theta_1}{a+l} = \frac{\theta_2}{b+m} = \frac{\Delta}{c+n}:$$

in other words, *the intermediate integral is given by the elimination of* ρ *and* σ *between the equations*

$$\left.\begin{aligned} v + ax + by + cz - (c+n)^2 \frac{\theta}{\Delta^2} &= a' \\ \frac{\theta_1}{a+l} = \frac{\theta_2}{b+m} &= \frac{\Delta}{c+n} \end{aligned}\right\}.$$

To obtain a complete primitive, we need the values of

$$\frac{\partial u}{\partial a},\quad \frac{\partial u}{\partial b},\quad \frac{\partial u}{\partial c}.$$

For this purpose, let a new quantity ξ be introduced by the definition

$$a + l = \theta_1 \xi;$$

then

$$b + m = \theta_2 \xi,$$

$$c + n = \Delta \xi;$$

and also

$$u=v+ax+by+cz-\theta\xi^2-a'.$$

We have

$$1=\theta_1\frac{\partial\xi}{\partial a}+\xi\left(\theta_{11}\frac{\partial\rho}{\partial a}+\theta_{12}\frac{\partial\sigma}{\partial a}\right),$$

$$0=\theta_2\frac{\partial\xi}{\partial a}+\xi\left(\theta_{12}\frac{\partial\rho}{\partial a}+\theta_{22}\frac{\partial\sigma}{\partial a}\right),$$

$$0=\Delta\frac{\partial\xi}{\partial a}+\xi\left(\Delta_1\frac{\partial\rho}{\partial a}+\Delta_2\frac{\partial\sigma}{\partial a}\right),$$

the suffixes 1 and 2 implying derivation with regard to ρ and σ respectively: multiplying the first of these by ρ, the second by σ, and adding to the third, we have

$$\rho=2\theta\frac{\partial\xi}{\partial a}+\xi\left(\theta_1\frac{\partial\rho}{\partial a}+\theta_2\frac{\partial\sigma}{\partial a}\right).$$

Now

$$\begin{aligned}\frac{\partial u}{\partial a}&=x-2\theta\xi\frac{\partial\xi}{\partial a}-\xi^2\left(\theta_1\frac{\partial\rho}{\partial a}+\theta_2\frac{\partial\sigma}{\partial a}\right)\\&=x-\rho\xi;\end{aligned}$$

and, similarly,

$$\frac{\partial u}{\partial b}=y-\sigma\xi,$$

$$\frac{\partial u}{\partial c}=z-\xi.$$

Hence, by the general theory, we eliminate ρ, σ, ξ among the equations

$$\begin{aligned}v+ax+by+cz-\theta\xi^2&=a',\\ x-\rho\xi&=\alpha,\\ y-\sigma\xi&=\beta,\\ z-\xi&=\gamma,\end{aligned}$$

the constant a' being unessential. The complete* primitive is of the form

$$g(x,\ y,\ z,\ v-a',\ a,\ b,\ c,\ \alpha,\ \beta,\ \gamma)=0.$$

The *general primitive* is obtained by the elimination of all the constants between the equations

$$\left.\begin{aligned}&g=0,\quad \frac{dg}{da}=0,\quad \frac{dg}{db}=0\\ &c=\phi(a,\ b),\quad \gamma=\chi(a,\ b)\\ &\alpha=-\chi(a,\ b)\frac{\partial\phi}{\partial a},\quad \beta=-\chi(a,\ b)\frac{\partial\phi}{\partial b}\end{aligned}\right\}.$$

*** It is the most complete primitive thus obtainable. But it is not the complete primitive in the customary sense; for it contains only seven, not nine, arbitrary constants.**

Ex. 2. Let it be required to find those equations of the specified type, which are provided by taking

$$2\theta = A\rho^2 + 2H\rho\sigma + B\sigma^2 + 2G\rho + 2F\sigma + C,$$

where A, H, B, G, F, C involve v, x, y, z, l, m, n, but no other variable quantities.

For such values of these magnitudes as satisfy the conditions, the differential equation of the second order is given by equating to zero the discriminant of the equation

$$(a-A)\rho^2 + 2(h-H)\rho\sigma + (b-B)\sigma^2 + 2(g-G)\rho + 2(f-F)\sigma + c - C = 0:$$

consequently, it is

$$\begin{vmatrix} a-A, & h-H, & g-G \\ h-H, & b-B, & f-F \\ g-G, & f-F, & c-C \end{vmatrix} = 0.$$

The equations for the intermediate integral are

$$p_2 + x_5 p_1 + p_7(A\rho + H\sigma + G) = 0,$$

that is,

$$p_2 + x_5 p_1 + Ap_5 + Hp_6 + Gp_7 = 0,$$

with

$$p_3 + x_6 p_1 + Hp_5 + Bp_6 + Fp_7 = 0,$$
$$p_4 + x_7 p_1 + Gp_5 + Fp_6 + Cp_7 = 0.$$

Let these be denoted by

$$\Delta(u) = \left(\frac{\partial}{\partial x} + l\frac{\partial}{\partial v} + A\frac{\partial}{\partial l} + H\frac{\partial}{\partial m} + G\frac{\partial}{\partial n}\right)u = 0,$$

$$\Delta'(u) = \left(\frac{\partial}{\partial y} + m\frac{\partial}{\partial v} + H\frac{\partial}{\partial l} + B\frac{\partial}{\partial m} + F\frac{\partial}{\partial n}\right)u = 0,$$

$$\Delta''(u) = \left(\frac{\partial}{\partial z} + n\frac{\partial}{\partial v} + G\frac{\partial}{\partial l} + F\frac{\partial}{\partial m} + C\frac{\partial}{\partial n}\right)u = 0.$$

As these constitute a complete system, the Poisson-Jacobi relations

$$(\Delta, \Delta') = 0, \quad (\Delta', \Delta'') = 0, \quad (\Delta'', \Delta) = 0,$$

must be satisfied without the introduction of any new equations for u. The necessary and sufficient conditions* are

$$\left.\begin{matrix} \Delta' A = \Delta H \\ \Delta'' A = \Delta G \end{matrix}\right\}, \quad \left.\begin{matrix} \Delta'' B = \Delta' F \\ \Delta B = \Delta' H \end{matrix}\right\}, \quad \left.\begin{matrix} \Delta C = \Delta'' G \\ \Delta' C = \Delta'' F \end{matrix}\right\},$$

$$\Delta F = \Delta' G = \Delta'' H:$$

we shall assume that they are satisfied.

In these circumstances, the system possesses four algebraically independent integrals: let them be u_1, u_2, u_3, u_4. Then the equations

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

* They agree with the conditions, otherwise obtained, in a paper by the author, *Camb. Phil. Trans.*, vol. XVI (1898), p. 198.

can be treated as coexistent, when we revert to the older variables and take

$$v,\ x,\ y,\ z=x_1,\ x_2,\ x_3,\ x_4,$$
$$l,\ m,\ n=x_5,\ x_6,\ x_7.$$

For

$$\begin{aligned}[u_1,\ u_2]= & \left(\frac{\partial u_1}{\partial x}+l\ \frac{\partial u_1}{\partial v}\right)\frac{\partial u_2}{\partial l}-\left(\frac{\partial u_2}{\partial x}+l\ \frac{\partial u_2}{\partial v}\right)\frac{\partial u_1}{\partial l} \\ & +\left(\frac{\partial u_1}{\partial y}+m\frac{\partial u_1}{\partial v}\right)\frac{\partial u_2}{\partial m}-\left(\frac{\partial u_2}{\partial y}+m\frac{\partial u_2}{\partial v}\right)\frac{\partial u_1}{\partial m} \\ & +\left(\frac{\partial u_1}{\partial z}+n\ \frac{\partial u_1}{\partial v}\right)\frac{\partial u_2}{\partial n}-\left(\frac{\partial u_2}{\partial z}+n\ \frac{\partial u_2}{\partial v}\right)\frac{\partial u_1}{\partial n} \\ = & \,0,\end{aligned}$$

on substituting from the equations which are satisfied by u_1 and u_2; and, similarly,

$$[u_i,\ u_j]=0,$$

for all the combinations $i, j=1, 2, 3, 4$. As we have four equations

$$u_1=a_1,\quad u_2=a_2,\quad u_3=a_3,\quad u_4=a_4,$$

which are coexistent with one another, and as the quantities u_1, u_2, u_3, u_4 are functionally independent of one another, it follows that l, m, n can be eliminated among the four equations: and the eliminant is of the form

$$g\,(v,\ x,\ y,\ z,\ a_1,\ a_2,\ a_3,\ a_4)=0,$$

which is a primitive involving four arbitrary constants.

The primitive thus obtained can be modified so as to give the general primitive. When we take

$$a_3=\phi\,(a_1,\ a_2),\quad a_4=\psi\,(a_1,\ a_2),$$

where ϕ and ψ are arbitrary functions, then *the equation, which results from the elimination of* a_1 *and* a_2 *between the equations*

$$\left.\begin{aligned} g\,(v,\ x,\ y,\ z,\ a_1,\ a_2,\ \phi,\ \psi)=0 \\ \frac{\partial g}{\partial a_1}+\frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial a_1}+\frac{\partial g}{\partial \psi}\frac{\partial \psi}{\partial a_1}=0 \\ \frac{\partial g}{\partial a_2}+\frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial a_2}+\frac{\partial g}{\partial \psi}\frac{\partial \psi}{\partial a_2}=0 \end{aligned}\right\},$$

furnishes the general primitive of the equation of the second order. This statement can be established easily by verifying that the value of v thus given does actually satisfy the equation. Suppose the integral equations resolved, so as to express v explicitly: the system then becomes

$$\left.\begin{aligned} v=k\,(x,\ y,\ z,\ a_1,\ a_2,\ \phi,\ \psi) \\ 0=\frac{\partial k}{\partial a_1}+\frac{\partial k}{\partial \phi}\frac{\partial \phi}{\partial a_1}+\frac{\partial k}{\partial \psi}\frac{\partial \psi}{\partial a_1}=\frac{dk}{da_1} \\ 0=\frac{\partial k}{\partial a_2}+\frac{\partial k}{\partial \phi}\frac{\partial \phi}{\partial a_2}+\frac{\partial k}{\partial \psi}\frac{\partial \psi}{\partial a_2}=\frac{dk}{da_2} \end{aligned}\right\},$$

and the quantities a_1, a_2, as assigned by the last two equations, are functions of x, y, z.

The first derivatives of v, as determined by the system, are

$$l=\frac{\partial k}{\partial x}+\frac{dk}{da_1}\frac{\partial a_1}{\partial x}+\frac{dk}{da_2}\frac{\partial a_2}{\partial x}=\frac{\partial k}{\partial x},$$

and, similarly,

$$m=\frac{\partial k}{\partial y}, \quad n=\frac{\partial k}{\partial z};$$

so that, in form, the first derivatives are the same, whether a_1 and a_2 be parametric or variable.

As regards the second derivatives of v, we have

$$\begin{aligned} a&=\frac{\partial l}{\partial x}\\ &=\frac{\partial^2 k}{\partial x^2}+\frac{\partial^2 k}{\partial a_1\partial x}\frac{\partial a_1}{\partial x}+\frac{\partial^2 k}{\partial a_2\partial x}\frac{\partial a_2}{\partial x};\end{aligned}$$

but from $\dfrac{dk}{da_1}=0$, $\dfrac{dk}{da_2}=0$, it follows that

$$0=\frac{\partial^2 k}{\partial a_1\partial x}+\frac{d^2k}{da_1^2}\frac{\partial a_1}{\partial x}+\frac{d^2k}{da_1 da_2}\frac{\partial a_2}{\partial x},$$

$$0=\frac{\partial^2 k}{\partial a_2\partial x}+\frac{d^2k}{da_1 da_2}\frac{\partial a_1}{\partial x}+\frac{d^2k}{da_2^2}\frac{\partial a_2}{\partial x};$$

and therefore

$$a=\frac{\partial^2 k}{\partial x^2}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)^2,$$

where

$$\alpha,\ \beta,\ \gamma=\frac{d^2k}{da_1^2},\ \frac{d^2k}{da_1 da_2},\ \frac{d^2k}{da_2^2},$$

respectively. Similarly,

$$h=\frac{\partial^2 k}{\partial x\partial y}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)\!\!\left(\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\right),$$

$$g=\frac{\partial^2 k}{\partial x\partial z}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)\!\!\left(\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right),$$

$$b=\frac{\partial^2 k}{\partial y^2}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\right)^2,$$

$$f=\frac{\partial^2 k}{\partial y\partial z}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\right)\!\!\left(\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right),$$

$$c=\frac{\partial^2 k}{\partial z^2}-\left(\alpha,\ \beta,\ \gamma\right)\!\!\left(\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right)^2.$$

Now, in connection with the equations

$$u_i=a_i,$$

for $i=1, 2, 3, 4$, we have

$$\frac{\partial u_i}{\partial x}+l\frac{\partial u_i}{\partial v}+a\frac{\partial u_i}{\partial l}+h\frac{\partial u_i}{\partial m}+g\frac{\partial u_i}{\partial n}=0;$$

and (regard being paid to the two notations), we have

$$\frac{\partial u_i}{\partial x}+l\frac{\partial u_i}{\partial v}+A\frac{\partial u_i}{\partial l}+H\frac{\partial u_i}{\partial m}+G\frac{\partial u_i}{\partial n}=0;$$

hence, when the quantities a_1, a_2, a_3, a_4 are constant,

$$(a-A)\frac{\partial u_i}{\partial l}+(h-H)\frac{\partial u_i}{\partial m}+(g-G)\frac{\partial u_i}{\partial n}=0,$$

for $i=1, 2, 3, 4$. Not all the quantities

$$J\left(\frac{u_1,\ u_2,\ u_3,\ u_4}{l,\ m,\ n}\right)$$

vanish: hence, when a_1, a_2, a_3, a_4 are constant,

$$A=a=\frac{\partial^2 k}{\partial x^2},$$

$$H=h=\frac{\partial^2 k}{\partial x\partial y},$$

$$G=g=\frac{\partial^2 k}{\partial x\partial z}.$$

Similarly,

$$B=b=\frac{\partial^2 k}{\partial y^2},$$

$$F=f=\frac{\partial^2 k}{\partial y\partial z},$$

$$C=c=\frac{\partial^2 k}{\partial z^2},$$

when a_1, a_2, a_3, a_4 are constant quantities. Thus the equations for a, b, c, f, g, h, when a_1, a_2, a_3, a_4 are made variable, acquire the forms

$$a-A=-\left(\alpha,\ \beta,\ \gamma\between\frac{\partial a_1}{\partial x},\ \frac{\partial a_2}{\partial x}\right)^2,$$

$$f-F=-\left(\alpha,\ \beta,\ \gamma\between\frac{\partial a_1}{\partial y},\ \frac{\partial a_2}{\partial y}\between\frac{\partial a_1}{\partial z},\ \frac{\partial a_2}{\partial z}\right);$$

and so for the others. If, therefore, the differential equation

$$\begin{vmatrix} a-A, & h-H, & g-G \\ h-H, & b-B, & f-F \\ g-G, & f-F, & c-C \end{vmatrix}=0$$

is to be satisfied when a_1, a_2, a_3, a_4 are variable, we must have the determinant

$$\begin{vmatrix} (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2)^2, & (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\eta_1,\ \eta_2), & (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\zeta_1,\ \zeta_2) \\ (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\eta_1,\ \eta_2), & (\alpha,\ \beta,\ \gamma\between\eta_1,\ \eta_2)^2, & (\alpha,\ \beta,\ \gamma\between\eta_1,\ \eta_2\between\zeta_1,\ \zeta_2) \\ (\alpha,\ \beta,\ \gamma\between\xi_1,\ \xi_2\between\zeta_1,\ \zeta_2), & (\alpha,\ \beta,\ \gamma\between\eta_1,\ \eta_2\between\zeta_1,\ \zeta_2), & (\alpha,\ \beta,\ \gamma\between\zeta_1,\ \zeta_2)^2 \end{vmatrix},$$

equal to zero, where ξ_1, η_1, ζ_1 are the derivatives of a_1, and ξ_2, η_2, ζ_2 are those of a_2; as the determinant is the product of

$$\begin{vmatrix} \xi_1, & \xi_2, & 0 \\ \eta_1, & \eta_2, & 0 \\ \zeta_1, & \zeta_2, & 0 \end{vmatrix} \text{ and } \begin{vmatrix} \alpha\xi_1+\beta\xi_2, & \beta\xi_1+\gamma\xi_2, & 0 \\ \alpha\eta_1+\beta\eta_2, & \beta\eta_1+\gamma\eta_2, & 0 \\ \alpha\zeta_1+\beta\zeta_2, & \beta\zeta_1+\gamma\zeta_2, & 0 \end{vmatrix},$$

it vanishes identically.

Hence the proposition is valid.

Note. Having obtained the general primitive, we need not now concern ourselves as to intermediate integrals: but it must be noticed that the construction of the general primitive depends upon the possibility of eliminating l, m, n, between the equations

$$u_1=a_1, \quad u_2=a_2, \quad u_3=a_3, \quad u_4=a_4,$$

with the result of giving a primitive that involves the four arbitrary constants.

If this elimination is not possible, or if the eliminant does not possess the assumed form, then we must proceed otherwise. It is easy to see that not more than two of the quantities u_1, u_2, u_3, u_4 can be free from the variables l, m, n; for if three of them, say u_1, u_2, u_3, are functions of x, y, z, v only, we should have

$$l\frac{\partial u_1}{\partial v}+\frac{\partial u_1}{\partial x}+\lambda=0,$$

$$l\frac{\partial u_2}{\partial v}+\frac{\partial u_2}{\partial x}+\lambda=0,$$

$$l\frac{\partial u_3}{\partial v}+\frac{\partial u_3}{\partial x}+\lambda=0,$$

where $\lambda=\lambda(x_1, \ldots, x_7, 0, 0, 0)$: and corresponding equations hold for derivatives with regard to y and to z. These equations imply that

$$u_1-u_2=\omega(u_2-u_3),$$

where ω is a functional form: the integrals u_1, u_2, u_3 are not then independent. Accordingly, at least two of the four quantities, say u_3 and u_4, involve some of the variables l, m, n; hence

$$u_3=\phi(u_1, u_2),$$

$$u_4=\psi(u_1, u_2),$$

are general intermediate integrals, ϕ and ψ being arbitrary functions. On account of the relations

$$[u_i, u_j]=0, \qquad (i, j=1, 2, 3, 4),$$

these intermediate integrals coexist: and the primitive can be obtained by integrating either of them, or by integrating both of them as a simultaneous system: the general integral of either, regarded as an equation of the first order, leads to a primitive.

Moreover, it will be found that a knowledge of the form of u_4 is of substantial assistance in the integration of the equations subsidiary to the integration of

$$u_3=\phi(u_1, u_2);$$

and, similarly, with the knowledge of u_3 in relation to the integration of

$$u_4=\psi(u_1, u_2):$$

these results are easily established by considering the characteristic of each of these equations of the first order.

Ex. 3. Prove that the equation

$$x^2z\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} - x^2n\,(ab-h^2) - 2z\,(xl-my)\,(bc-f^2) + 4xzm\,(ch-fg)$$

$$= 4xmnh - 2n\,(xl-my)\,b + 4m^2zc - 4m^2n,$$

has an integral

$$v = a_1x^3 + a_2x^2y + a_3z^2 + a_4\,;$$

and deduce the general primitive. (Vivanti.)

Ex. 4. Verify that the conditions in the preceding discussion for the equation

$$\begin{vmatrix} a-A, & h-H, & g-G \\ h-H, & b-B, & f-F \\ g-G, & f-F, & c-C \end{vmatrix} = 0$$

are satisfied, (i), when

$$A=\lambda,\quad B=\mu,\quad F=a,\quad G=\beta,\quad H=\gamma,\quad C=\begin{vmatrix} l, & m, & n \\ \lambda, & \gamma, & \beta \\ \gamma, & \mu, & a \end{vmatrix},$$

where λ, μ, a, β, γ are constants: (ii) also, when

$$A=\frac{1+l^2}{v},\quad B=\frac{1+m^2}{v},\quad C=\frac{1+n^2}{v},$$

$$F=\frac{mn}{v},\quad G=\frac{nl}{v},\quad H=\frac{lm}{v}\,;$$

(iii) also, when*

$$A=\frac{l}{x},\quad B=\frac{m}{y},\quad C=\frac{n}{z},\quad F=0,\quad G=0,\quad H=0.$$

Obtain the general primitive in the respective cases.

Cases when the Three Relations are not a Complete System.

320. We still have to consider the case in which the relations

$$(L, M)=0,\quad (M, N)=0,\quad (N, L)=0,$$

are satisfied, though not identically; then they are equations additional to $L=0$, $M=0$, $N=0$. They are of the form

$$\nu_1p_1+\nu_2=0,$$

$$\lambda_1p_1+\lambda_2=0,$$

$$\mu_1p_1+\mu_2=0,$$

* This third example is due to Tanner, *Proc. L. M. S.*, t. VII (1876), p. 89.

where

$$\nu_1 = \frac{\partial \mu}{\partial p_5} - \frac{\partial \lambda}{\partial p_6}, \quad \lambda_1 = \frac{\partial \nu}{\partial p_6} - \frac{\partial \mu}{\partial p_7}, \quad \mu_1 = \frac{\partial \lambda}{\partial p_7} - \frac{\partial \nu}{\partial p_5}.$$

The most important case arises when they are equivalent to only a single additional equation: and this can occur in three kinds of ways, viz.

(i) two of the conditions may be satisfied identically, and the remaining condition then gives the new equation:

(ii) one of the conditions may be satisfied identically, and the other two give new equations which are equivalent to one another:

(iii) no one of the conditions may be satisfied identically, but the three are equivalent to one another.

Let the new equation be $P = 0$. Then the relations

$$(P, L) = 0, \quad (P, M) = 0, \quad (P, N) = 0,$$

must also be satisfied, either identically or in virtue of the equations of the system

$$L = 0, \quad M = 0, \quad N = 0, \quad P = 0.$$

We shall assume that this requirement is actually met without the association of other new equations. The system is a complete Jacobian system; as it involves seven variables, it possesses three algebraically independent integrals, a set of which may be denoted by u_1, u_2, u_3.

We then resolve the equations

$$u_1 = a_1', \quad u_2 = a_2', \quad u_3 = a_3',$$

together with the four equations of the complete system, so as to give the values of $p_1, \ldots, p_7$, the quantities a_1', a_2', a_3' being constants. The values are substituted in

$$du = \sum_{i=1}^{7} p_i dx_i;$$

and quadrature is effected, giving an equation of the form

$$u = \omega(v, x, y, z, l, m, n, a_1', a_2', a_3') + a_4',$$

where a_4' is an arbitrary constant. Now $u = 0$ is the intermediate integral: adopting this value of u, and dividing out by one of the

arbitrary constants, say by a_3', we have the intermediate integral in a form

$$\omega(v, x, y, z, l, m, n, a_1, a_2) + a_3 = 0.$$

This may be called a *complete intermediate integral*, as it contains the greatest number of arbitrary independent constants which generally can be eliminated from the equations

$$\omega + a_3 = 0, \quad \frac{d\omega}{dx} = 0, \quad \frac{d\omega}{dy} = 0, \quad \frac{d\omega}{dz} = 0:$$

the eliminant is the differential equation required.

A general intermediate integral, obtained in the usual manner from the complete intermediate integral, is given by the elimination of a_1 and a_2 between

$$\left.\begin{aligned} \omega(v, x, y, z, l, m, n, a_1, a_2) + \phi(a_1, a_2) &= 0 \\ \frac{\partial\omega}{\partial a_1} + \frac{\partial\phi}{\partial a_1} &= 0 \\ \frac{\partial\omega}{\partial a_2} + \frac{\partial\phi}{\partial a_2} &= 0 \end{aligned}\right\},$$

where ϕ is an arbitrary constant.

In order to proceed to the primitive, we integrate either of the intermediate integrals as an equation of the first order: its general integral will be a general primitive of the differential equation of the second order.

This primitive has been obtained from the set of equations $L = 0$, $M = 0$, $N = 0$, which may be only one of several sets of equations deduced from the original conditions. When there are other sets, each of them must be discussed: and each may lead to a primitive. The various primitives are so many branches of the final primitive.

Generalisation of Monge's Equation.

321. One of the simplest classes of equations is constituted by the generalisation of equations which belong to the type considered by Monge. Let θ, ϕ, ψ denote three algebraically independent functions of v, x, y, z, l, m, n: then an equation of the first order is given by

$$F(\theta, \phi, \psi) = 0,$$

where F is an arbitrary function. Also, let

$$\frac{d}{dx}=\frac{\partial}{\partial x}+l\frac{\partial}{\partial v}+a\frac{\partial}{\partial l}+h\frac{\partial}{\partial m}+g\frac{\partial}{\partial n},$$

$$\frac{d}{dy}=\frac{\partial}{\partial y}+m\frac{\partial}{\partial v}+h\frac{\partial}{\partial l}+b\frac{\partial}{\partial m}+f\frac{\partial}{\partial n},$$

$$\frac{d}{dz}=\frac{\partial}{\partial z}+n\frac{\partial}{\partial v}+g\frac{\partial}{\partial l}+f\frac{\partial}{\partial m}+c\frac{\partial}{\partial n}:$$

then, in order to construct an equation of the second order which has $F=0$ for an intermediate integral (and which therefore will be of the class under consideration), it is sufficient to eliminate the derivatives of F between the equations

$$\frac{\partial F}{\partial\theta}\frac{d\theta}{dx}+\frac{\partial F}{\partial\phi}\frac{d\phi}{dx}+\frac{\partial F}{\partial\psi}\frac{d\psi}{dx}=0,$$

$$\frac{\partial F}{\partial\theta}\frac{d\theta}{dy}+\frac{\partial F}{\partial\phi}\frac{d\phi}{dy}+\frac{\partial F}{\partial\psi}\frac{d\psi}{dy}=0,$$

$$\frac{\partial F}{\partial\theta}\frac{d\theta}{dz}+\frac{\partial F}{\partial\phi}\frac{d\phi}{dz}+\frac{\partial F}{\partial\psi}\frac{d\psi}{dz}=0.$$

Obviously, the equation is

$$\begin{vmatrix} \frac{d\theta}{dx}, & \frac{d\phi}{dx}, & \frac{d\psi}{dx} \\ \frac{d\theta}{dy}, & \frac{d\phi}{dy}, & \frac{d\psi}{dy} \\ \frac{d\theta}{dz}, & \frac{d\phi}{dz}, & \frac{d\psi}{dz} \end{vmatrix}=0,$$

which, when expanded in full, is

$$\begin{aligned} D\Delta+PA+QB+RC+2SF+2TG+2UH& \\ +Ia+Jb+Kc+2Lf+2Mg+2Nh&=W, \end{aligned}$$

where

$$\Delta=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2,$$

$$A=bc-f^2,\quad B=ca-g^2,\quad C=ab-h^2,$$

$$F=gh-af,\quad G=hf-bg,\quad H=fg-ch,$$

and the coefficients are various combinations of the derivatives of θ, ϕ, ψ. As there are fourteen coefficients in the equation, all dependent upon θ, ϕ, ψ, it is manifest that a considerable number of relations among them must be satisfied.

In particular,

$$D = J\left(\frac{\theta,\ \phi,\ \psi}{l,\ m,\ n}\right), \quad W = J\left(\frac{\theta,\ \phi,\ \psi}{x,\ y,\ z}\right).$$

Now the form of equation thus obtained is the only possible form when an intermediate integral of the assumed functional form exists: but an equation of that form does not necessarily possess such an intermediate integral, for (as we have indicated) certain conditions must be satisfied. The conditions may be obtained as follows.

322. Assuming that the equation of the second order has an intermediate integral

$$u(v,\ x,\ y,\ z,\ l,\ m,\ n) = 0,$$

and having regard to its relation to the equation, we know that when the equations

$$\begin{aligned} u_x + au_l + hu_m + gu_n &= 0,\\ u_y + hu_l + bu_m + fu_n &= 0,\\ u_z + gu_l + fu_m + cu_n &= 0,\end{aligned}$$

are used to eliminate three derivatives of the second order (say a, b, c) from

$$\begin{aligned} D\Delta + PA + QB + RC + 2SF + 2TG + 2UH&\\ + Ia + Jb + Kc + 2Lf + 2Mg + 2Nh &= W,\end{aligned}$$

the resulting equation must be evanescent. Now

$$a = -\frac{u_m u_n}{u_l}\left(\frac{u_x}{u_m u_n} + \frac{g}{u_m} + \frac{h}{u_n}\right):$$

writing

$$X = \frac{u_x}{u_m u_n}, \quad Y = \frac{u_y}{u_n u_l}, \quad Z = \frac{u_z}{u_l u_m},$$

$$\phi = \frac{f}{u_l}, \quad \gamma = \frac{g}{u_m}, \quad \eta = \frac{h}{u_n},$$

we have

$$a = -\frac{u_m u_n}{u_l}(X \qquad + \gamma + \eta);$$

similarly,

$$b = -\frac{u_n u_l}{u_m}(Y + \phi \qquad + \eta),$$

$$c = -\frac{u_l u_m}{u_n}(Z + \phi + \gamma \qquad).$$

Again,

$$A = bc - f^2$$
$$= u_l^2\{YZ + \phi(Y+Z) + \gamma Y + \eta Z + \phi\gamma + \gamma\eta + \eta\phi\},$$

with similar expressions for B and C: also,

$$F = gh - af$$
$$= u_m u_n(X\phi + \phi\gamma + \gamma\eta + \eta\phi),$$

with similar expressions for G and H. Lastly,

$$\Delta = -u_l u_m u_n\{XYZ + (\phi\gamma + \gamma\eta + \eta\phi)(X + Y + Z)$$
$$+ \phi(XY + XZ) + \gamma(YX + YZ) + \eta(ZX + ZY)\}.$$

Substituting these values in the given equation, and equating to zero the coefficients of the various combinations of ϕ, γ, η, so that the modified equation is evanescent, we have various relations.

The coefficient of $\phi\gamma + \gamma\eta + \eta\phi$ yields the relation

$$-Du_l u_m u_n(X + Y + Z) + Pu_l^2 + Qu_m^2 + Ru_n^2$$
$$+ 2Su_m u_n + 2Tu_n u_l + 2Uu_l u_m = 0.$$

The coefficients of ϕ, of γ, and of η, yield the relations

$$-Du_l u_m u_n(XY + XZ) - J\frac{u_n u_l}{u_m} - K\frac{u_l u_m}{u_n} + 2Lu_l$$
$$+ P(Y + Z)u_l^2 + QXu_m^2 + RXu_n^2 + 2SXu_m u_n = 0,$$

$$-Du_l u_m u_n(YX + YZ) - I\frac{u_m u_n}{u_l} - K\frac{u_l u_m}{u_n} + 2Mu_m$$
$$+ PYu_l^2 + Q(X + Z)u_m^2 + RYu_n^2 + 2TYu_n u_l = 0,$$

$$-Du_l u_m u_n(ZX + ZY) - I\frac{u_m u_n}{u_l} - J\frac{u_n u_l}{u_m} + 2Nu_n$$
$$+ PZu_l^2 + QZu_m^2 + R(X + Y)u_n^2 + 2UZu_l u_m = 0,$$

respectively; and the terms, independent of ϕ, γ, η, yield the relation

$$-Du_l u_m u_n XYZ - IX\frac{u_m u_n}{u_l} - JY\frac{u_n u_l}{u_m} - KZ\frac{u_l u_m}{u_n}$$
$$+ PYZu_l^2 + QZXu_m^2 + RXYu_n^2 = W.$$

Apparently, there are five relations involving the quantities u_x, u_y, u_z, u_l, u_m, u_n: they can, under conditions, be reduced to a set (or to sets) of three equations.

The forms of these relations (and the remembrance of the subsidiary equations in Boole's method in the case of two independent variables) suggest the homogeneous linear forms

$$\left.\begin{aligned} Du_x &= Pu_l + \gamma' u_m + \beta u_n \\ Du_y &= \gamma u_l + Qu_m + \alpha' u_n \\ Du_z &= \beta' u_l + \alpha u_m + Ru_n \end{aligned}\right\}:$$

in these at this stage, we shall regard α, α', β, β', γ, γ' as six quantities to be determined and as independent of u_l, u_m, u_n.

The first of the preceding five relations is then satisfied identically, if

$$\left.\begin{aligned} \gamma + \gamma' &= 2U \\ \beta + \beta' &= 2T \\ \alpha + \alpha' &= 2S \end{aligned}\right\},$$

which accordingly will be regarded as three equations for the determination of the six quantities.

The second of the five relations is satisfied identically, if

$$\beta\beta' = PR - DJ,$$
$$\gamma\gamma' = PQ - DK,$$
$$\beta\gamma + \beta'\gamma' = 2(LD + SP);$$

the third is satisfied identically if, further,

$$\alpha\alpha' = QR - DI,$$
$$\alpha\gamma + \alpha'\gamma' = 2(MD + TQ);$$

the fourth is satisfied identically if, further,

$$\alpha\beta + \alpha'\beta' = 2(ND + UR);$$

and the fifth is satisfied identically if, further,

$$\alpha\beta\gamma + \alpha'\beta'\gamma' = 2PQR - PID - QJD - RKD - D^2W.$$

These equations for α, α', β, β', γ, γ' imply relations among the coefficients of the differential equation, which must be satisfied if it is to possess an intermediate integral of the assumed type.

Moreover, these linear forms in the derivatives of u secure that the equation is satisfied: for, substituting from the equation

$$u_x + au_l + hu_m + gu_n = 0,$$

in the equation

$$Du_x = Pu_l + \gamma' u_m + \beta u_n,$$

we have

$$(aD + P)\, u_l + (hD + \gamma')\, u_m + (gD + \beta)\, u_n = 0\,;$$

with two similar equations. Eliminating u_l, u_m, u_n, we have

$$\begin{vmatrix} aD + P, & hD + \gamma', & gD + \beta \\ hD + \gamma, & bD + Q, & fD + \alpha' \\ gD + \beta', & fD + \alpha, & cD + R \end{vmatrix} = 0\,;$$

when this determinant is expanded, and the relations among the quantities α, α', β, β', γ, γ' are used, the initial equation is reproduced.

323. Accordingly, we shall assume that the appropriate relations among the coefficients of the given equation are satisfied; so that α, α', β, β', γ, γ' are determinate, and have one set (or several sets) of values. The equations for u then have the form

$$\Delta_1(u) = \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v} - \frac{1}{D}\left(P\frac{\partial u}{\partial l} + \gamma'\frac{\partial u}{\partial m} + \beta\frac{\partial u}{\partial n}\right),$$

$$\Delta_2(u) = \frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v} - \frac{1}{D}\left(\gamma\frac{\partial u}{\partial l} + Q\frac{\partial u}{\partial m} + \alpha'\frac{\partial u}{\partial n}\right),$$

$$\Delta_3(u) = \frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v} - \frac{1}{D}\left(\beta'\frac{\partial u}{\partial l} + \alpha\frac{\partial u}{\partial m} + R\frac{\partial u}{\partial n}\right),$$

on the assumption that D is not zero. They must satisfy the Poisson-Jacobi conditions

$$(\Delta_1, \Delta_2) = 0, \quad (\Delta_2, \Delta_3) = 0, \quad (\Delta_3, \Delta_1) = 0\,;$$

these are

$$0 = \frac{\gamma - \gamma'}{D}\frac{\partial u}{\partial v} + \left\{\Delta_2\left(\frac{P}{D}\right) - \Delta_1\left(\frac{\gamma}{D}\right)\right\}\frac{\partial u}{\partial l} + \left\{\Delta_2\left(\frac{\gamma'}{D}\right) - \Delta_1\left(\frac{Q}{D}\right)\right\}\frac{\partial u}{\partial m}$$
$$+ \left\{\Delta_2\left(\frac{\beta}{D}\right) - \Delta_1\left(\frac{\alpha'}{D}\right)\right\}\frac{\partial u}{\partial n},$$

$$0 = \frac{\alpha - \alpha'}{D}\frac{\partial u}{\partial v} + \left\{\Delta_3\left(\frac{\gamma}{D}\right) - \Delta_2\left(\frac{\beta'}{D}\right)\right\}\frac{\partial u}{\partial l} + \left\{\Delta_3\left(\frac{Q}{D}\right) - \Delta_2\left(\frac{\alpha}{D}\right)\right\}\frac{\partial u}{\partial m}$$
$$+ \left\{\Delta_3\left(\frac{\alpha'}{D}\right) - \Delta_2\left(\frac{R}{D}\right)\right\}\frac{\partial u}{\partial n},$$

$$0 = \frac{\beta - \beta'}{D}\frac{\partial u}{\partial v} + \left\{\Delta_1\left(\frac{\beta'}{D}\right) - \Delta_3\left(\frac{P}{D}\right)\right\}\frac{\partial u}{\partial l} + \left\{\Delta_1\left(\frac{\alpha}{D}\right) - \Delta_3\left(\frac{\gamma'}{D}\right)\right\}\frac{\partial u}{\partial m}$$
$$+ \left\{\Delta_1\left(\frac{R}{D}\right) - \Delta_3\left(\frac{\beta}{D}\right)\right\}\frac{\partial u}{\partial n},$$

respectively. They clearly are not satisfied in virtue of

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0.$$

If they are satisfied identically, then

$$\alpha = \alpha' = S, \quad \beta = \beta' = T, \quad \gamma = \gamma' = U,$$

$$\left.\begin{aligned} \Delta_2\left(\frac{P}{D}\right) &= \Delta_1\left(\frac{S}{D}\right) \\ \Delta_3\left(\frac{P}{D}\right) &= \Delta_1\left(\frac{T}{D}\right) \end{aligned}\right\}, \quad \left.\begin{aligned} \Delta_1\left(\frac{Q}{D}\right) &= \Delta_2\left(\frac{U}{D}\right) \\ \Delta_3\left(\frac{Q}{D}\right) &= \Delta_2\left(\frac{S}{D}\right) \end{aligned}\right\}, \quad \left.\begin{aligned} \Delta_1\left(\frac{R}{D}\right) &= \Delta_3\left(\frac{T}{D}\right) \\ \Delta_2\left(\frac{R}{D}\right) &= \Delta_3\left(\frac{S}{D}\right) \end{aligned}\right\},$$

$$\Delta_1\left(\frac{S}{D}\right) = \Delta_2\left(\frac{T}{D}\right) = \Delta_3\left(\frac{U}{D}\right);$$

and the differential equation is

$$\begin{vmatrix} aD + P, & hD + U, & gD + T \\ hD + U, & bD + Q, & fD + S \\ gD + T, & fD + S, & cD + R \end{vmatrix} = 0.$$

The case has already been discussed (§ 319, Ex. 2). We shall assume the alternative hypothesis, that the Poisson-Jacobi conditions are not satisfied identically: they therefore provide a new equation or new equations. We shall suppose that they provide one new equation in one or other of the three kinds of ways above indicated: let it be

$$\Delta_4 = 0.$$

The presence of this additional equation requires that the additional Poisson-Jacobi conditions

$$(\Delta_1, \Delta_4) = 0, \quad (\Delta_2, \Delta_4) = 0, \quad (\Delta_3, \Delta_4) = 0,$$

shall be satisfied: we shall assume that they are satisfied without the association of any new equations. The system of equations

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0, \quad \Delta_4 = 0,$$

is complete; as it involves seven independent variables, it possesses three algebraically independent integrals which may be denoted by u_1, u_2, u_3. Then

$$\phi(u_1, u_2, u_3) = 0,$$

where ϕ is any arbitrary function of its arguments, is an integral of the system: it manifestly also is a general intermediate integral of the original equation.

This integral is provided by a subsidiary system associated with one set of values of α, α', β, β', γ, γ'. It might happen that another intermediate integral is provided by the subsidiary system associated with a different set of values of α, α', β, β', γ, γ', the additional conditions of course being supposed to be satisfied: let it be denoted by

$$\psi(u_1', u_2', u_3') = 0,$$

where ψ is arbitrary, and u_1', u_2', u_3' are the three algebraically independent integrals of the new subsidiary system.

It is easy to assign the circumstances which allow these two intermediate integrals (if obtainable) to be treated as simultaneous. The quantities u_1, u_2, u_3 are integrals of the first system, each equation in which is homogeneous and linear in the derivatives of u: hence ϕ is also an integral, and we have

$$\left.\begin{aligned} D\phi_x &= P\phi_l + \gamma'\phi_m + \beta\phi_n \\ D\phi_y &= \gamma\phi_l + Q\phi_m + \alpha'\phi_n \\ D\phi_z &= \beta'\phi_l + \alpha\phi_m + R\phi_n \end{aligned}\right\}.$$

Similarly, as u_1', u_2', u_3' are integrals of an alternative system, ψ also is an integral of that system which may be taken to have the form

$$\left.\begin{aligned} D\psi_x &= P\psi_l + \Gamma'\psi_m + B\psi_n \\ D\psi_y &= \Gamma\psi_l + Q\psi_m + A'\psi_n \\ D\psi_z &= B'\psi_l + A\psi_m + R\psi_n \end{aligned}\right\},$$

where

A, A' are either α, α'; or α', α:

B, B' .... β, β'; or β', β:

Γ, Γ' .... γ, γ'; or γ', γ:

the set of first alternatives throughout giving the system for ϕ. In order that the equations

$$\phi = 0, \quad \psi = 0,$$

may coexist, the relation

$$[\phi, \psi] = 0$$

must be satisfied, that is, we must have

$$\phi_x\psi_l - \psi_x\phi_l + \phi_y\psi_m - \psi_y\phi_m + \phi_z\psi_n - \psi_z\phi_n = 0.$$

Substituting for $\phi_x, \phi_y, \phi_z, \psi_x, \psi_y, \psi_z$ from the equations in which they respectively occur and collecting terms, we have

$$\begin{aligned}(\gamma' - \Gamma)\,\phi_m\psi_l + (\gamma - \Gamma')\,\psi_m\phi_l \\ + (\alpha' - A)\,\phi_n\psi_m + (\alpha - A')\,\psi_n\phi_m \\ + (\beta' - B)\,\phi_l\psi_n + (\beta - B')\,\psi_l\phi_n = 0.\end{aligned}$$

Evidently this is satisfied identically, when

$$\begin{aligned} A &= \alpha', \quad A' = \alpha, \\ B &= \beta', \quad B' = \beta, \\ \Gamma &= \gamma', \quad \Gamma' = \gamma: \end{aligned}$$

and therefore we have the result:

If all the conditions for the possession of three algebraically independent integrals be satisfied for each of the systems

$$\left.\begin{aligned} D\phi_x &= P\phi_l + \gamma'\phi_m + \beta\phi_n \\ D\phi_y &= \gamma\phi_l + Q\phi_m + \alpha'\phi_n \\ D\phi_z &= \beta'\phi_l + \alpha\phi_m + R\phi_n \end{aligned}\right\}, \qquad \left.\begin{aligned} D\psi_x &= P\psi_l + \gamma\psi_m + \beta'\psi_n \\ D\psi_y &= \gamma'\psi_l + Q\psi_m + \alpha\psi_n \\ D\psi_z &= \beta\psi_l + \alpha'\psi_m + R\psi_n \end{aligned}\right\},$$

then any intermediate integral of the differential equation provided by the first system can be associated with any intermediate integral provided by the second system.

Moreover, when we proceed to integrate either intermediate integral as an equation of the first order so as to obtain the primitive, the subsidiary equations for that integration include the equations subsidiary to the construction of an integral of the other system. When any integrals of the other system are known, they can be used to simplify the integration that leads to the primitive.

Other theorems, analogous to the corresponding theorems for equations in two independent variables, can similarly be obtained. We shall not enter into the further development of the details connected with the type of equation under consideration: the method has been sufficiently outlined to allow of application to any particular equation.

Ex. 1. Integrate the equation

$$\begin{vmatrix} a+a_1, & h+a_6, & g+a_5 \\ h+a_6, & b+a_2, & f+a_4 \\ g+a_5, & f+a_4, & c+a_3 \end{vmatrix} + a_7^2(c+a_8) = 0,$$

where all the quantities $a_1, \ldots, a_7$ are constants, shewing that, when certain conditions of inequality are satisfied, there are two subsidiary systems.

Ex. 2. Integrate the equation

$$\begin{vmatrix} a, & h\,, & g, & l \\ h, & b\,, & f, & m \\ g, & f\,, & c, & n \\ l\,, & m, & n, & v \end{vmatrix} = 0.$$

The Linear Equation.

324. In the preceding illustrations of the theory, it has been assumed that D is not zero. When D is zero, it is simpler, in any particular case, to re-apply the process from the beginning than to modify the equations of the more general form. By way of illustration, we shall re-apply it to the equation

$$Aa + 2Hh + Bb + 2Gg + 2Ff + Cc = K,$$

where the quantities A, B, C, F, G, H, K are functions of x, y, z, v, l, m, n, and do not involve any derivative of the second order.

Substituting from the equations

$$u_x + au_l + hu_m + gu_n = 0,$$

$$u_y + hu_l + bu_m + fu_n = 0,$$

$$u_z + gu_l + fu_m + cu_n = 0,$$

for a, b, c in terms of f, g, h, and making the transformed equation evanescent, we have

$$2F - B\frac{u_n}{u_m} - C\frac{u_m}{u_n} = 0,$$

$$2G - A\frac{u_n}{u_l} - C\frac{u_l}{u_n} = 0,$$

$$2H - A\frac{u_m}{u_l} - B\frac{u_l}{u_m} = 0,$$

$$K + A\frac{u_x}{u_l} + B\frac{u_y}{u_m} + C\frac{u_z}{u_n} = 0,$$

apparently four equations. But, on writing

$$u_l = \theta u_n, \quad u_m = \phi u_n,$$

the first three equations become

$$2F - \frac{B}{\phi} - C\phi = 0,$$

$$2G - \frac{A}{\theta} - C\theta = 0,$$

$$2H - A\frac{\phi}{\theta} - B\frac{\theta}{\phi} = 0:$$

hence

$$\begin{aligned} 4(GH - AF) &= A^2\frac{\phi}{\theta^2} + BC\frac{\theta^2}{\phi} - AC\phi - AB\theta \\ &= \left(A\frac{\phi}{\theta} - B\frac{\theta}{\phi}\right)\left(\frac{A}{\theta} - C\theta\right) \\ &= 4(H^2 - AB)^{\frac{1}{2}}(G^2 - AC)^{\frac{1}{2}}; \end{aligned}$$

and therefore, assuming that A does not vanish, we have

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0.$$

Thus one purely algebraical relation among the coefficients A, B, C, F, G, H must be satisfied*: and then the equations for u are three in number, viz.

$$\left.\begin{aligned} u_l &= \theta u_n \\ u_m &= \phi u_n \\ \frac{A}{\theta}u_x + \frac{B}{\phi}u_y + Cu_z + Ku_n &= 0 \end{aligned}\right\},$$

with possibly two values for θ and possibly two values for ϕ.

Now that the equations for u have been obtained, the same method as before can be used for the construction of the intermediate integral (when it exists) and for the consequent derivation of the primitive.

Ex. 1. As a single application, let it be required to find the general primitive of

$$x^2a + 2xyh + y^2b + 2xzg + 2yzf + z^2c = 0.$$

The condition

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0$$

* It was first given by Euler, though not from the point of view of this method of integration: *Inst. Calc.*, t. III, p. 448.

is obviously satisfied: and the equations for u are easily found to consist of the single system

$$\left.\begin{array}{c}\dfrac{u_l}{x}=\dfrac{u_m}{y}=\dfrac{u_n}{z}\\ xu_x+yu_y+zu_z=0\end{array}\right\}.$$

This is a complete Jacobian system as it stands: hence it possesses four algebraically independent integrals, and these can be taken in the form

$$\frac{y}{x},\ \frac{z}{x},\ \frac{lx+my+nz}{x},\ v-(lx+my+nz).$$

Here, no one of the quantities u_1, u_2, u_3, u_4 involves derivatives of u with regard to x, y, z, v, l, m, n. We adopt the method explained in the Note at the end of Ex. 2 in § 319: we have two general intermediate integrals

$$\frac{lx+my+nz}{x}=\phi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

$$v-(lx+my+nz)=\psi\left(\frac{y}{x},\ \frac{z}{x}\right),$$

where ϕ and ψ are arbitrary functions of their arguments. The general primitive of the original equation is at once given by treating these equations as simultaneous, which is known to be permissible: that primitive is

$$v=x\phi\left(\frac{y}{x},\ \frac{z}{x}\right)+\psi\left(\frac{y}{x},\ \frac{z}{x}\right).$$

Ex. 2. Integrate the equation

$$x^2a+2xyh+y^2b+2xzg+2yzf+z^2c=\alpha(xl+ym+zn)+\beta v,$$

where α and β are constants.

Ex. 3. Integrate the equation

$$7a+2b+c+9h+8g+3f=\frac{l+m+n}{x+y-z}. \qquad \text{(Vivanti.)}$$

Ex. 4. Deduce the primitive of the equation

$$l^2(bc-f^2)+2lm(fg-ch)+m^2(ac-g^2)$$
$$+2ln(fh-bg)+2mn(gh-af)+n^2(ab-h^2)=0,$$

by applying a contact-transformation to the equation in the preceding Ex. 1; or otherwise integrate the equation.

Subsidiary System in Differential Elements.

325. The preceding investigation depends upon the integration of simultaneous partial differential equations of the first order; and this integration, as usual, ultimately depends upon the integration of a system or of systems of simultaneous ordinary equations. There is an alternative method of proceeding,

which uses ordinary equations as directly subsidiary to the construction of the intermediate integral when it exists: they are the equations of the characteristics of the first order. The two methods bear to one another the same relation as do the corresponding methods applied to equations of the second order in two independent variables.

It will be sufficient for the present purpose if the method is applied to the equation

$$\begin{aligned}D\Delta + PA + QB + RC + 2SF + 2TG + 2UH&\\ + Ia + Jb + Kc + 2Lf + 2Mg + 2Nh &= W,\end{aligned}$$

already (§ 322) considered by the other method. Writing

$$\begin{aligned}dl &= a\,dx + h\,dy + g\,dz,\\ dm &= h\,dx + b\,dy + f\,dz,\\ dn &= g\,dx + f\,dy + c\,dz,\end{aligned}$$

and substituting, in the differential equation, values of a, b, c derived from these equations in the form

$$\begin{aligned}a &= \frac{dydz}{dx}\left(\frac{dl}{dydz} - \frac{g}{dy} - \frac{h}{dz}\right),\\ b &= \frac{dzdx}{dy}\left(\frac{dm}{dxdz} - \frac{f}{dx} - \frac{h}{dz}\right),\\ c &= \frac{dxdy}{dz}\left(\frac{dn}{dxdy} - \frac{f}{dx} - \frac{g}{dy}\right),\end{aligned}$$

we obtain the equations of the characteristic by making the resulting equation evanescent*. Hence, as

$$\begin{aligned}A &= bc - f^2\\ &= \left\{\frac{dmdn}{dxdzdydz} - \frac{f}{dx}\left(\frac{dm}{dxdz} + \frac{dn}{dxdy}\right) - \frac{g}{dy}\frac{dm}{dxdz} - \frac{h}{dz}\frac{dn}{dxdy}\right.\\ &\qquad\qquad \left. + \frac{fg}{dxdy} + \frac{gh}{dydz} + \frac{fh}{dxdz}\right\} dx^2,\end{aligned}$$

with similar expressions for B and C, and

$$\begin{aligned}F &= gh - af\\ &= dydz\left(\frac{fg}{dxdy} + \frac{gh}{dydz} + \frac{fh}{dxdz} - \frac{f}{dx}\frac{dl}{dydz}\right),\end{aligned}$$

* This is only a statement as to actual results: the argument is similar to that in the case of two variables (§ 233) and need not be repeated here.

with similar expressions for G and H, and

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

$$= \left\{ \frac{dl}{dydz} \frac{dm}{dzdx} \frac{dn}{dxdy} \right.$$

$$+ \left(\frac{fg}{dxdy} + \frac{gh}{dydz} + \frac{fh}{dxdz} \right) \left(\frac{dl}{dydz} + \frac{dm}{dzdx} + \frac{dn}{dxdy} \right)$$

$$- \frac{f}{dx} \frac{dl}{dydz} \left(\frac{dm}{dzdx} + \frac{dn}{dxdy} \right)$$

$$- \frac{g}{dy} \frac{dm}{dxdz} \left(\frac{dl}{dydz} + \frac{dn}{dxdy} \right)$$

$$\left. - \frac{h}{dz} \frac{dn}{dxdy} \left(\frac{dl}{dydz} + \frac{dm}{dxdz} \right) \right\} dxdydz,$$

we have

$$D(dxdl + dydm + dzdn)$$
$$+ Pdx^2 + Qdy^2 + Rdz^2 + 2Sdydz + 2Tdxdz + 2Udxdy = 0,$$

from the evanescence of the terms in fg, gh, hf:

$$Ddl(dydm + dzdn) + Pdx(dydm + dzdn)$$
$$+ dl(Qdy^2 + 2Sdydz + Rdz^2) + dx(Jdz^2 - 2Ldydz + Kdy^2) = 0,$$

from the evanescence of the term in f:

$$Ddm(dzdn + dxdl) + Qdy(dzdn + dxdl)$$
$$+ dm(Rdz^2 + 2Tdzdx + Pdx^2) + dy(Kdx^2 - 2Mdxdz + Idz^2) = 0,$$

from the evanescence of the term in g:

$$Ddn(dxdl + dydm) + Rdz(dxdl + dydm)$$
$$+ dn(Pdx^2 + 2Udxdy + Qdy^2) + dz(Idy^2 - 2Ndxdy + Jdx^2) = 0,$$

from the evanescence of the term in h: and

$$Ddldmdn + Idldydz + Jdmdzdx + Kdndxdy$$
$$+ Pdmdndx + Qdndldy + Rdldmdz = Wdxdydz,$$

being the aggregate of the remaining terms.

Proceeding as before, we reduce these five equations to three, conditionally on certain relations among the coefficients being

satisfied*: these relations are sufficient to secure the possibility of determining six quantities α, α', β, β', γ, γ', such that

$$\left.\begin{aligned}\alpha+\alpha'&=2S\\ \alpha\alpha'&=QR-DI\end{aligned}\right\},\quad \left.\begin{aligned}\beta+\beta'&=2T\\ \beta\beta'&=PR-DJ\end{aligned}\right\},\quad \left.\begin{aligned}\gamma+\gamma'&=2U\\ \gamma\gamma'&=PQ-DK\end{aligned}\right\},$$

$$\begin{aligned}\beta\gamma+\beta'\gamma'&=2(LD+SP),\\ \gamma\alpha+\gamma'\alpha'&=2(MD+TQ),\\ \alpha\beta+\alpha'\beta'&=2(ND+UR),\\ \alpha\beta\gamma+\alpha'\beta'\gamma'&=2PQR-PID-QJD-RKD-D^2W.\end{aligned}$$

We shall assume the conditions satisfied. When the differential relations are resolved, we have

$$\left.\begin{aligned}Ddl+Pdx+\gamma dy+\beta' dz&=0\\ Ddm+\gamma' dx+Qdy+\alpha dz&=0\\ Ddn+\beta dx+\alpha' dy+Rdz&=0\\ dv-ldx-mdy-ndz&=0\end{aligned}\right\},$$

as the equations of the characteristic of the first order.

It is easy to see that, if

$$du(x, y, z, v, l, m, n)=0$$

is a linear integrable combination of these equations, then

$$\begin{aligned}Du_x&=Pu_l+\gamma' u_m+\beta u_n,\\ Du_y&=\gamma u_l+Qu_m+\alpha' u_n,\\ Du_z&=\beta' u_l+\alpha u_m+Ru_n,\end{aligned}$$

being the former set (§ 322) of equations for u.

Ex. 1. When the coefficients D, P, Q, R, S, T, U vanish, so that the equation has the linear form

$$Ia+Jb+Kc+2Lf+2Mg+2Nh=W,$$

the preceding equations are evanescent: prove that the equations of the characteristic of the first order are

$$\left.\begin{aligned}dy=\theta\, dx,\quad dz=\phi\, dx\\ Idl+\frac{J}{\theta}dm+\frac{K}{\phi}dn=Wdx\end{aligned}\right\},$$

* They are due to the fact that all the coefficients are certain functional combinations of the derivatives of three quantities, as explained in § 321; the explanation is similarly set out in Vivanti's memoir, quoted at the beginning of this chapter (p. 490).

where

$$I\theta^2-2N\theta+J=0, \quad I\phi^2-2M\phi+K=0;$$

and prove that

$$\begin{vmatrix} I, & N, & M \\ N, & J, & L \\ M, & L, & K \end{vmatrix}=0,$$

in order that there may be a characteristic of the first order.

(See, for comparison, § 324.)

Ex. 2. Obtain the equations of the characteristics of the first order of the equation

$$PA+QB+RC+2SF+2TG+2UH+Ia+Jb+Kc+2Lf+2Mg+2Nh=W,$$

obtaining the preliminary algebraic relations among the coefficients which must hold if there is to be a characteristic of the first order.

Ex. 3. Integrate the equation

$$a-h-g+f=0.$$

Ex. 4. A dependent variable z is a function of four independent variables x_1, x_2, x_3, x_4; and the derivatives of the first order and the second order are denoted by p_i, for $i=1, 2, 3, 4$, and by p_{ij}, for $i, j=1, 2, 3, 4$. Shew that, if the equation

$$\sum_{i=1}^{4}\sum_{j=1}^{4} A_{ij}\,p_{ij}=U$$

possesses an intermediate integral of the first order, where the coefficients A_{ij} are functions of the variables and the first derivatives only, and where $A_{ij}=A_{ji}$, then the minor of every term in the diagonal of the determinant

$$\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14} \\ A_{21}, & A_{22}, & A_{23}, & A_{24} \\ A_{31}, & A_{32}, & A_{33}, & A_{34} \\ A_{41}, & A_{42}, & A_{43}, & A_{44} \end{vmatrix}=0$$

vanishes: and obtain the equations of the characteristic of the first order.

Shew that the conditions are satisfied for the equation

$$\sum_{i=1}^{4}\sum_{j=1}^{4} x_i x_j\,p_{ij}=U,$$

where U involves $z, x_1, x_2, x_3, x_4, p_1\ p_2, p_3, p_4$ at the utmost: and construct the primitive in the cases

(i) $U=0$, (ii) $U=az$, (iii) $U=c\,(x_1p_1+x_2p_2+x_3p_3+x_4p_4)$,

where a and c are constants.

CHAPTER XXIV.

EQUATIONS OF THE SECOND ORDER IN THREE INDEPENDENT VARIABLES, NOT NECESSARILY HAVING AN INTERMEDIATE INTEGRAL.

THE present chapter, like the preceding chapter, is devoted to the extension, to equations involving a number of independent variables greater than two, of methods applicable to equations in only two independent variables. As before, the results are given and the notation is specially devised for equations in three independent variables: but many of the results can obviously be generalised to the case when the number of independent variables is n, though it has not seemed necessary to state them in the general form.

As regards the range of the chapter, no assumption is made (as was done in the preceding chapter) that an intermediate integral exists: and the particular methods, generalised from equations in two independent variables, are those of Ampère and of Darboux. The chapter is mainly based upon a memoir by the author*.

Some illustrations of the theory, in the case of n independent variables, are to be found in another memoir by the author†: they belong to the theory of symmetrical algebra.

Moreover, it is to be understood that only the general theory of the partial equations is considered: there is no attempt to construct and coordinate the properties of particular equations, however important these may be in mathematical physics‡. Similarly, there is no discussion of the integrals of particular equations as determined by so-called boundary conditions§.

* *Phil. Trans.*, vol. 191 (1898), pp. 1—86.

† *Camb. Phil. Trans.*, vol. XVI (1898), pp. 291—325.

‡ Such equations, together with applications to mathematical physics, are discussed in Weber's edition (in two volumes, 1900) of Riemann's lectures *Die partiellen Differentialgleichungen der mathematischen Physik.*

§ Full references will be found in Sommerfeld's article on this part of the subject, *Encyclopädie der mathematischen Wissenschaften*, t. II, pp. 504—570.

326. We now proceed to discuss the possibility of obtaining an integral of an equation

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

when no assumption is made concerning the existence of an intermediate integral. If a method of any generality can be devised, it should implicitly or explicitly lead to an intermediate integral (if any such exists).

By Cauchy's theorem, the equation usually possesses an integral which is determined by the value of v and of one of its first derivatives for an assigned relation between x, y, z. Regarding this relation as the equation of a surface, we can consider that Cauchy's theorem gives the values of v and l over the surface. If the surface be

$$S(x, y, z) = 0,$$

then, for all variations subject to the relation

$$\frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = 0,$$

we know the value of

$$l\,dx + m\,dy + n\,dz;$$

but l is known over the surface, hence m and n are known over the surface. Thus v, l, m, n can be regarded as known all over the surface, in connection with Cauchy's theorem.

Now consider the higher derivatives at points on the surface. Denoting the derivatives of z with regard to x and to y along the surface by p and q, we have

$$dl = a\,dx + h\,dy + g\,dz = (a + pg)\,dx + (h + qg)\,dy,$$
$$dm = h\,dx + b\,dy + f\,dz = (h + pf)\,dx + (b + qf)\,dy,$$
$$dn = g\,dx + f\,dy + c\,dz = (g + pc)\,dx + (f + qc)\,dy,$$

along the surface so that, as l, m, n are known everywhere on the surface, the quantities on the right-hand sides of the equations

$$a + pg = \frac{dl}{dx},\quad h + qg = \frac{dl}{dy},$$
$$h + pf = \frac{dm}{dx},\quad b + qf = \frac{dm}{dy},$$
$$g + pc = \frac{dn}{dx},\quad f + qc = \frac{dn}{dy},$$

are known along the surface. It is obvious that, if the values thus given are to be consistent with one another, we must have

$$\frac{dl}{dy} + p\frac{dn}{dy} = \frac{dm}{dx} + q\frac{dn}{dx}:$$

consequently, the equations are equivalent to five only, so that they can determine five of the quantities a, b, c, f, g, h, in terms of the remaining one, say

$$f = \frac{dn}{dy} - qc,$$

$$b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2c,$$

$$g = \frac{dn}{dx} - pc,$$

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2c,$$

$$h = \frac{dm}{dx} - p\frac{dn}{dy} + pqc = \frac{dl}{dy} - q\frac{dn}{dx} + pqc.$$

We also have

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

so that there are six equations to determine the six derivatives of the second order. These generally will be sufficient for the purpose; and therefore the six derivatives can be regarded as known along the surface.

Similarly, the derivatives of v of all orders can be regarded as known along the surface, being determinable in the same manner as a, b, c, f, g, h. Then, taking any point x_0, y_0, z_0 on the surface as an initial point, provided only that it is an ordinary point for the equation as given, we can expand v as a series of powers of $x - x_0$, $y - y_0$, $z - z_0$, the coefficients in which can be regarded as known. The convergence of the series can be established as in the proof of Cauchy's general theorem; and we then have the integral as established by that theorem.

This conclusion, however, is justified only if the six equations do actually determine a set or sets of values of a, b, c, f, g, h: it will fail of establishment, if sets of values are not determinately derivable. Such a result occurs for instance when $\phi = 0$ becomes

evanescent on the substitution of the values of a, b, f, g, h in terms of c: one condition then is

$$p^2\frac{\partial\phi}{\partial a}+pq\frac{\partial\phi}{\partial h}+q^2\frac{\partial\phi}{\partial b}-p\frac{\partial\phi}{\partial g}-q\frac{\partial\phi}{\partial f}+\frac{\partial\phi}{\partial c}=0,$$

but this, of course, is only one among a number of such equations. Moreover, even when the integral has been obtained, it is found only in the form of infinite power-series: consequently, it is desirable to possess other methods of proceeding to an integral.

Extension of Ampère's Method.

327. One such method, applicable to equations whose integrals are expressible in finite form without partial quadratures, is to be found in a generalisation of Ampère's method devised in connection with equations involving only two independent variables. Let u be an argument in an arbitrary function occurring in the integral; and let the independent variables be changed from x, y, z to x, y, u, on the supposition that u is not independent of z. Then

$$\begin{aligned} l\,dx+m\,dy+n\,dz &= dv \\ &= \frac{dv}{dx}dx+\frac{dv}{dy}dy+\frac{dv}{du}du: \end{aligned}$$

hence, if p and q denote the derivatives of z with regard to x and to y respectively when u is constant, we have

$$\left.\begin{aligned} l+np &= \frac{dv}{dx} \\ m+nq &= \frac{dv}{dy} \\ n\frac{\partial z}{\partial u} &= \frac{dv}{du} \end{aligned}\right\}.$$

Similarly,

$$\left.\begin{aligned} a+gp &= \frac{dl}{dx} \\ h+gq &= \frac{dl}{dy} \\ g\frac{\partial z}{\partial u} &= \frac{dl}{du} \end{aligned}\right\},\qquad \left.\begin{aligned} h+fp &= \frac{dm}{dx} \\ b+fq &= \frac{dm}{dy} \\ f\frac{\partial z}{\partial u} &= \frac{dm}{du} \end{aligned}\right\},\qquad \left.\begin{aligned} g+cp &= \frac{dn}{dx} \\ f+cq &= \frac{dn}{dy} \\ c\frac{\partial z}{\partial u} &= \frac{dn}{du} \end{aligned}\right\}.$$

As before, we have

$$\frac{dl}{dy} + p\frac{dn}{dy} = \frac{dm}{dx} + q\frac{dn}{dx},$$

a universal relation; and then, using the first two equations out of each of the three sets in order to express a, b, f, g, h in terms of c, we have

$$f = \frac{dn}{dy} - qc,$$

$$b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2c,$$

$$g = \frac{dn}{dx} - pc,$$

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2c,$$

$$h = \frac{dm}{dx} - p\frac{dn}{dy} + pqc = \frac{dl}{dy} - q\frac{dn}{dx} + pqc,$$

and we take

$$c = \frac{\dfrac{dn}{du}}{\dfrac{dz}{du}}, \quad n = \frac{\dfrac{dv}{du}}{\dfrac{dz}{du}}.$$

When these values of a, b, f, g, h are substituted in

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

it becomes an equation involving c: and when we suppose (as now will be assumed) that ϕ is a polynomial function of its various arguments a, b, c, f, g, h, of order μ', the transformed equation in c can be arranged in powers of c. It will take a form

$$E_0 + E_1c + \dots + E_\mu c^\mu = 0,$$

where μ is not greater than μ', and may be less than μ' if particular combinations of the derivatives of the second order occur in ϕ.

When the value of v is substituted, the equation must be satisfied identically. Now, in the expression

$$\frac{dn}{du} \div \frac{dz}{du},$$

being the value of c, the u-derivative of the arbitrary function which has u for an argument is of order higher than that of any other u-derivative contained in any of the quantities which make up $E_0, E_1, \ldots, E_\mu$: hence the equation can be satisfied identically, only if

$$E_\mu = 0, \quad E_{\mu-1} = 0, \ldots, E_1 = 0, \quad E_0 = 0.$$

If $\mu = \mu'$, then the equation $E_\mu = 0$ is the equivalent of the equation

$$p^2 \frac{\partial \phi}{\partial a} + pq \frac{\partial \phi}{\partial h} + q^2 \frac{\partial \phi}{\partial b} - p \frac{\partial \phi}{\partial g} - q \frac{\partial \phi}{\partial f} + \frac{\partial \phi}{\partial c} = 0;$$

if $\mu < \mu'$, this equation still is satisfied in virtue of the equations $E_\mu = 0, \ldots, E_0 = 0$: in either case, it is satisfied. Now, as p and q are the derivatives of z with regard to x and y when u is constant, we have

$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} = 0:$$

substituting for p and q, we have

$$A \left(\frac{\partial u}{\partial x}\right)^2 + H \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + B \left(\frac{\partial u}{\partial y}\right)^2 + G \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + F \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + C \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

where

$$A, B, C, F, G, H = \frac{\partial \phi}{\partial a}, \frac{\partial \phi}{\partial b}, \frac{\partial \phi}{\partial c}, \frac{\partial \phi}{\partial f}, \frac{\partial \phi}{\partial g}, \frac{\partial \phi}{\partial h}.$$

This relation must therefore be satisfied by an argument of an arbitrary function which occurs in the primitive of the original equation.

Characteristic Invariant.

328. But, further, this relation is invariantive for all changes of the independent variables. Suppose them changed according to the law

$$x' = \xi(x, y, z),$$

$$y' = \eta(x, y, z),$$

$$z' = \zeta(x, y, z);$$

and let $l', m', n', a', b', c', f', g', h'$ be the respective derivatives of v with regard to the new variables. Then

$$l, m, n = \left(\begin{array}{ccc} \xi_x, & \eta_x, & \zeta_x \\ \xi_y, & \eta_y, & \zeta_y \\ \xi_z, & \eta_z, & \zeta_z \end{array} \middle\} l', m', n' \right),$$

$$a = (a', b', c', f', g', h' \between \xi_x, \eta_x, \zeta_x)^2 + \dots,$$

$$h = (a', b', c', f', g', h' \between \xi_x, \eta_x, \zeta_x \between \xi_y, \eta_y, \zeta_y) + \dots,$$

$$\vdots$$

the omitted terms represented by $+ \dots$ being terms which involve derivatives of the first order only. Applying these transformations, let $\phi(v, x, y, z, l, m, n, a, b, c, f, g, h)$ become $\phi'(v, x', y', z', l', m', n', a', b', c', f', g', h')$; then

$$A' = \frac{\partial \phi'}{\partial a'} = A\xi_x^2 + H\xi_x\xi_y + B\xi_y^2 + G\xi_x\xi_z + F\xi_y\xi_z + C\xi_z^2,$$

$$H' = \frac{\partial \phi'}{\partial h'} = 2A\xi_x\eta_x + H(\xi_x\eta_y + \eta_x\xi_y) + 2B\xi_y\eta_y$$
$$+ G(\xi_x\eta_z + \xi_z\eta_x) + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z,$$

and so on: hence, if

$$u(x, y, z) = u'(x', y', z'),$$

so that

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} = \left(\begin{array}{ccc} \xi_x, & \eta_x, & \zeta_x \\ \xi_y, & \eta_y, & \zeta_y \\ \xi_z, & \eta_z, & \zeta_z \end{array} \middle\} \frac{\partial u'}{\partial x'}, \frac{\partial u'}{\partial y'}, \frac{\partial u'}{\partial z'} \right),$$

we have

$$A\left(\frac{\partial u}{\partial x}\right)^2 + H\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \dots = A'\left(\frac{\partial u'}{\partial x'}\right)^2 + H'\frac{\partial u'}{\partial x'}\frac{\partial u'}{\partial y'} + \dots,$$

after substitution and collection of terms.

The alleged property is thus established: on account of the property, the relation satisfied by u is called the *characteristic invariant* of the given differential equation.

329. The equations

$$E_0 = 0, \quad E_1 = 0, \dots, E_\mu = 0,$$

are resolved into sets as simple as possible: and then we seek integrable combinations, as many as possible, of each particular

set. In effecting the integrations, it is to be borne in mind that the quantity u is latent: and in order to take account of it, the equations

$$\frac{1}{n}\frac{dv}{du}=\frac{1}{g}\frac{dl}{du}=\frac{1}{f}\frac{dm}{du}=\frac{1}{c}\frac{dn}{du}=\frac{dz}{du}$$

(which, except for one part of the discussion, have been left on one side) must be used and be satisfied. When all the operations have been completed and limitations upon functional forms as required by the equations have been imposed, a number of relations will result.

The importance of the process lies in the fact that *the relations thus obtained satisfying all these subsidiary equations constitute an integral of the original equation.* For suppose that, in the expression which v acquires from the relations, we consider u eliminated in favour of z: then

$$\begin{aligned}\frac{dv}{dx}dx+\frac{dv}{dy}dy+\frac{dv}{du}du&=dv\\&=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}\left(p\,dx+q\,dy+\frac{dz}{du}du\right),\end{aligned}$$

whence

$$\frac{dv}{du}=\frac{\partial v}{\partial z}\frac{dz}{du},$$

$$\frac{dv}{dx}=\frac{\partial v}{\partial z}p+\frac{\partial v}{\partial x},$$

$$\frac{dv}{dy}=\frac{\partial v}{\partial z}q+\frac{\partial v}{\partial y}.$$

Comparing these with the equations of the system that led to the integral relation, we have

$$l=\frac{\partial v}{\partial x},\quad m=\frac{\partial v}{\partial y},\quad n=\frac{\partial v}{\partial z}.$$

Next, take a quantity c defined by the equation

$$c=\frac{\dfrac{dn}{du}}{\dfrac{dz}{du}},$$

so that, in virtue of the integral system, its value is determinate. Since the equations

$$\frac{dv}{dx}=l+np,\quad \frac{dv}{du}=n\frac{dz}{du},$$

are satisfied by the integral system (for they are members of the subsidiary system), we have

$$\frac{d}{du}(l + np) = \frac{d}{dx}\left(n\frac{dz}{du}\right),$$

and therefore

$$\frac{dl}{du} + p\frac{dn}{du} = \frac{dn}{dx}\frac{dz}{du},$$

the terms $n\dfrac{d^2z}{dxdu}$ on both sides cancelling: consequently,

$$\begin{aligned}\frac{dl}{du} &= \frac{dn}{dx}\frac{dz}{du} - p\frac{dn}{du}\\ &= \left(\frac{dn}{dx} - pc\right)\frac{dz}{du}\\ &= g\frac{dz}{du}.\end{aligned}$$

Similarly, from

$$\frac{dv}{dy} = m + nq, \quad \frac{dv}{du} = n\frac{dz}{du},$$

we can prove that the integral relations lead to

$$\frac{dm}{du} = f\frac{dz}{du}.$$

Again, we have

$$\begin{aligned}&\frac{\partial^2 v}{\partial x^2}dx + \frac{\partial^2 v}{\partial x\partial y}dy + \frac{\partial^2 v}{\partial x\partial z}\left(p\,dx + q\,dy + \frac{dz}{du}du\right)\\ &= d\left(\frac{\partial v}{\partial x}\right)\\ &= dl\\ &= \frac{dl}{dx}dx + \frac{dl}{dy}dy + \frac{dl}{du}du,\end{aligned}$$

and therefore

$$\begin{aligned}\frac{dl}{du} &= \frac{dz}{du}\frac{\partial^2 v}{\partial x\partial z},\\ \frac{dl}{dx} &= p\frac{\partial^2 v}{\partial x\partial z} + \frac{\partial^2 v}{\partial x^2},\\ \frac{dl}{dy} &= q\frac{\partial^2 v}{\partial x\partial z} + \frac{\partial^2 v}{\partial x\partial y}.\end{aligned}$$

Comparing these with the former equations, we have

$$a = \frac{\partial^2 v}{\partial x^2}, \quad h = \frac{\partial^2 v}{\partial x\partial y}, \quad g = \frac{\partial^2 v}{\partial x\partial z}.$$

Similarly, we find

$$b=\frac{\partial^2 v}{\partial y^2}, \quad f=\frac{\partial^2 v}{\partial y \partial z}, \quad c=\frac{\partial^2 v}{\partial z^2}.$$

Again, when we take the combination

$$E_0 + cE_1 + \ldots + c^{\mu} E_{\mu} = 0,$$

and eliminate from this equation the derivatives of l, m, n with regard to x and y by means of the other equations, we have

$$\phi(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0:$$

and the quantities $l, m, n, a, b, c, f, g, h$ are such that

$$l,\ m,\ n = \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial z},$$

$$a,\ b,\ c,\ f,\ g,\ h = \frac{\partial^2 v}{\partial x^2},\ \frac{\partial^2 v}{\partial y^2},\ \frac{\partial^2 v}{\partial z^2},\ \frac{\partial^2 v}{\partial y \partial z},\ \frac{\partial^2 v}{\partial z \partial x},\ \frac{\partial^2 v}{\partial x \partial y}:$$

that is, v is an integral of the original differential equation. We therefore may summarise the result as follows:

When an equation of the second order $\phi = 0$ is transformed into

$$E_0 + E_1 c + \ldots + E_{\mu} c^{\mu} = 0,$$

by the elimination of a, b, f, g, h through the equations

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2 c, \quad b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2 c,$$

$$g = \frac{dn}{dx} - pc, \quad f = \frac{dn}{dy} - qc,$$

$$h = \frac{dm}{dx} - p\frac{dn}{dy} + pqc = \frac{dl}{dy} - q\frac{dn}{dx} + pqc,$$

and when, in the integral equivalent of the simultaneous system

$$\left.\begin{array}{c} E_0 = 0, \quad E_1 = 0, \ \ldots, \ E_{\mu} = 0 \\ \dfrac{dm}{dx} - p\dfrac{dn}{dy} = \dfrac{dl}{dy} - q\dfrac{dn}{dx} \\ \dfrac{dv}{dx} = l + np, \quad \dfrac{dv}{dy} = m + nq \\ \dfrac{dz}{dx} = p, \quad \dfrac{dz}{dy} = q \end{array}\right\},$$

all arbitrary constants are made functions of u, and all arbitrary functions are made to involve u, subject to the equation

$$\frac{dv}{du} = n\frac{dz}{du},$$

the value of v so provided is an integral of the original differential equation.

The integer μ is never less than unity: the equation

$$\frac{dm}{dx} - p\frac{dn}{dy} = \frac{dl}{dy} - q\frac{dn}{dx}$$

is a functional consequence of the two equations

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq,$$

being a necessity to their coexistence: thus there are at least six equations for the determination of five quantities l, m, n, v, z as functions of x and y, the quantity u being latent throughout these equations. There is no perfectly general process, the universal application of which is subject only to the difficulties of quadrature, that can be applied to the system: and indeed, no such process can be expected which does not essentially involve the hypothesis made as to the character of the integral—that it is expressible in finite terms without partial quadratures.

When an integral equivalent of the system has been obtained, the relation of the argument u to the other variables in that equivalent can be settled by means of the equation

$$\frac{dv}{du} = n\frac{dz}{du},$$

which must be satisfied identically. The arbitrary elements that then survive will indicate how far the integral can be regarded as coinciding with the integral in Cauchy's theorem.

A detailed example will shew the working of the preceding theory in connection with the explanations.

Ex. Consider the equation

$$b = f - g + h,$$

which, as a matter of fact, does possess intermediate integrals. The characteristic invariant is

$$-pq + q^2 - p + q = 0,$$

that is,

$$(p-q)(q+1)=0;$$

so that two sets of subsidiary equations arise for consideration: they are given by

$$p-q=0, \quad q+1=0.$$

When we substitute the values of b, f, g, h, given in terms of c by the subsidiary equations, and take account of the characteristic invariant, we find that (in the general notation) $\mu=1$, that $E_1=0$ is the characteristic invariant, and that the equation $E_0=0$ is

$$\frac{dm}{dy}+(p-q-1)\frac{dn}{dy}+\frac{dn}{dx}-\frac{dm}{dx}=0.$$

First, let

$$p-q=0,$$

that is, on integration

$$z=\theta(x+y, u),$$

where θ is an arbitrary function. Also, using the relation $p-q=0$, the preceding subsidiary equation is

$$\frac{d(m-n)}{dy}-\frac{d(m-n)}{dx}=0,$$

so that

$$m-n=\psi(x+y, u),$$

where ψ is an arbitrary function: hence, taking account of the preceding equation that defines z, we have

$$m-n=g(x+y, z),$$

where g is an arbitrary function. This equation is manifestly an intermediate integral: when integrated, it leads to a primitive

$$v=G(x+y, z)+H(y+z, x),$$

where G and H are arbitrary functions.

Next, let

$$q+1=0,$$

that is, on integration

$$y+z=\vartheta(x, u'),$$

where ϑ is an arbitrary function, and u' is a new argument. The other subsidiary equation is

$$\frac{dm}{dy}-\frac{dm}{dx}+\frac{dn}{dx}+p\frac{dn}{dy}=0:$$

hence, in connection with the universal equation

$$\frac{dm}{dx}-p\frac{dn}{dy}-\frac{dl}{dy}+q\frac{dn}{dx}=0,$$

together with the relation $q+1=0$, we have

$$\frac{dm}{dy}-\frac{dl}{dy}=0,$$

so that

$$l-m=\chi(x, u'),$$

where χ is an arbitrary function. Taking account of the preceding equation that defines u', we have

$$l-m=k(y+z,\ x),$$

where k is an arbitrary function. Again, this equation is an intermediate integral: when integrated, it leads to the same primitive

$$v=G(x+y,\ z)+H(y+z,\ x)$$

as before.

Application of Darboux's Method.

330. When it happens that no integrable combination of the preceding subsidiary equations is obtainable, we pass from the preceding generalisation of Ampère's method for equations in two independent variables to a generalisation of Darboux's method, whereby we seek to obtain equations (if any) that are compatible with a given equation and are not of lower order.

When the method is applied to an equation of the second order

$$\phi(v,\ x,\ y,\ z,\ l,\ m,\ n,\ a,\ b,\ c,\ f,\ g,\ h)=0,$$

we need to take account of derivatives of the third order. Let

$$\alpha_0=\frac{\partial^3 v}{\partial x^3},\qquad \alpha_1=\frac{\partial^3 v}{\partial x^2\partial z},\qquad \alpha_2=\frac{\partial^3 v}{\partial x\partial z^2},\qquad \alpha_3=\frac{\partial^3 v}{\partial z^3},$$

$$\beta_0=\frac{\partial^3 v}{\partial x^2\partial y},\quad \beta_1=\frac{\partial^3 v}{\partial x\partial y\partial z},\quad \beta_2=\frac{\partial^3 v}{\partial y\partial z^2},$$

$$\gamma_0=\frac{\partial^3 v}{\partial x\partial y^2},\quad \gamma_1=\frac{\partial^3 v}{\partial y^2\partial z},$$

$$\delta_0=\frac{\partial^3 v}{\partial y^3},$$

so that

$$\left.\begin{aligned}
da &= \alpha_0 dx+\beta_0 dy+\alpha_1 dz\\
dh &= \beta_0 dx+\gamma_0 dy+\beta_1 dz\\
db &= \gamma_0 dx+\delta_0 dy+\gamma_1 dz\\
dg &= \alpha_1 dx+\beta_1 dy+\alpha_2 dz\\
df &= \beta_1 dx+\gamma_1 dy+\beta_2 dz\\
dc &= \alpha_2 dx+\beta_2 dy+\alpha_3 dz
\end{aligned}\right\}.$$

The equation $\phi = 0$ must be satisfied identically when the appropriate value of v is substituted: hence, writing

$$X = \frac{\partial\phi}{\partial x} + l\,\frac{\partial\phi}{\partial v} + a\,\frac{\partial\phi}{\partial l} + h\,\frac{\partial\phi}{\partial m} + g\,\frac{\partial\phi}{\partial n},$$

$$Y = \frac{\partial\phi}{\partial y} + m\,\frac{\partial\phi}{\partial v} + h\,\frac{\partial\phi}{\partial l} + b\,\frac{\partial\phi}{\partial m} + f\,\frac{\partial\phi}{\partial n},$$

$$Z = \frac{\partial\phi}{\partial z} + n\,\frac{\partial\phi}{\partial v} + g\,\frac{\partial\phi}{\partial l} + f\,\frac{\partial\phi}{\partial m} + c\,\frac{\partial\phi}{\partial n},$$

and keeping the former notation for derivatives of ϕ with regard to a, b, c, f, g, h, we have

$$\left.\begin{aligned} X + A\alpha_0 + H\beta_0 + B\gamma_0 + G\alpha_1 + F\beta_1 + C\alpha_2 &= 0 \\ Y + A\beta_0 + H\gamma_0 + B\delta_0 + G\beta_1 + F\gamma_1 + C\beta_2 &= 0 \\ Z + A\alpha_1 + H\beta_1 + B\gamma_1 + G\alpha_2 + F\beta_2 + C\alpha_3 &= 0 \end{aligned}\right\},$$

in connection with the value of v provided by the integral sought.

Now suppose that the independent variables are changed from x, y, z to x, y, u, where the only assumption made at the moment is that u certainly involves z: then, keeping the former significance for p and q, the derivatives of a, b, c, f, g, h with regard to x, y, u are given by the equations

$$\frac{da}{dx} = \alpha_0 + \alpha_1 p,\quad \frac{da}{dy} = \beta_0 + \alpha_1 q,\quad \frac{da}{du} = \alpha_1\frac{dz}{du},$$

$$\frac{dh}{dx} = \beta_0 + \beta_1 p,\quad \frac{dh}{dy} = \gamma_0 + \beta_1 q,\quad \frac{dh}{du} = \beta_1\frac{dz}{du},$$

$$\frac{db}{dx} = \gamma_0 + \gamma_1 p,\quad \frac{db}{dy} = \delta_0 + \gamma_1 q,\quad \frac{db}{du} = \gamma_1\frac{dz}{du},$$

$$\frac{dg}{dx} = \alpha_1 + \alpha_2 p,\quad \frac{dg}{dy} = \beta_1 + \alpha_2 q,\quad \frac{dg}{du} = \alpha_2\frac{dz}{du},$$

$$\frac{df}{dx} = \beta_1 + \beta_2 p,\quad \frac{df}{dy} = \gamma_1 + \beta_2 q,\quad \frac{df}{du} = \beta_2\frac{dz}{du},$$

$$\frac{dc}{dx} = \alpha_2 + \alpha_3 p,\quad \frac{dc}{dy} = \beta_2 + \alpha_3 q,\quad \frac{dc}{du} = \alpha_3\frac{dz}{du}.$$

The following relations, derived from these equations, obviously subsist among the derivatives of a, b, c, f, g, h with regard to x and y, viz.

$$\frac{dh}{dx}+q\frac{dg}{dx}=\frac{da}{dy}+p\frac{dg}{dy},$$

$$\frac{db}{dx}+q\frac{df}{dx}=\frac{dh}{dy}+p\frac{df}{dy},$$

$$\frac{df}{dx}+q\frac{dc}{dx}=\frac{dg}{dy}+p\frac{dc}{dy},$$

each of which is free from the derivatives of the third order. Again, the equations can be used to express all but one of the derivatives of the third order in terms of that one: expressing them in terms of β_1, we find

$$\alpha_0=\frac{da}{dx}-\frac{p}{q}\left(\frac{da}{dy}-\frac{dh}{dx}\right)-\beta_1\frac{p^2}{q},$$

$$\alpha_1=\frac{1}{q}\left(\frac{da}{dy}-\frac{dh}{dx}\right)+\beta_1\frac{p}{q},$$

$$\alpha_2=\frac{1}{q}\frac{dq}{dy}-\beta_1\frac{1}{q},$$

$$\alpha_3=\frac{1}{p}\frac{dc}{dx}-\frac{1}{pq}\frac{dg}{dy}+\beta_1\frac{1}{pq},$$

$$\beta_0=\frac{dh}{dx}-\beta_1 p,$$

$$\beta_2=\frac{1}{p}\frac{df}{dx}-\beta_1\frac{1}{p},$$

$$\gamma_0=\frac{dh}{dy}-\beta_1 q,$$

$$\gamma_1=\frac{1}{p}\left(\frac{db}{dx}-\frac{dh}{dy}\right)+\beta_1\frac{q}{p},$$

$$\delta_0=\frac{db}{dy}-\frac{q}{p}\left(\frac{db}{dx}-\frac{dh}{dy}\right)-\beta_1\frac{q^2}{p}.$$

Other expressions are obtainable: but they are equivalent to this set, in virtue of the earlier three relations. Moreover, the value of β_1 is taken to be

$$\frac{dh}{du}\div\frac{dz}{du}.$$

When the values of the derivatives of v of the third order, as expressed in terms of β_1, are substituted in the three equations which arise as first derivatives of $\phi = 0$, we have

$$X + A\left\{\frac{da}{dx} - \frac{p}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right)\right\} + H\frac{dh}{dx} + B\frac{dh}{dy}$$
$$+ G\frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + C\frac{1}{q}\frac{dq}{dy} - \frac{\Delta}{q}\beta_1 = 0,$$
$$Y + A\frac{dh}{dx} + H\frac{dh}{dy} + B\left\{\frac{db}{dy} - \frac{q}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right)\right\}$$
$$+ F\frac{1}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right) + C\frac{1}{p}\frac{df}{dx} - \frac{\Delta}{p}\beta_1 = 0,$$
$$Z + A\frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + B\frac{1}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right)$$
$$+ G\frac{1}{q}\frac{dq}{dy} + F\frac{1}{p}\frac{df}{dx} + C\left(\frac{1}{p}\frac{dc}{dx} - \frac{1}{pq}\frac{dg}{dy}\right) + \frac{\Delta}{pq}\beta_1 = 0,$$

where

$$\Delta = Ap^2 + Hpq + Bq^2 - Gp - Fq + C;$$

and, in each of the three equations, β_1 has the previously mentioned value.

331. At this stage, there are two distinct courses of reasoning which, while leading to the same subsidiary equations, give divided significance to the equations.

According to the considerations in the first of these courses, we use the freedom, given by the fact that the new variable u has remained arbitrary, to impose a condition: we suppose u to be chosen so that $\Delta = 0$, that is, we have

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0.$$

The term in β_1 disappears from the three equations: and these equations, after slight modifications, can be made to assume the forms

$$E_1 = X + A\left(\frac{da}{dx} - p\frac{dg}{dx}\right) + H\left(\frac{dh}{dx} - p\frac{dg}{dy}\right) + B\left(\frac{dh}{dy} - q\frac{dg}{dy}\right)$$
$$+ G\frac{dg}{dx} + F\frac{dg}{dy} = 0,$$
$$E_2 = Y + A\left(\frac{dh}{dx} - p\frac{df}{dx}\right) + H\left(\frac{db}{dx} - p\frac{df}{dy}\right) + B\left(\frac{db}{dy} - q\frac{df}{dy}\right)$$
$$+ G\frac{df}{dx} + F\frac{df}{dy} = 0,$$

$$E_3 = Z + A\left(\frac{dg}{dx} - p\frac{dc}{dx}\right) + H\left(\frac{df}{dx} - p\frac{dc}{dy}\right) + B\left(\frac{df}{dy} - q\frac{dc}{dy}\right)$$
$$+ G\frac{dc}{dx} + F\frac{dc}{dy} = 0.$$

According to the considerations in the second course of reasoning, we assume that the equations of the integral can be expressed in finite form and that the variable u is an argument of an arbitrary function in the integral. Then, as $\frac{dh}{du}$ in the value of β_1 involves a differentiation with regard to u higher than any which occurs in any other term, and as the equations are to be satisfied identically, the term in β_1 must disappear in and by itself from each equation: hence

$$\Delta = 0,$$

being the same conclusion as before. The remaining parts of the equations must also vanish: they have already been given in the forms

$$E_1 = 0, \quad E_2 = 0, \quad E_3 = 0.$$

332. The quantities, which occur in these simultaneous subsidiary equations and which have to be determined for our present purpose, are eleven in number, viz., $a, b, c, f, g, h, l, m, n, v, z$: they are functions of x, y, u. Omitting initially those equations in which derivatives with regard to u occur, the eleven quantities are to be functions of x and y. The constants, which arise in the integration, are made functions of u; and arbitrary functions, which are introduced, involve u: the forms of the functions must be such that the equations containing derivatives of u are satisfied.

The equations for the determination of the eleven unknown quantities are simultaneous partial equations of the first order. Among them, we have

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq,$$
$$\frac{dl}{dx} = a + gp, \quad \frac{dl}{dy} = h + gq,$$
$$\frac{dm}{dx} = h + fp, \quad \frac{dm}{dy} = b + fq,$$
$$\frac{dn}{dx} = g + cp, \quad \frac{dn}{dy} = f + cq,$$

which are equivalent to seven in all, because the relation

$$\frac{dl}{dy}+p\frac{dn}{dy}=\frac{dm}{dx}+q\frac{dn}{dx}$$

is identically satisfied by the foregoing values of the derivatives of l, m, n. Moreover, the relations

$$\frac{dh}{dx}+q\frac{dg}{dx}=\frac{da}{dy}+p\frac{dg}{dy},$$

$$\frac{db}{dx}+q\frac{df}{dx}=\frac{dh}{dy}+p\frac{df}{dy},$$

$$\frac{df}{dx}+q\frac{dc}{dx}=\frac{dg}{dy}+p\frac{dc}{dy},$$

are deducible from the preceding relations by substituting in

$$\frac{d}{dy}\left(\frac{dl}{dx}\right)=\frac{d}{dx}\left(\frac{dl}{dy}\right),\quad \frac{d}{dy}\left(\frac{dm}{dx}\right)=\frac{d}{dx}\left(\frac{dm}{dy}\right),\quad \frac{d}{dy}\left(\frac{dn}{dx}\right)=\frac{d}{dx}\left(\frac{dn}{dy}\right):$$

so that they are not independent equations. Consequently, on the score of this set of equations, there are seven which are independent of one another: they are a universal set, belonging to all equations of the type under consideration.

The remaining equations in the system belong specially to the equation $\phi=0$: they are

$$\Delta=0,\quad E_1=0,\quad E_2=0,\quad E_3=0,$$

being four in number.

Hence the tale of independent equations in the subsidiary system is eleven, the same as the number of quantities to be determined at this stage.

The original equation $\phi=0$ is an integral of the system. For the effective solution of the system, ten other integrals would be required: in particular cases, the process can be appreciably shortened.

Ex. 1. Consider the equation

$$a+f-g-h+\frac{2l-m-n}{y+z}=0,$$

which has no intermediate integral. Here

$$\Delta=p^2-pq+p-q=0,$$

so that we have two cases to consider, viz.

$$p+1=0,\quad p-q=0.$$

The first of these gives

$$y = \text{function of } x+z,$$

when the appropriate argument u is constant: the second gives

$$z = \text{function of } x+y,$$

when the other appropriate argument u is constant.

The three equations $E_1=0$, $E_2=0$, $E_3=0$, are

$$\frac{da}{dx} - \frac{dh}{dx} - (p+1)\left(\frac{dg}{dx} - \frac{dg}{dy}\right) + \frac{2a-h-g}{y+z} = 0,$$

$$\frac{dh}{dx} - \frac{db}{dx} - (p+1)\left(\frac{df}{dx} - \frac{df}{dy}\right) + \frac{2h-b-f}{y+z} - \frac{2l-m-n}{(y+z)^2} = 0,$$

$$\frac{dg}{dx} - \frac{df}{dx} - (p+1)\left(\frac{dc}{dx} - \frac{dc}{dy}\right) + \frac{2g-f-c}{y+z} - \frac{2l-m-n}{(y+z)^2} = 0.$$

First, let

$$p+1=0:$$

then, from the first and the third of the equations, we have

$$\frac{d}{dx}(a-h-g+f) + \frac{2a-h-g-2g+f+c}{y+z} + \frac{2l-m-n}{(y+z)^2} = 0.$$

Now

$$\begin{aligned} 2a-h-g-2g+f+c &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)(2l-m-n) \\ &= \left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z}\right)(2l-m-n) \\ &= \frac{d}{dx}(2l-m-n), \end{aligned}$$

and

$$\frac{d}{dx}\left(\frac{1}{y+z}\right) = -p\frac{1}{(y+z)^2} = \frac{1}{(y+z)^2};$$

hence

$$\frac{d}{dx}(a-h-g+f) + \frac{d}{dx}\left(\frac{2l-m-n}{y+z}\right) = 0.$$

We thus recover the original differential equation which is an integral of the system: the arbitrary function, which otherwise would enter, is made definite by the equation: in fact,

$$a-h-g+f+\frac{2l-m-n}{y+z} = 0.$$

When this integral is used to eliminate $2l-m-n$ from the second equation, the latter becomes

$$\frac{d}{dx}(h-b) + \frac{a+h-b-g}{y+z} = 0.$$

Combining this with the first equation, we have

$$\frac{d}{dx}(a-2h+b) + \frac{a-2h+b}{y+z} = 0:$$

hence, as $p+1=0$,

$$\frac{d}{dx}\left(\frac{a-2h+b}{y+z}\right)=0.$$

Consequently,

$$\frac{a-2h+b}{y+z}=\text{arbitrary function of } u:$$

and, from $p+1=0$, we have

$$y=\text{function of } x+z,\ u:$$

hence, eliminating u, we have

$$\frac{a-2h+b}{y+z}=\theta(x+z,\ y),$$

where θ is any arbitrary function. Let another arbitrary function χ of $x+z$ and y be chosen, such that

$$\chi=\chi(x+z,\ y)=\chi(\eta,\ y),$$

and

$$\theta=\frac{\partial^3\chi}{\partial\eta^3}-3\frac{\partial^3\chi}{\partial\eta^2\partial y}+3\frac{\partial^3\chi}{\partial\eta\partial y^2}-\frac{\partial^3\chi}{\partial y^3}:$$

then our integral is

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(l-m)=(y+z)\left\{\frac{\partial^3\chi}{\partial\eta^3}-3\frac{\partial^3\chi}{\partial\eta^2\partial y}+3\frac{\partial^3\chi}{\partial\eta\partial y^2}-\frac{\partial^3\chi}{\partial y^3}\right\}.$$

A first integral of this equation is

$$l-m=(y+z)\left(\frac{\partial^2\chi}{\partial\eta^2}-2\frac{\partial^2\chi}{\partial\eta\partial y}+\frac{\partial^2\chi}{\partial y^2}\right)+\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\psi(x+y,\ z),$$

where ψ is an arbitrary function of its arguments. This is an equation of the first order: but it is not an intermediate integral of the original equation in the ordinary sense of that term, for it contains two arbitrary functions: and the original equation cannot be derived from this equation alone.

Further integration leads to the relation

$$v=(y+z)\left(\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}\right)+2\chi+x\psi(x+y,\ z)+\Psi(x+y,\ z),$$

where Ψ is an arbitrary function of its arguments. This equation is of the nature of a primitive: but the number of arbitrary functions is three, being too great* by one unit: and there are various ways of reducing the number. Perhaps the simplest of these ways is to notice that the original equation is symmetrical in y and z, so that the integral can be expected to have the same symmetry. Accordingly, let

$$x+y=\zeta,$$
$$\vartheta=\vartheta(\zeta,\ z);$$

* Their presence is due to the fact that the equation really is the primitive of the equation of the third order

$$\frac{d}{dx}\left(\frac{a-2h+b}{y+z}\right)=0,$$

which is compatible with the given equation.

then, taking

$$\Psi(\zeta, z)+\{\zeta+z-(y+z)\}\,\psi(\zeta, z)$$
$$=(y+z)\left(\frac{\partial\vartheta}{\partial\zeta}-\frac{\partial\vartheta}{\partial z}\right)+2\vartheta,$$

we have

$$v=2\chi+2\vartheta+(y+z)\left(\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\frac{\partial\vartheta}{\partial\zeta}-\frac{\partial\vartheta}{\partial z}\right),$$

where χ and ϑ are arbitrary functions.

Next, consider the equation

$$p-q=0:$$

then the three equations of general identity take the form

$$\frac{dh}{dx}+p\frac{dg}{dx}=\frac{da}{dy}+p\frac{dg}{dy},$$
$$\frac{db}{dx}+p\frac{df}{dx}=\frac{dh}{dy}+p\frac{df}{dy},$$
$$\frac{df}{dx}+p\frac{dc}{dx}=\frac{dg}{dy}+p\frac{dc}{dy}.$$

Using these relations to eliminate the terms in p from the equations which are particular to the present example, we have

$$\frac{d}{dx}(a-g)-\frac{d}{dy}(a-g)+\frac{2a-h-g}{y+z}=0,$$
$$\frac{d}{dx}(h-f)-\frac{d}{dy}(h-f)+\frac{2h-b-f}{y+z}-\frac{2l-m-n}{(y+z)^2}=0,$$
$$\frac{d}{dx}(g-c)-\frac{d}{dy}(g-c)+\frac{2g-f-c}{y+z}-\frac{2l-m-n}{(y+z)^2}=0.$$

Proceeding as before, we can recover the original differential equation. When it is used to eliminate the term in $2l-m-n$ from the second and the third of these equations, they become

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(h-f)+\frac{a+h-b-g}{y+z}=0,$$
$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(g-c)+\frac{a-h+g-c}{y+z}=0.$$

From the first of the former three and the second of the latter two, we find

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(a-2g+c)+\frac{a-2g+c}{y+z}=0.$$

But as $p=q$, we have

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)(y+z)=-1,$$

and therefore

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)\left(\frac{a-2g+c}{y+z}\right)=0,$$

so that

$$\frac{a-2g+c}{y+z}=\text{arbitrary function of } x+y,\ u',$$

where

$$x+y=\text{arbitrary function of } z,\ u'.$$

Eliminating u', we have

$$\frac{a-2g+c}{y+z}=\mu(x+y,\ z),$$

where μ is an arbitrary function of its arguments. Let

$$x+y=\zeta,$$

and introduce a new arbitrary function $\psi'(\zeta,\ z)$, such that

$$\mu=\frac{\partial^3\psi'}{\partial\zeta^3}-3\frac{\partial^3\psi'}{\partial\zeta^2\partial z}+3\frac{\partial^3\psi'}{\partial\zeta\partial z^2}-\frac{\partial^3\psi'}{\partial z^3}:$$

then our integral is

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right)(l-n)=(y+z)\left\{\frac{\partial^3\psi'}{\partial\zeta^3}-3\frac{\partial^3\psi'}{\partial\zeta^2\partial z}+3\frac{\partial^3\psi'}{\partial\zeta\partial z^2}-\frac{\partial^3\psi'}{\partial z^3}\right\}.$$

A first integral of this equation is

$$l-n=(y+z)\left(\frac{\partial^2\psi'}{\partial\zeta^2}-2\frac{\partial^2\psi'}{\partial\zeta\partial z}+\frac{\partial^2\psi'}{\partial z^2}\right)+\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}+\sigma(x+z,\ y),$$

where σ is an arbitrary function of its arguments. This is an equation of the first order: but, for precisely the same reasons as were applied in the earlier case, it is not an intermediate integral of the equation in the ordinary sense of the term.

Further integration leads to the relation

$$v=(y+z)\left(\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial z}\right)+2\psi'+x\sigma(x+z,\ y)+\Sigma(x+z,\ y),$$

where Σ is an arbitrary function of its arguments. This equation is of the nature of a primitive: as before, the number* of arbitrary functions is too great by one unit. Effecting the necessary reductions by making the integral symmetrical in y and z, because the original equation is symmetrical in those variables, let

$$x+z=\eta$$

$$\chi=\chi(\eta,\ y);$$

then, taking

$$\Sigma(\eta,\ y)+\{\eta+y-(y+z)\}\,\sigma(\eta,\ y)$$
$$=(y+z)\left(\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}\right)+2\chi,$$

* The equation is really the primitive of

$$\left(\frac{d}{dx}-\frac{d}{dy}\right)\left(\frac{a-2g+c}{y+z}\right)=0,$$

which is an equation of the third order, compatible with the given equation and having three arbitrary functions in its primitive.

we have

$$v=2\chi+2\psi'+(y+z)\left(\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial z}+\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}\right).$$

Another way of proceeding is as follows. The two equations of the first order are

$$l-m=(y+z)\left(\frac{\partial^2\chi}{\partial\eta^2}-2\frac{\partial^2\chi}{\partial\eta\partial y}+\frac{\partial^2\chi}{\partial y^2}\right)+\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\psi(x+y,\ z),$$

$$l-n=(y+z)\left(\frac{\partial^2\psi'}{\partial\zeta^2}-2\frac{\partial^2\psi'}{\partial\zeta\partial z}+\frac{\partial^2\psi'}{\partial z^2}\right)+\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}+\sigma(x+z,\ y);$$

in order that they may coexist, they must satisfy the Poisson-Jacobi condition which, when developed, gives

$$\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y}+\psi+\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}+\sigma=0.$$

Taking account of the arguments of the functions and of the fact that the functions are arbitrary, we see that this condition can be satisfied only if

$$-\sigma=\frac{\partial\chi}{\partial\eta}-\frac{\partial\chi}{\partial y},$$

$$-\psi=\frac{\partial\psi'}{\partial\zeta}-\frac{\partial\psi'}{\partial y}.$$

Assuming these relations satisfied, we have the two equations of the first order coexisting with one another in the form

$$l-m=(y+z)(\chi_{11}-2\chi_{12}+\chi_{22})+\chi_1-\chi_2-(\psi_1'-\psi_2'),$$

$$l-n=(y+z)(\psi'_{11}-2\psi'_{12}+\psi'_{22})+\psi_1'-\psi_2'-(\chi_1-\chi_2),$$

where

$$\chi_1=\frac{\partial\chi}{\partial\eta},\quad \chi_{11}=\frac{\partial^2\chi}{\partial\eta^2},\quad \chi_2=\frac{\partial\chi}{\partial y},$$

and so on: and

$$\chi=\chi(\eta,\ y)=\chi(x+z,\ y),$$

$$\psi'=\psi(\zeta,\ z)=\psi(x+y,\ z).$$

Even so, neither of the equations is an intermediate integral.

Taken as a pair of equations in a single dependent variable, their common integral can be obtained by any of the regular methods in Chapter IV: it is found to be

$$v=2\chi+2\psi'+(y+z)(\chi_1-\chi_2+\psi_1-\psi_2),$$

in the preceding notation.

Ex. 2. Obtain the primitive of the equation

$$a-h-g+f+\frac{l-n+\beta(l-m)}{a(x+y+z)+y+\beta z}=0,$$

where a and β are constants.

Influence, upon the Integral, of the Resolubility of the Characteristic Invariant.

333. In the two examples that have been given, the characteristic invariant $\Delta = 0$ has been resoluble into two linear equations: these, taken in turn with other equations of the system, have led to integrable forms. There is, however, no indication of systematic method to be pursued in the quest of such forms: and some systematic method must be devised if the process is to be effective. Before proceeding to the discussion of such a method, it is advisable to indicate a classification of equations of the second order determined by the resolubility or the non-resolubility of $\Delta = 0$ into two linear equations.

When $\Delta = 0$ is resoluble into two linear equations, the subsidiary equations are of Lagrange's linear form so far as concerns derivatives of a, b, c, f, g, h: their integral is such that some combination θ of variables can be an arbitrary function of some other combination χ. But the equations themselves subsist on the condition that some other combination ψ of the variables is constant, and so this combination ψ is latent in the preceding relation. When explicit account is taken of it, the integral has a form

$$\theta = \Theta(\chi, \psi),$$

where Θ is an arbitrary function: this equation can coexist with the original equation. Hence, when the method is effective because the integral of the equation is expressible in finite terms, we infer that, *if $\Delta = 0$ is a resoluble equation, arbitrary functions of two arguments occur in the most general integral equivalent of the original equation in three independent variables.*

The converse also is true: *if an integral relation in three independent variables involves at least one arbitrary function of two distinct arguments and if it is equivalent to a partial differential equation of the second order free from arbitrary functional forms, the characteristic invariant can be resolved into two linear equations.*

Let ξ and η denote two independent functions of x, y, z, so that not more than one of the three quantities

$$\xi_x\eta_y - \xi_y\eta_x, \quad \xi_y\eta_z - \xi_z\eta_y, \quad \xi_z\eta_x - \xi_x\eta_z$$

can vanish. We assume that ξ and η are the arguments of an arbitrary function which occurs in the integral: let it occur in the form

$$v = \Theta\{\ldots, \rho(\xi, \eta), \ldots\},$$

where ρ denotes the derivative of the arbitrary function in Θ which is of the highest order. Then

$$l = \frac{\partial\Theta}{\partial\rho}(\rho_1\xi_x + \rho_2\eta_x) + \text{other terms},$$

with similar expressions for m and n: also,

$$a = \frac{\partial\Theta}{\partial\rho}(\rho_{11}\xi_x^2 + 2\rho_{12}\xi_x\eta_x + \rho_{22}\eta_x^2) + \ldots,$$

$$f = \frac{\partial\Theta}{\partial\rho}\{\rho_{11}\xi_y\xi_z + \rho_{12}(\xi_y\eta_z + \xi_z\eta_y) + \rho_{22}\eta_y\eta_z\} + \ldots,$$

where the unexpressed terms involve derivatives of ρ of order lower than those expressed: and there are similar values for b, g, c, h. By hypothesis, the integral equation is to be equivalent to a partial differential equation of the second order

$$\phi(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0,$$

free from all arbitrary functional forms. Hence, when the preceding values of the derivatives are substituted in this equation which must be satisfied, the terms involving the various combinations of the arbitrary functions must disappear. Thus the highest power of ρ_{11} must disappear of itself: it disappears in the combination $\rho_{11}\dfrac{\partial\Theta}{\partial\rho}$: and the necessary condition is

$$\xi_x^2\frac{\partial\phi}{\partial a} + \xi_x\xi_y\frac{\partial\phi}{\partial h} + \xi_x\xi_z\frac{\partial\phi}{\partial g} + \xi_y^2\frac{\partial\phi}{\partial b} + \xi_y\xi_z\frac{\partial\phi}{\partial f} + \xi_z^2\frac{\partial\phi}{\partial c} = 0,$$

or, in the old notation,

$$A\xi_x^2 + H\xi_x\xi_y + G\xi_x\xi_z + B\xi_y^2 + F\xi_y\xi_z + C\xi_z^2 = 0.$$

Similarly, the highest power of ρ_{12} must disappear: it disappears in the combination $\rho_{12}\dfrac{\partial\Theta}{\partial\rho}$: and the necessary condition is

$$2A\xi_x\eta_x + H(\xi_x\eta_y + \xi_y\eta_x) + G(\xi_x\eta_z + \xi_z\eta_x) + 2B\xi_y\eta_y \\ + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z = 0.$$

Similarly, the highest power of ρ_{22} must disappear, also in a combination $\rho_{22}\dfrac{\partial\Theta}{\partial\rho}$: the necessary condition is

$$A\eta_x^2 + H\eta_x\eta_y + G\eta_x\eta_z + B\eta_y^2 + F\eta_y\eta_z + C\eta_z^2 = 0.$$

Now these conditions give

$$(A\xi_x + \tfrac{1}{2}H\xi_y + \tfrac{1}{2}G\xi_z)^2$$
$$= (\tfrac{1}{4}H^2 - AB)\,\xi_y^2 + 2\,(\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,\xi_y\xi_z + (\tfrac{1}{4}G^2 - AC)\,\xi_z^2,$$

$$(A\xi_x + \tfrac{1}{2}H\xi_y + \tfrac{1}{2}G\xi_z)\,(A\eta_x + \tfrac{1}{2}H\eta_y + \tfrac{1}{2}G\eta_z)$$
$$= (\tfrac{1}{4}H^2 - AB)\,\xi_y\eta_y + (\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,(\xi_y\eta_z + \xi_z\eta_y) + (\tfrac{1}{4}G^2 - AC)\,\xi_z\eta_z,$$

and

$$(A\eta_x + \tfrac{1}{2}H\eta_y + \tfrac{1}{2}G\eta_z)^2$$
$$= (\tfrac{1}{4}H^2 - AB)\,\eta_y^2 + 2\,(\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,\eta_y\eta_z + (\tfrac{1}{4}G^2 - AC)\,\eta_z^2,$$

respectively. Squaring the middle equation, and subtracting the product of the first and third from that square, reducing, and removing a factor A, we have

$$I\,(\xi_y\eta_z - \xi_z\eta_y)^2 = 0,$$

where I is the discriminant of Δ.

Similarly, modifying the equations so as to obtain expressions for the squares and the product of

$$\tfrac{1}{2}H\xi_x + B\xi_y + \tfrac{1}{2}F\xi_z, \quad \tfrac{1}{2}H\eta_x + B\eta_y + \tfrac{1}{2}F\eta_z,$$

and proceeding as before, we should find

$$I\,(\xi_x\eta_z - \xi_z\eta_x)^2 = 0:$$

and a corresponding treatment of the three equations, with reference to

$$\tfrac{1}{2}G\xi_x + \tfrac{1}{2}F\xi_y + C\xi_z, \quad \tfrac{1}{2}G\eta_x + \tfrac{1}{2}F\eta_y + C\eta_z,$$

leads to a relation

$$I\,(\xi_x\eta_y - \xi_y\eta_x)^2 = 0.$$

We know that not more than one of the quantities

$$\xi_y\eta_z - \xi_z\eta_y, \quad \xi_x\eta_z - \xi_z\eta_x, \quad \xi_x\eta_y - \xi_y\eta_x,$$

can vanish: hence we must have

$$I = 0.$$

Consequently, $\Delta = 0$ is resoluble into two linear equations. The proposition is therefore established.

If however ρ, instead of being a function of two arguments ξ and η, is a function of only a single argument u, then we can deduce only a single equation

$$Au_x^2 + Hu_xu_y + Gu_xu_z + Bu_y^2 + Fu_yu_z + Cu_z^2 = 0;$$

and we cannot prove that

$$I = 0.$$

It does not follow that $\Delta = 0$ is not resoluble in particular cases: we cannot affirm that the circumstances in general do provide a resoluble characteristic invariant. Hence *when* $\Delta = 0$ *cannot be resolved into two equations linear in* p *and* q, *we infer that each of the arbitrary functions, which occur in the integral equivalent, has only a single argument.*

It thus appears that equations of the second order in three independent variables, whose integrals can be expressed in finite terms without essential partial quadratures, belong to one or other of two classes, according as the characteristic invariant $\Delta = 0$ can or cannot be resolved into two equations linear in p and q.

Equations having a Resoluble Characteristic Invariant.

334. Consider now more particularly those equations for which $\Delta = 0$ is resoluble into two linear equations or into a repeated linear equation. If a subsidiary system possesses an integrable combination, it is desirable to have a general method of determining the combination: and the method should indicate whether an integrable combination does or does not exist.

Assuming that the discriminant of $\Delta = 0$ vanishes, we first obtain the two linear equations into which the characteristic invariant is resolved. It will be sufficient for our purpose to assume that a occurs in the original equation $\phi = 0$, so that A does not vanish. We have, by $\Delta = 0$,

$$(Ap + \tfrac{1}{2}Hq - \tfrac{1}{2}G)^2 = (\tfrac{1}{4}H^2 - AB)q^2 + (AF - \tfrac{1}{2}GH)q + \tfrac{1}{4}G^2 - AC;$$

so that, writing

$$\tfrac{1}{4}H^2 - AB = \theta^2,$$
$$AF - \tfrac{1}{2}GH = -2\theta^2\vartheta,$$

and therefore

$$\tfrac{1}{4}G^2 - AC = \theta^2\vartheta^2,$$

on account of the vanishing of the discriminant, we find

$$Ap + \tfrac{1}{2}Hq - \tfrac{1}{2}G = \pm\,\theta\,(q - \vartheta).$$

Thus the two equations equivalent to $\Delta = 0$ are

$$\left.\begin{array}{l} Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0 \\ Ap + (\tfrac{1}{2}H + \theta)\,q - (\tfrac{1}{2}G + \theta\vartheta) = 0 \end{array}\right\},$$

where

$$\theta^2 = \tfrac{1}{4}H^2 - AB, \qquad \vartheta = \frac{GH - 2AF}{H^2 - 4AB}.$$

Consider the subsidiary system associated with the linear equation

$$Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0.$$

Take the equation $E_1 = 0$ of § 331: it is

$$X + A\frac{da}{dx} + H\frac{dh}{dx} + B\frac{dh}{dy} + G\frac{dg}{dx} + F\frac{dg}{dy}$$
$$- p\left(A\frac{dg}{dx} + H\frac{dg}{dy}\right) - qB\frac{dg}{dy} = 0.$$

Take also the first of the three equations of identity of § 332 in the form

$$\frac{da}{dy} - \frac{dh}{dx} + p\frac{dg}{dy} - q\frac{dg}{dx} = 0.$$

Multiply the latter by $\tfrac{1}{2}H - \theta$, and add to the former. In the resulting equation, the coefficient of $\dfrac{dg}{dx}$

$$= G - Ap - (\tfrac{1}{2}H - \theta)\,q$$
$$= \tfrac{1}{2}G + \theta\vartheta;$$

the coefficient of $\dfrac{dg}{dy}$

$$= F - Hp - Bq + (\tfrac{1}{2}H - \theta)\,p$$
$$= F - \{(\tfrac{1}{2}H + \theta)\,p + Bq\}$$
$$= - F - \frac{\tfrac{1}{2}H + \theta}{A}\{Ap + (\tfrac{1}{2}H - \theta)\,q\}$$
$$= \frac{\tfrac{1}{2}GH - 2\theta^2\vartheta}{A} - \frac{\tfrac{1}{2}H + \theta}{A}(\tfrac{1}{2}G - \theta\vartheta)$$
$$= \frac{1}{A}(\tfrac{1}{2}G + \theta\vartheta)(\tfrac{1}{2}H - \theta);$$

and therefore the terms involving derivatives of g are

$$(\tfrac{1}{2}G + \theta\vartheta)\left\{\frac{dg}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{dg}{dy}\right\}.$$

The terms involving derivatives of a are

$$A\left\{\frac{da}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{da}{dy}\right\};$$

and those involving derivatives of h are

$$(\tfrac{1}{2}H + \theta)\frac{dh}{dx} + B\frac{dh}{dy}$$

$$= (\tfrac{1}{2}H + \theta)\left\{\frac{dh}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{dh}{dy}\right\}.$$

Hence, when we write

$$\delta = \frac{d}{dx} + \frac{1}{A}(\tfrac{1}{2}H - \theta)\frac{d}{dy},$$

the equation becomes

$$X + A\delta a + (\tfrac{1}{2}H + \theta)\delta h + (\tfrac{1}{2}G + \theta\vartheta)\delta g = 0.$$

The other equations $E_2 = 0$, $E_3 = 0$, can be similarly treated with the respective relations of identity: and so, connected with the linear equation

$$Ap + (\tfrac{1}{2}H - \theta)q - (\tfrac{1}{2}G - \theta\vartheta) = 0,$$

we have a system of three subsidiary equations free from p and q in the form

$$\left.\begin{array}{l} X + A\delta a + (\tfrac{1}{2}H + \theta)\delta h + (\tfrac{1}{2}G + \theta\vartheta)\delta g = 0 \\ Y + A\delta h + (\tfrac{1}{2}H + \theta)\delta b + (\tfrac{1}{2}G + \theta\vartheta)\delta f = 0 \\ Z + A\delta g + (\tfrac{1}{2}H + \theta)\delta f + (\tfrac{1}{2}G + \theta\vartheta)\delta c = 0 \end{array}\right\},$$

with the foregoing definition of δ.

Now suppose that

$$u(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

is an integrable combination of these equations: then

$$\frac{du}{dx} = 0, \qquad \frac{du}{dy} = 0,$$

that is,

$$\frac{\partial u}{\partial a}\frac{da}{dx}+\ldots+\frac{\partial u}{\partial h}\frac{dh}{dx}+(a+gp)\frac{\partial u}{\partial l}+(h+fp)\frac{\partial u}{\partial m}+(g+cp)\frac{\partial u}{\partial n}$$
$$+(l+np)\frac{\partial u}{\partial v}+\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}=0,$$

$$\frac{\partial u}{\partial a}\frac{da}{dy}+\ldots+\frac{\partial u}{\partial h}\frac{dh}{dy}+(h+gq)\frac{\partial u}{\partial l}+(b+fq)\frac{\partial u}{\partial m}+(f+cq)\frac{\partial u}{\partial n}$$
$$+(m+nq)\frac{\partial u}{\partial v}+\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}=0.$$

Multiply the latter by $\frac{1}{A}(\frac{1}{2}H-\theta)$ and add to the former: where p and q occur in the resulting equation, it is in a combination

$$p+\frac{1}{A}(\tfrac{1}{2}H-\theta)\,q,$$

which (on account of the linear equation) we replace by

$$\frac{1}{A}(\tfrac{1}{2}G-\theta\vartheta);$$

the resulting equation then is

$$\frac{\partial u}{\partial a}\delta a+\frac{\partial u}{\partial b}\delta b+\frac{\partial u}{\partial c}\delta c+\frac{\partial u}{\partial f}\delta f+\frac{\partial u}{\partial g}\delta g+\frac{\partial u}{\partial h}\delta h$$
$$+\frac{\partial u}{\partial x}+\frac{1}{A}(\tfrac{1}{2}H-\theta)\frac{\partial u}{\partial y}+\frac{1}{A}(\tfrac{1}{2}G-\theta\vartheta)\frac{\partial u}{\partial z}$$
$$+\left\{l+\frac{m}{A}(\tfrac{1}{2}H-\theta)+\frac{n}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}\frac{\partial u}{\partial v}$$
$$+\frac{\partial u}{\partial l}\left\{a+\frac{h}{A}(\tfrac{1}{2}H-\theta)+\frac{g}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}$$
$$+\left\{h+\frac{b}{A}(\tfrac{1}{2}H-\theta)+\frac{f}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}\frac{\partial u}{\partial m}$$
$$+\left\{g+\frac{f}{A}(\tfrac{1}{2}H-\theta)+\frac{c}{A}(\tfrac{1}{2}G-\theta\vartheta)\right\}\frac{\partial u}{\partial n}=0.$$

Substitute from the three earlier equations for δa, δb, δc, in terms of δf, δg, δh; then, as u is an integrable combination of those equations, the modified equation must become evanescent, so that

the coefficients of δf, δg, and δh, as well as the aggregate of terms independent of them, must vanish. Thus

$$\left.\begin{aligned}
0 &= \frac{\partial u}{\partial h} - \frac{\tfrac{1}{2}H + \theta}{A}\frac{\partial u}{\partial a} - \frac{A}{\tfrac{1}{2}H + \theta}\frac{\partial u}{\partial b} \\
0 &= \frac{\partial u}{\partial g} - \frac{\tfrac{1}{2}G + \theta\vartheta}{A}\frac{\partial u}{\partial a} - \frac{A}{\tfrac{1}{2}G + \theta\vartheta}\frac{\partial u}{\partial c} \\
0 &= \frac{\partial u}{\partial f} - \frac{\tfrac{1}{2}G + \theta\vartheta}{\tfrac{1}{2}H + \theta}\frac{\partial u}{\partial b} - \frac{\tfrac{1}{2}H + \theta}{\tfrac{1}{2}G + \theta\vartheta}\frac{\partial u}{\partial c} \\
0 &= -\frac{X}{A} - \frac{Y}{\tfrac{1}{2}H + \theta} - \frac{Z}{\tfrac{1}{2}G + \theta\vartheta} + \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v} + a\frac{\partial u}{\partial l} + h\frac{\partial u}{\partial m} + g\frac{\partial u}{\partial n} \\
&\quad + \frac{\tfrac{1}{2}H - \theta}{A}\left(\frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v} + h\frac{\partial u}{\partial l} + b\frac{\partial u}{\partial m} + f\frac{\partial u}{\partial n}\right) \\
&\quad + \frac{\tfrac{1}{2}G - \theta\vartheta}{A}\left(\frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v} + g\frac{\partial u}{\partial l} + f\frac{\partial u}{\partial m} + c\frac{\partial u}{\partial n}\right)
\end{aligned}\right\},$$

being a set of simultaneous equations for u, associated with

$$Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\vartheta) = 0.$$

This system of four equations for u, each of which is linear of the first order, must be made a complete Jacobian system: when this is done, let it contain M equations. There are thirteen possible arguments for u: hence the system would possess

$$13 - M$$

algebraically independent integrals. Among these integrals must be included

$$\phi\,(a, b, c, f, g, h, l, m, n, v, x, y, z)$$

from the original equation, as well as the two distinct integrals of the set

$$\frac{dx}{A} = \frac{dy}{\tfrac{1}{2}H - \theta} = \frac{dz}{\tfrac{1}{2}G - \theta\vartheta},$$

say ξ, η. Putting these on one side, we should have to obtain

$$10 - M$$

algebraically independent integrals of the complete Jacobian system: they would be obtainable by known processes.

If $M = 10$, there is no integrable combination. If $M = 9$, which is a not uncommon case when the method is effective, there is one such integral. Let it be u: then

$$u = \psi\,(\xi, \eta),$$

where ψ is an arbitrary function, is an equation of the second order coexistent with the original equation

$$\phi = 0.$$

335. The preceding system is associated with one of the two linear equations into which $\Delta = 0$ is resoluble. A corresponding system is associated with the other equation

$$Ap + (\tfrac{1}{2}H + \theta)\, q - (\tfrac{1}{2}G + \theta\vartheta) = 0:$$

formally, it can be derived from the preceding system by changing the sign of θ. The process of constructing the integrable combination (if any) is the same as before: when an integral exists, it is given as an integral common to a complete Jacobian system of equations of the first order.

Ex. We shall briefly set out the subsidiary equations that arise in connection with

$$a - h - g + f + \frac{2l - m - n}{y + z} = 0,$$

which has already been considered.

In the case of this equation, $\Delta = 0$ is resoluble: it is

$$(p - q)(p + 1) = 0.$$

First, let the equation

$$p + 1 = 0$$

be taken: here

$$A = 1, \quad H = -1, \quad G = -1:$$

thus

$$\theta = -\tfrac{1}{2}, \quad \vartheta = -1.$$

With the notation of the general investigation, we have

$$\delta = \frac{d}{dx},$$

$$\tfrac{1}{2}H + \theta = -1, \quad \tfrac{1}{2}G + \theta\vartheta = 0:$$

and so three of the subsidiary equations are

$$X + \frac{da}{dx} - \frac{dh}{dx} = 0,$$

$$Y + \frac{dh}{dx} - \frac{db}{dx} = 0,$$

$$Z + \frac{dg}{dx} - \frac{df}{dx} = 0,$$

where

$$X=\frac{2a-h-g}{y+z},$$
$$Y=\frac{2h-b-f}{y+z}-\frac{2l-m-n}{(y+z)^2},$$
$$Z=\frac{2g-f-c}{y+z}-\frac{2l-m-n}{(y+z)^2}.$$

If

$$u=u(a,\ b,\ c,\ f,\ g,\ h,\ l,\ m,\ n,\ v,\ x,\ y,\ z)=0$$

be an integral combination of the above equations, it must happen that

$$\frac{du}{dx}=0$$

(with $p+1=0$) is a linear combination of the three equations: as q is undetermined, it is useless to consider $\frac{du}{dy}=0$. When therefore we substitute

$$\frac{da}{dx}=\frac{dh}{dx}-X,\quad \frac{db}{dx}=\frac{dh}{dx}+Y,\quad \frac{df}{dx}=\frac{dg}{dx}+Z,$$

in the equation

$$\frac{du}{dx}=0,$$

the modified form of the latter must become evanescent. The necessary and sufficient conditions are

$$\frac{\partial u}{\partial c}=0,\quad \frac{\partial u}{\partial f}+\frac{\partial u}{\partial g}=0,\quad \frac{\partial u}{\partial a}+\frac{\partial u}{\partial h}+\frac{\partial u}{\partial b}=0,$$
$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial z}+(a-g)\frac{\partial u}{\partial l}+(h-f)\frac{\partial u}{\partial m}+(g-c)\frac{\partial u}{\partial n}+(l-n)\frac{\partial u}{\partial v}$$
$$=X\frac{\partial u}{\partial a}-Y\frac{\partial u}{\partial b}-Z\frac{\partial u}{\partial f}.$$

This set of four simultaneous equations must be rendered complete by associating with it all the Poisson-Jacobi conditions which provide new equations. When thus made complete by the ordinary processes, it is equivalent to the set of nine equations

$$\frac{\partial u}{\partial v}=0,\quad \frac{\partial u}{\partial c}=0,\quad -\frac{\partial u}{\partial h}=\frac{\partial u}{\partial a}+\frac{\partial u}{\partial b},$$
$$\tfrac{1}{2}\frac{\partial u}{\partial l}=-\frac{\partial u}{\partial m}=-\frac{\partial u}{\partial n}=\frac{1}{y+z}\frac{\partial u}{\partial f}=-\frac{1}{y+z}\frac{\partial u}{\partial g}=\frac{1}{y+z}\left(\frac{\partial u}{\partial a}-\frac{\partial u}{\partial b}\right),$$
$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial z}-\frac{2l-m-n}{(y+z)^2}\frac{\partial u}{\partial a}-\frac{2a-3h+b+f-g}{y+z}\frac{\partial u}{\partial b}=0.$$

This system involves thirteen variables: consequently, it possesses four algebraically independent integrals. They can be taken in the form

$$y,\quad x+z,$$
$$\frac{2a-3h+b+f-g}{y+z}+\frac{2l-m-n}{(y+z)^2},$$
$$a-h-g+f+\frac{2l-m-n}{y+z}.$$

The last is zero, owing to the original differential equation: when it is use to modify the third, the latter can be replaced by

$$\frac{a-2h+b}{y+z}.$$

Consequently, the most general integral of the subsidiary system is

$$\Phi\left(\frac{a-2h+b}{y+z}, \ x+z, \ y\right)=0,$$

where Φ is arbitrary: an equivalent form is

$$\frac{a-2h+b}{y+z}=\theta\,(x+z, \ y),$$

where θ is arbitrary.

Next, take the alternative linear equation arising out of $\Delta=0$: it is

$$p-q=0.$$

Proceeding similarly to the construction of a system to be associated wit this relation, we find that, if an equation

$$U(a, \ b, \ c, \ f, \ g, \ h, \ l, \ m, \ n, \ v, \ x, \ y, \ z)=0$$

exists simultaneously with the given equation, U is determined by th system

$$\frac{\partial U}{\partial b}=0, \quad \frac{\partial U}{\partial f}+\frac{\partial U}{\partial h}=0, \quad \frac{\partial U}{\partial a}+\frac{\partial U}{\partial g}+\frac{\partial U}{\partial c}=0,$$

$$\frac{\partial U}{\partial x}-\frac{\partial U}{\partial y}+(l-m)\frac{\partial U}{\partial v}+(a-h)\frac{\partial U}{\partial l}+(h-g)\frac{\partial U}{\partial m}+(g-f)\frac{\partial U}{\partial n}$$

$$=X\frac{\partial U}{\partial a}+Y\frac{\partial U}{\partial h}-Z\frac{\partial U}{\partial c},$$

with the earlier values of X, Y, Z. This system is to be made complete when complete, it is found to possess

$$\frac{a-2g+c}{y+z}=\psi\,(x+y, \ z),$$

where ψ is arbitrary, as its most general integral.

336. When either of the subsidiary systems leads to a nev equation of the second order, compatible with the original equa tion and involving one arbitrary function, it can be used as a equation initially propounded for integration. Frequently it wil possess an intermediate integral involving one arbitrary functio more than itself, that is, an equation of the first order involvin a couple of arbitrary functions: and indeed, it may almost b expected, even though the original equation itself possesses n

intermediate integral. Through the desired primitive, the dependent variable v (and therefore also its derivatives) involves a couple of arbitrary functions: hence any combination of v, l, m, n, will generally be expressible in terms of two arbitrary functions; and the preceding equation of the first order is of this type.

The integration of this equation of the first order leads to a primitive, often involving one other arbitrary function: substitution in the differential equation leads to relations between the arbitrary functions which reduce their number to two.

When both the subsidiary systems lead to new equations of the second order, we proceed similarly with each of them.

Examples have already been given.

It may however happen that neither of the subsidiary systems admits of an integral (other than the given equation) of the kind desired: the inference is that no equation of the second order containing only a single arbitrary function can be associated or is compatible, with the given equation. It may however be the case that some new equation of the third order—new, that is to say, in the sense that it is algebraically independent of the derivatives of the given equation—can be associated with the equation, and is such that its expression involves an arbitrary function: and this may occur for each of the two linear equations into which $\Delta = 0$ can be resolved.

Similarly, if there is no new equation of the third order, there may be a new equation of the fourth order of the kind desired, associable with (but algebraically independent of) the given equation, and involving an arbitrary function. And so on, precisely as in Darboux's method (Chapter XVIII) for partial equations of the second order in two independent variables: the object of the process is to find a new equation of finite order, which involves an arbitrary function, and which can be associated with the given equation while it is algebraically independent of the derivatives of the given equation.

We have assumed that the equation possesses no intermediate integral: if it did, the derivatives of that integral of appropriate order would arise, at each of the stages contemplated, as integrals at those stages. If, then, no new equation of finite order is

compatible with the given equation*, the method ceases to be effective. In that case, the only result generally attainable at present seems to be that which occurs in the establishment of Cauchy's existence-theorem: the integral certainly contains two arbitrary functions, but its expression (in the form of a converging series) is not finite.

The process will be sufficiently illustrated by the analysis adapted to the construction of a new equation of the third order, independent of the derivatives of the given equation: we shall limit the discussion to that aim.

COMPATIBLE EQUATIONS OF THE THIRD ORDER CONSTRUCTED BY DARBOUX'S METHOD.

337. Accordingly, we assume that the equation

$$\phi(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

is such that no equation of the first order and no equation of the second order can be found, compatible with $\phi = 0$ and involving only one arbitrary function: we have to find the subsidiary system or systems, which will give an equation (if any) of the third order, compatible with $\phi = 0$ and involving one arbitrary function. This new equation, when it exists, must be compatible with, and algebraically independent of, the three derivatives of $\phi = 0$ which, in the notation of § 330, are

$$X + A\alpha_0 + H\beta_0 + B\gamma_0 + G\alpha_1 + F\beta_1 + C\alpha_2 = 0,$$
$$Y + A\beta_0 + H\gamma_0 + B\delta_0 + G\beta_1 + F\gamma_1 + C\beta_2 = 0,$$
$$Z + A\alpha_1 + H\beta_1 + B\gamma_1 + G\alpha_2 + F\beta_2 + C\alpha_3 = 0.$$

The actual details are only an amplification of those given in the earlier cases: and so the explanations can be abbreviated.

When the proper values of v and of the deduced derivatives are substituted in $\phi = 0$, the new form is an identity: so that, when the latter is differentiated with regard to the independent variables, the results are identities. Having regard to the hypothesis

* A simple instance is given by the equation

$$a - h - g + f + \lambda\,\frac{2l - m - n}{y + z} = 0,$$

where λ is a positive constant other than an integer.

that first derivatives of $\phi = 0$ have not led to an effective end, we form all the second derivatives: these are

$$\frac{d^2\phi}{dx^2} = 0, \quad \frac{d^2\phi}{dy^2} = 0, \quad \frac{d^2\phi}{dz^2} = 0,$$
$$\frac{d^2\phi}{dy\,dz} = 0, \quad \frac{d^2\phi}{dz\,dx} = 0, \quad \frac{d^2\phi}{dx\,dy} = 0.$$

These equations involve derivatives of v of the fourth order. These derivatives are fifteen in number: let them be denoted by $r_1, \ldots, r_{15}$, according to the scheme

$$r_1 = \frac{\partial^4 v}{\partial x^4}, \quad r_3 = \frac{\partial^4 v}{\partial x^3 \partial z}, \quad r_6 = \frac{\partial^4 v}{\partial x^2 \partial z^2}, \quad r_{10} = \frac{\partial^4 v}{\partial x \partial z^3}, \quad r_{15} = \frac{\partial^4 v}{\partial z^4},$$
$$r_2 = \frac{\partial^4 v}{\partial x^3 \partial y}, \quad r_5 = \frac{\partial^4 v}{\partial x^2 \partial y \partial z}, \quad r_9 = \frac{\partial^4 v}{\partial x \partial y \partial z^2}, \quad r_{14} = \frac{\partial^4 v}{\partial y \partial z^3},$$
$$r_4 = \frac{\partial^4 v}{\partial x^2 \partial y^2}, \quad r_8 = \frac{\partial^4 v}{\partial x \partial y^2 \partial z}, \quad r_{13} = \frac{\partial^4 v}{\partial y^2 \partial z^2},$$
$$r_7 = \frac{\partial^4 v}{\partial x \partial y^3}, \quad r_{12} = \frac{\partial^4 v}{\partial y^3 \partial z}, \quad r_{11} = \frac{\partial^4 v}{\partial y^4}.$$

Further, let (XX) denote the portion of $\frac{d^2\phi}{dx^2}$ which is free from derivatives of v of the fourth order, (XY) the corresponding portion of $\frac{d^2\phi}{dx\,dy}$, and so on. Then the six equations are

$$\begin{aligned}
(XX) + Ar_1 + Hr_2 + Gr_3 + Br_4 + Fr_5 + Cr_6 &= 0,\\
(XY) + Ar_2 + Hr_4 + Gr_5 + Br_7 + Fr_8 + Cr_9 &= 0,\\
(XZ) + Ar_3 + Hr_5 + Gr_6 + Br_8 + Fr_9 + Cr_{10} &= 0,\\
(YY) + Ar_4 + Hr_7 + Gr_8 + Br_{11} + Fr_{12} + Cr_{13} &= 0,\\
(YZ) + Ar_5 + Hr_8 + Gr_9 + Br_{12} + Fr_{13} + Cr_{14} &= 0,\\
(ZZ) + Ar_6 + Hr_9 + Gr_{10} + Br_{13} + Fr_{14} + Cr_{15} &= 0.
\end{aligned}$$

As before, let the independent variables be changed from x, y, z, to x, y, u, where u is a function of x, y, z, as yet undetermined. Thus z is a function of x, y, u: partial derivatives with regard to the new variables will be denoted by $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{du}$; and, in particular, we write

$$p = \frac{dz}{dx}, \quad q = \frac{dz}{dy}.$$

Now

$$\begin{aligned}
d\alpha_0 &= r_1 dx + r_2 dy + r_3 dz\\
&= (r_1 + r_3 p)\, dx + (r_2 + r_3 q)\, dy + r_3 \frac{dz}{du} du,
\end{aligned}$$

and similarly for the others: hence

$$\frac{d\alpha_0}{dx} = r_1 + r_3 p, \quad \frac{d\alpha_0}{dy} = r_2 + r_3 q, \quad \frac{d\alpha_0}{dz} = r_3 \frac{dz}{du},$$

$$\frac{d\alpha_1}{dx} = r_3 + r_6 p, \quad \frac{d\alpha_1}{dy} = r_5 + r_6 q, \quad \frac{d\alpha_1}{dz} = r_6 \frac{dz}{du},$$

$$\frac{d\alpha_2}{dx} = r_6 + r_{10} p, \quad \frac{d\alpha_2}{dy} = r_9 + r_{10} q, \quad \frac{d\alpha_2}{dz} = r_{10} \frac{dz}{du},$$

$$\frac{d\alpha_3}{dx} = r_{10} + r_{15} p, \quad \frac{d\alpha_3}{dy} = r_{14} + r_{15} q, \quad \frac{d\alpha_3}{dz} = r_{15} \frac{dz}{du},$$

$$\frac{d\beta_0}{dx} = r_2 + r_5 p, \quad \frac{d\beta_0}{dy} = r_4 + r_5 q, \quad \frac{d\beta_0}{dz} = r_5 \frac{dz}{du},$$

$$\frac{d\beta_1}{dx} = r_5 + r_9 p, \quad \frac{d\beta_1}{dy} = r_8 + r_9 q, \quad \frac{d\beta_1}{dz} = r_9 \frac{dz}{du},$$

$$\frac{d\beta_2}{dx} = r_9 + r_{14} p, \quad \frac{d\beta_2}{dy} = r_{13} + r_{14} q, \quad \frac{d\beta_2}{dz} = r_{14} \frac{dz}{du},$$

$$\frac{d\gamma_0}{dx} = r_4 + r_8 p, \quad \frac{d\gamma_0}{dy} = r_7 + r_8 q, \quad \frac{d\gamma_0}{dz} = r_8 \frac{dz}{du},$$

$$\frac{d\gamma_1}{dx} = r_8 + r_{13} p, \quad \frac{d\gamma_1}{dy} = r_{12} + r_{13} q, \quad \frac{d\gamma_1}{dz} = r_{13} \frac{dz}{du},$$

$$\frac{d\delta_0}{dx} = r_7 + r_{12} p, \quad \frac{d\delta_0}{dy} = r_{11} + r_{12} q, \quad \frac{d\delta_0}{dz} = r_{12} \frac{dz}{du}.$$

There are twenty equations in the first two columns, involving the fifteen derivatives r; but they are equivalent to only fourteen independent equations in these derivatives, owing to the relations

$$\frac{d\beta_0}{dx} - \frac{d\alpha_0}{dy} = p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx},$$

$$\frac{d\beta_1}{dx} - \frac{d\alpha_1}{dy} = p\frac{d\alpha_2}{dy} - q\frac{d\alpha_2}{dx},$$

$$\frac{d\beta_2}{dx} - \frac{d\alpha_2}{dy} = p\frac{d\alpha_3}{dy} - q\frac{d\alpha_3}{dx},$$

$$\frac{d\gamma_0}{dx} - \frac{d\beta_0}{dy} = p\frac{d\beta_1}{dy} - q\frac{d\beta_1}{dx},$$

$$\frac{d\gamma_1}{dx} - \frac{d\beta_1}{dy} = p\frac{d\beta_2}{dy} - q\frac{d\beta_2}{dx},$$

$$\frac{d\delta_0}{dx} - \frac{d\gamma_0}{dy} = p\frac{d\gamma_1}{dy} - q\frac{d\gamma_1}{dx},$$

which are free from the derivatives r, and also are free from all derivatives with regard to u. The fourteen independent equations can then be used to express all but one of the derivatives r in terms of that one: the simplest forms of expression occur when the one is r_5, or r_8, or r_9: we choose r_5, and we have

$$
\begin{aligned}
r_1 &= -r_5\frac{p^2}{q} + \frac{p}{q}\left(p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx}\right) + \frac{d\alpha_0}{dx}, \\
r_2 &= -r_5\, p + \frac{d\beta_0}{dx}, \\
r_3 &= r_5\frac{p}{q} - \frac{1}{q}\left(p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx}\right), \\
r_4 &= -r_5\, q + \frac{d\beta_0}{dy}, \\
r_6 &= -r_5\frac{1}{q} + \frac{1}{q}\frac{d\alpha_1}{dy}, \\
r_7 &= -r_5\frac{q^2}{p} + \frac{q}{p}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{d\gamma_0}{dy}, \\
r_8 &= r_5\frac{q}{p} - \frac{1}{p}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right), \\
r_9 &= -r_5\frac{1}{p} + \frac{1}{p}\frac{d\beta_1}{dx}, \\
r_{10} &= r_5\frac{1}{pq} - \frac{1}{pq}\frac{d\alpha_1}{dy} + \frac{1}{p}\frac{d\alpha_2}{dx}, \\
r_{11} &= -r_5\frac{q^3}{p^2} + \frac{q^2}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{q}{p}\left(q\frac{d\gamma_1}{dx} - p\frac{d\gamma_1}{dy}\right) + \frac{d\delta_0}{dy}, \\
r_{12} &= r_5\frac{q^2}{p^2} - \frac{q}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) - \frac{1}{p}\left(q\frac{d\gamma_1}{dx} - p\frac{d\gamma_1}{dy}\right), \\
r_{13} &= -r_5\frac{q}{p^2} + \frac{1}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{1}{p}\frac{d\gamma_1}{dx}, \\
r_{14} &= r_5\frac{1}{p^2} - \frac{1}{p^2}\frac{d\beta_1}{dx} + \frac{1}{p}\frac{d\beta_2}{dx}, \\
r_{15} &= -r_5\frac{1}{qp^2} + \frac{1}{qp^2}\frac{d\beta_1}{dx} - \frac{1}{qp}\frac{d\beta_2}{dx} + \frac{1}{q}\frac{d\alpha_3}{dy}.
\end{aligned}
$$

When these values of all the derivatives r, in terms of r_5, are substituted in the six equations arising through the second

derivatives of $\phi = 0$, and when terms are collected, it appears that the terms in r_5 in the six equations are

$$-r_5\frac{1}{q}\Delta,\quad -r_5\frac{1}{p}\Delta,\quad r_5\frac{1}{pq}\Delta,\quad -r_5\frac{q}{p^2}\Delta,\quad r_5\frac{1}{p^2}\Delta,\quad -r_5\frac{1}{p^2q}\Delta,$$

respectively, where Δ is the same quantity as before. For the same reasons in two different courses of reasoning as in the former case (§ 331), so that they need not be repeated here, the term in r_5 is made to disappear from each of the equations: thus

$$\Delta = Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0,$$

which is the characteristic invariant. And then the six equations, after some simple changes, are

$$(XX) + A\left(\frac{d\alpha_0}{dx} - p\frac{d\alpha_1}{dx}\right) + H\left(\frac{d\beta_0}{dx} - p\frac{d\alpha_1}{dy}\right) + B\left(\frac{d\beta_0}{dy} - q\frac{d\alpha_1}{dy}\right)$$
$$+ G\frac{d\alpha_1}{dx} + F\frac{d\alpha_1}{dy} = 0,$$

$$(XY) + A\left(\frac{d\beta_0}{dx} - p\frac{d\beta_1}{dx}\right) + H\left(\frac{d\gamma_0}{dx} - p\frac{d\beta_1}{dy}\right) + B\left(\frac{d\gamma_0}{dy} - q\frac{d\beta_1}{dy}\right)$$
$$+ G\frac{d\beta_1}{dx} + F\frac{d\beta_1}{dy} = 0,$$

$$(XZ) + A\left(\frac{d\alpha_1}{dx} - p\frac{d\alpha_2}{dx}\right) + H\left(\frac{d\beta_1}{dx} - p\frac{d\alpha_2}{dy}\right) + B\left(\frac{d\beta_1}{dy} - q\frac{d\alpha_2}{dy}\right)$$
$$+ G\frac{d\alpha_2}{dx} + F\frac{d\alpha_2}{dy} = 0,$$

$$(YY) + A\left(\frac{d\gamma_0}{dx} - p\frac{d\gamma_1}{dx}\right) + H\left(\frac{d\delta_0}{dx} - p\frac{d\gamma_1}{dy}\right) + B\left(\frac{d\delta_0}{dy} - q\frac{d\gamma_1}{dy}\right)$$
$$+ G\frac{d\gamma_1}{dx} + F\frac{d\gamma_1}{dy} = 0,$$

$$(YZ) + A\left(\frac{d\beta_1}{dx} - p\frac{d\beta_2}{dx}\right) + H\left(\frac{d\gamma_1}{dx} - p\frac{d\beta_2}{dy}\right) + B\left(\frac{d\gamma_1}{dy} - q\frac{d\beta_2}{dy}\right)$$
$$+ G\frac{d\beta_2}{dx} + F\frac{d\beta_2}{dy} = 0,$$

$$(ZZ) + A\left(\frac{d\alpha_2}{dx} - p\frac{d\alpha_3}{dx}\right) + H\left(\frac{d\beta_2}{dx} - p\frac{d\alpha_3}{dy}\right) + B\left(\frac{d\beta_2}{dy} - q\frac{d\alpha_3}{dy}\right)$$
$$+ G\frac{d\alpha_3}{dx} + F\frac{d\alpha_3}{dy} = 0.$$

As regards the aggregate of differential equations in the system, we have all these new equations, as well as all the equations that arose in the earlier stage. The total number of quantities occurring as dependent variables is 21, made up of v, z, l, m, n, the six derivatives of v of the second order, and the ten derivatives of v of the third order. Of the total system of equations, we definitely know four integrals, viz.

$$\phi, \quad \frac{d\phi}{dx}, \quad \frac{d\phi}{dy}, \quad \frac{d\phi}{dz},$$

each of which vanishes on account of the initial equation. What is wanted is, if possible, some new integrable combination which is algebraically independent of these four.

338. Thus far the reasoning has not been influenced by the character of the invariant $\Delta = 0$, and consequently it applies whether the invariant be resoluble or not. Now suppose that $\Delta = 0$ is resoluble into two linear equations: and let one of them be

$$Ap + (\tfrac{1}{2}H - \theta)\, q - (\tfrac{1}{2}G - \theta\vartheta) = 0,$$

where

$$\theta^2 = \tfrac{1}{4}H^2 - AB, \qquad \vartheta = \frac{GH - 2AF}{H^2 - 4AB}$$

Then combining the equations of identity with the equations particular to $\phi = 0$ in the same way as before in § 334, and writing

$$\delta = \frac{d}{dx} + \frac{\tfrac{1}{2}H - \theta}{A}\frac{d}{dy},$$

we have

$$\begin{aligned}
(XX) + A\,\delta\alpha_0 + (\tfrac{1}{2}H + \theta)\,\delta\beta_0 + (\tfrac{1}{2}G + \theta\vartheta)\,\delta\alpha_1 &= 0,\\
(XY) + A\,\delta\beta_0 + (\tfrac{1}{2}H + \theta)\,\delta\gamma_0 + (\tfrac{1}{2}G + \theta\vartheta)\,\delta\beta_1 &= 0,\\
(XZ) + A\,\delta\alpha_1 + (\tfrac{1}{2}H + \theta)\,\delta\beta_1 + (\tfrac{1}{2}G + \theta\vartheta)\,\delta\alpha_2 &= 0,\\
(YY) + A\,\delta\gamma_0 + (\tfrac{1}{2}H + \theta)\,\delta\delta_0 + (\tfrac{1}{2}G + \theta\vartheta)\,\delta\gamma_1 &= 0,\\
(YZ) + A\,\delta\beta_1 + (\tfrac{1}{2}H + \theta)\,\delta\gamma_1 + (\tfrac{1}{2}G + \theta\vartheta)\,\delta\beta_2 &= 0,\\
(ZZ) + A\,\delta\alpha_2 + (\tfrac{1}{2}H + \theta)\,\delta\beta_2 + (\tfrac{1}{2}G + \theta\vartheta)\,\delta\alpha_3 &= 0,
\end{aligned}$$

as an aggregate of equations included in the subsidiary system belonging to the linear equation that arises out of $\Delta = 0$. Now let

$$E\,(\alpha_0, \ldots, \delta_0, a, \ldots, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination of the subsidiary system: then we must have

$$\Sigma \frac{\partial E}{\partial \alpha_0} \frac{d\alpha_0}{dx} + \Sigma \frac{\partial E}{\partial a} \frac{da}{dx} + \Sigma \frac{\partial E}{\partial l} \frac{dl}{dx} + \frac{\partial E}{\partial v} (l + np) + \frac{\partial E}{\partial x} + p \frac{\partial E}{\partial z} = 0,$$

$$\Sigma \frac{\partial E}{\partial \alpha_0} \frac{d\alpha_0}{dy} + \Sigma \frac{\partial E}{\partial a} \frac{da}{dy} + \Sigma \frac{\partial E}{\partial l} \frac{dl}{dy} + \frac{\partial E}{\partial v} (m + nq) + \frac{\partial E}{\partial y} + q \frac{\partial E}{\partial z} = 0,$$

where the respective summations extend over all the derivatives of the same order as the typical term. Multiplying the former by A, the latter by $\frac{1}{2}H - \theta$, and paying regard to the linear equation in p and q, we have

$$A \Sigma \frac{\partial E}{\partial \alpha_0} \delta\alpha_0 + A (E, x) + (\tfrac{1}{2}H - \theta)(E, y) + (\tfrac{1}{2}G - \theta\vartheta)(E, z) = 0,$$

where

$$(E, x) = \left(\frac{\partial}{\partial x} + l \frac{\partial}{\partial v} + a \frac{\partial}{\partial l} + h \frac{\partial}{\partial m} + g \frac{\partial}{\partial n}\right.$$
$$\left. + \alpha_0 \frac{\partial}{\partial a} + \beta_0 \frac{\partial}{\partial h} + \gamma_0 \frac{\partial}{\partial b} + \alpha_1 \frac{\partial}{\partial g} + \beta_1 \frac{\partial}{\partial f} + \alpha_2 \frac{\partial}{\partial c}\right) E,$$

$$(E, y) = \left(\frac{\partial}{\partial y} + m \frac{\partial}{\partial v} + h \frac{\partial}{\partial l} + b \frac{\partial}{\partial m} + f \frac{\partial}{\partial n}\right.$$
$$\left. + \beta_0 \frac{\partial}{\partial a} + \gamma_0 \frac{\partial}{\partial h} + \delta_0 \frac{\partial}{\partial b} + \beta_1 \frac{\partial}{\partial g} + \gamma_1 \frac{\partial}{\partial f} + \beta_2 \frac{\partial}{\partial c}\right) E,$$

$$(E, z) = \left(\frac{\partial}{\partial z} + n \frac{\partial}{\partial v} + g \frac{\partial}{\partial l} + f \frac{\partial}{\partial m} + c \frac{\partial}{\partial n}\right.$$
$$\left. + \alpha_1 \frac{\partial}{\partial a} + \beta_1 \frac{\partial}{\partial h} + \gamma_1 \frac{\partial}{\partial b} + \alpha_2 \frac{\partial}{\partial g} + \beta_2 \frac{\partial}{\partial f} + \alpha_3 \frac{\partial}{\partial c}\right) E.$$

As $E = 0$ is an integrable combination of the subsidiary system, and as the deduced equation satisfied by E does not contain either p or q, this deduced equation must be a linear combination of the preceding six equations, so that it must be expressible in a form

$$\lambda_1 \{(XX) + \ldots\} + \lambda_2 \{(XY) + \ldots\} + \lambda_3 \{(XZ) + \ldots\}$$
$$+ \lambda_4 \{(YY) + \ldots\} + \lambda_5 \{(YZ) + \ldots\} + \lambda_6 \{(ZZ) + \ldots\} = 0,$$

with appropriate values of the indeterminate multipliers $\lambda_1, \ldots, \lambda_6$. The conditions, necessary and sufficient to secure this result, are

$$0 = \rho^3 \frac{\partial E}{\partial \alpha_0} - \rho^2 \frac{\partial E}{\partial \beta_0} + \rho \frac{\partial E}{\partial \gamma_0} - \frac{\partial E}{\partial \delta_0},$$

$$0 = \sigma^3 \frac{\partial E}{\partial \alpha_0} - \sigma^2 \frac{\partial E}{\partial \alpha_1} + \sigma \frac{\partial E}{\partial \alpha_2} - \frac{\partial E}{\partial \alpha_3},$$

$$0 = \sigma^3 \frac{\partial E}{\partial \delta_0} - \sigma^2 \rho \frac{\partial E}{\partial \gamma_1} + \sigma \rho^2 \frac{\partial E}{\partial \beta_2} - \rho^3 \frac{\partial E}{\partial \alpha_3},$$

$$0 = 2\rho\sigma \frac{\partial E}{\partial \alpha_0} - \sigma \frac{\partial E}{\partial \beta_0} - \rho \frac{\partial E}{\partial \alpha_1} + \frac{\partial E}{\partial \beta_1} - \frac{1}{\rho}\frac{\partial E}{\partial \gamma_1} + \sigma \frac{\partial E}{\partial \delta_0},$$

$$0 = (XX)\frac{\partial E}{\partial \alpha_0} + (XY)\left\{\frac{\partial E}{\partial \beta_0} - \rho \frac{\partial E}{\partial \alpha_0}\right\} + (XZ)\left\{\frac{\partial E}{\partial \alpha_1} - \sigma \frac{\partial E}{\partial \alpha_0}\right\}$$
$$+ (YY)\frac{1}{\rho}\frac{\partial E}{\partial \delta_0} + (YZ)\left\{\frac{1}{\sigma}\frac{\partial E}{\partial \beta_2} - \frac{\rho}{\sigma^2}\frac{\partial E}{\partial \alpha_3}\right\} + (ZZ)\frac{1}{\sigma}\frac{\partial E}{\partial \alpha_3},$$

where

$$\rho A = \tfrac{1}{2}H + \theta, \quad \sigma A = \tfrac{1}{2}G + \theta\vartheta.$$

This system of simultaneous equations of the first order must be made complete through the addition of all new equations which are provided by the Poisson-Jacobi conditions. The number of variables, which can occur in E, is 23: hence, if the complete system contains μ equations, the number of algebraically independent integrals is

$$23 - \mu.$$

Among these integrals will be found:

(i) the quantity ϕ which occurs in the original differential equation $\phi = 0$;

(ii) the three derivatives $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, all of which vanish in virtue of that differential equation;

(iii) the two integrals (say ξ and η) of the equations

$$\frac{dx}{A} = \frac{dy}{\tfrac{1}{2}H - \theta} = \frac{dz}{\tfrac{1}{2}G - \theta\vartheta},$$

which are subsidiary to the integration of the linear component of $\Delta = 0$.

Putting these on one side, there remain $17 - \mu$ new algebraically independent integrals: thus μ cannot be greater than 16, if the

process is to be effective at this stage. If, when $\mu=16$, the new integral be denoted by u, then

$$u=\psi(\xi, \eta),$$

ψ being an arbitrary function of its arguments, is an equation of the third order that can be associated with the given equation.

Note I. The use made of this equation of the third order is similar to that made of the new equation of the second order in § 336: it can lead to a primitive, the excessive number of arbitrary functions, which it contains initially, being reduced to their proper tale of two by means of the differential equation.

Note II. The preceding analysis is associated with one of the two linear equations which are supposed to be the equivalent of $\Delta=0$: the corresponding set of subsidiary equations for the other linear equation is obtainable by changing the sign of θ throughout.

When this subsidiary system leads to a new equation of the third order, it can be treated in the same way as the preceding equation. If it should happen that each subsidiary system leads to such an equation, and if each such equation leads to an equation of the first order involving a superfluous number of arbitrary functions, the Poisson-Jacobi condition of coexistence affords a means of making the due reduction in the number.

Ex. 1. Apply the preceding process to the equation

$$a-h-g+f+2\,\frac{2l-m-n}{y+z}=0,$$

obtaining the equations of the third order

$$\alpha_0-3\beta_0+3\gamma_0-\delta_0=(y+z)\,\phi\,(x+z,\ y),$$
$$a_0-3a_1+3a_2-a_3=(y+z)\,\psi\,(x+y,\ z),$$

where ϕ and ψ are arbitrary, these equations being associable with the original equation of the second order.

Construct the general primitive.

Ex. 2. Denoting $F(a, \beta)$ and $G(a', \beta')$ respectively by F and G, where

$$a=x+y,\quad \beta=z,\quad a'=x+z,\quad \beta'=y\,;$$

also denoting by δ and by δ' respectively the operations

$$\delta=\frac{\partial}{\partial a}-\frac{\partial}{\partial \beta},\quad \delta'=\frac{\partial}{\partial a'}-\frac{\partial}{\partial \beta'}\,;$$

prove that the integral equation

$$v = \sum_{s=0}^{\mu} \left\{ \frac{\mu!\,(2\mu - s)!}{(\mu - s)!\; s!\; 2\mu!} (y+z)^s (\delta^s F + \delta'^s G) \right\},$$

where μ is a positive integer, satisfies the partial equation of the second order

$$a - h - g + f + \mu \frac{2l - m - n}{y + z} = 0.$$

Ex. 3. Discuss the equation

$$a - h - g + f + \mu \frac{2l - m - n}{y + z} = 0,$$

where μ is a negative integer: and obtain a primitive in the particular case when $\mu = -1$.

339. When the characteristic invariant $\Delta = 0$ cannot be resolved into linear equations, it is not possible to give so detailed a development of the subsidiary equations as in the preceding sections. We must fall back upon the theorem in § 329 and obtain, if possible, an integral of the subsidiary equations there given, or an integral of the equations in § 337 which hold whether $\Delta = 0$ be resoluble or not.

No general process, at present known, will apply to the simultaneous partial equations in a number of dependent variables: the equations, either in form or in number, do not admit of the application of Hamburger's process; and until some process can be devised, which is generally effective for such systems of equations, each attempt to construct an integral of them must be special to the equation under consideration.

We proceed to one or two examples, selecting equations in mathematical physics for this purpose.

Application to Laplace's Equation.

340. Consider Laplace's equation

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0,$$

which, in the notation of the present chapter, is

$$a + b + c = 0.$$

The characteristic invariant, being

$$p^2 + q^2 + 1 = 0,$$

is irresoluble.

When we substitute

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2c, \quad b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2c,$$

in the differential equation and make the resulting form evanescent quà equation in c, we have

$$\frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} = 0,$$

$$p^2 + q^2 + 1 = 0:$$

also, we have generally

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq, \quad \frac{dv}{du} = n\frac{dz}{du},$$

$$\frac{dm}{dx} + q\frac{dn}{dx} = \frac{dl}{dy} + p\frac{dn}{dy};$$

these are the aggregate of the subsidiary equations. Now a primitive of the equation

$$p^2 + q^2 + 1 = 0$$

is

$$z + ix\cos\alpha + iy\sin\alpha = \beta,$$

where α and β are constants: but the unknown variable u is constant throughout the system; hence we take

$$z + ix\cos u + iy\sin u = f(u),$$

where f is arbitrary, so far as this equation is concerned.

Instead of proceeding at once with the other equations, we utilise the fact, given by the general theory, that u will be an argument of an arbitrary function: and, for this purpose, we make x, y, u the independent variables in the differential equation. Let

$$\tau = f'(u) + ix\sin u - iy\cos u,$$

where obviously

$$\tau\frac{\partial u}{\partial z} = 1;$$

then

$$l = \frac{dv}{dx} + \frac{i\cos u}{\tau}\frac{dv}{du},$$

$$m = \frac{dv}{dy} + \frac{i\sin u}{\tau}\frac{dv}{du},$$

$$n = \qquad \frac{1}{\tau}\frac{dv}{du},$$

$$a = \frac{d^2v}{dx^2} + 2\frac{i\cos u}{\tau}\frac{d^2v}{dx\,du} - \frac{\cos^2 u}{\tau^2}\frac{d^2v}{du^2}$$
$$+ \frac{dv}{du}\left\{2\frac{\cos u \sin u}{\tau^2} + (f'' + ix\cos u + iy\sin u)\frac{\cos^2 u}{\tau^3}\right\},$$

$$b = \frac{d^2v}{dy^2} + 2\frac{i\sin u}{\tau}\frac{d^2v}{dy\,du} - \frac{\sin^2 u}{\tau^2}\frac{d^2v}{du^2}$$
$$+ \frac{dv}{du}\left\{-2\frac{\cos u \sin u}{\tau^2} + (f'' + ix\cos u + iy\sin u)\frac{\sin^2 u}{\tau^3}\right\},$$

$$c = \frac{1}{\tau^2}\frac{d^2v}{du^2} - \frac{1}{\tau^3}\frac{dv}{du}(f'' + ix\cos u + iy\sin u).$$

Hence, substituting in the equation

$$a + b + c = 0,$$

we have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{2i}{\tau}\left(\cos u\frac{d^2v}{dx\,du} + \sin u\frac{d^2v}{dy\,du}\right) = 0.$$

It is clear that the equation will be satisfied by taking v equal to any function of u: we proceed to obtain integrals expressible in terms of u and τ. Writing

$$v = \theta(u, \tau),$$

we have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = -\frac{d^2\theta}{d\tau^2},$$

$$\cos u\frac{d^2v}{dx\,du} + \sin u\frac{d^2v}{dy\,du} = i\frac{d\theta}{d\tau};$$

the preceding equation becomes

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{\tau}\frac{d\theta}{d\tau} = 0.$$

Hence

$$\theta = F(u) + \frac{G(u)}{\tau},$$

where F and G are arbitrary functions. Consequently, a primitive of Laplace's equation

$$\nabla^2 v = 0$$

is given by

$$v = F(u) + \frac{G(u)}{\tau},$$

where u is determined by the equation

$$z + ix\cos u + iy\sin u = f(u);$$

the quantity τ is given by

$$\tau = f'(u) + ix \sin u - iy \cos u,$$

and F and G are arbitrary functions.

When we proceed with the subsidiary equations, which are constructed on the supposition that the partial derivations are effected with a constant u, and when we use the integral of $\Delta = 0$ in the form

$$z + ix \cos u + iy \sin u = f(u),$$

so that

$$p = -i \cos u, \quad q = -i \sin u,$$

the latter quantities being therefore constant in the subsidiary equations, we infer the further relation

$$\frac{d}{dx}(l - np) + \frac{d}{dy}(m - nq) = 0.$$

Hence there is some function ξ of x and y (and possibly involving u), such that

$$l - np = \frac{d\xi}{dy}, \quad m - nq = -\frac{d\xi}{dx}.$$

Moreover, we have

$$l + np = \frac{dv}{dx}, \quad m + nq = \frac{dv}{dy};$$

consequently,

$$2l = \frac{dv}{dx} + \frac{d\xi}{dy},$$

$$2m = \frac{dv}{dy} - \frac{d\xi}{dx},$$

$$2np = \frac{dv}{dx} - \frac{d\xi}{dy}, \quad 2nq = \frac{dv}{dy} + \frac{d\xi}{dx}.$$

The last two equations give

$$q\left(\frac{dv}{dx} - \frac{d\xi}{dy}\right) = p\left(\frac{dv}{dy} + \frac{d\xi}{dx}\right),$$

so that

$$\frac{d}{dx}(qv - p\xi) = \frac{d}{dy}(pv + q\xi).$$

Hence some function w of x and y (and possibly involving u) exists, such that

$$pv + q\xi = -\frac{dw}{dx},$$

$$qv - p\xi = -\frac{dw}{dy},$$

and therefore

$$v = p\frac{dw}{dx} + q\frac{dw}{dy}, \quad \xi = q\frac{dw}{dx} - p\frac{dw}{dy}.$$

Thus, from the equation

$$2np = \frac{dv}{dx} - \frac{d\xi}{dy},$$

we have

$$2n = \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2};$$

and therefore, as

$$\frac{dv}{du} = n\frac{dz}{du},$$

while

$$\frac{dz}{du} = f'(u) + ix\sin u - iy\cos u = \tau,$$

we have

$$2\frac{d}{du}\left(p\frac{dw}{dx} + q\frac{dw}{dy}\right) = \tau\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right).$$

This is the equation limiting the form of w: if its general integral were known, the general value of v could be deduced.

Ex. 1. Let u be defined as a function of x, y, z, by means of the relation

$$z + ix\cos u + iy\sin u = f(u),$$

where f is arbitrary; and write

$$\tau = f'(u) + ix\sin u - iy\cos u.$$

Prove that a primitive of the equation

$$a + b + c + \kappa^2 v = 0,$$

where κ is a constant, is given by

$$\tau v = e^{\kappa(x\sin u - y\cos u)}\phi(u) + e^{-\kappa(x\sin u - y\cos u)}\psi(u),$$

ϕ and ψ being arbitrary functions.

Ex. 2. Prove that a primitive of the equation

$$b + c = \mu h,$$

where μ is a constant, cannot be obtained in finite terms when free from partial quadratures: and obtain a primitive in the form

$$v = F + 4\frac{x}{\mu}\frac{\partial^2 F}{\partial\xi\partial\eta} + \left(4\frac{x}{\mu}\right)^2\frac{1}{2!}\frac{\partial^4 F}{\partial\xi^2\partial\eta^2} + \dots,$$

where $F = F(\xi, \eta)$, and

$$\xi = z + iy, \quad \eta = z - iy.$$

Ex. 3. All the preceding results, by a slight change of notation, can be modified so as to give corresponding results for the equation

$$\frac{d^2v}{dt^2}=\kappa^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),$$

κ being a constant. Let a quantity u be defined by the equation

$$x\cos u+y\sin u-\kappa t=f(u),$$

where u is arbitrary, and let

$$\theta=f'(u)+x\sin u-y\cos u:$$

then a primitive of the equation is given by

$$v=F(u)+\frac{G(u)}{\theta},$$

where F and G are arbitrary functions.

A primitive, in the shape of a definite integral, is easily obtainable in the form

$$v=\int_0^{2\pi} g(x\cos u+y\sin u-\kappa t,\ u)\,du,$$

where g is an arbitrary function of its two arguments.

The form of integral, which is most useful in the applications to mathematical physics, arises by taking

$$ve^{cti}=w,$$

where w is independent of t: we then have

$$\kappa^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=c^2w.$$

The detailed interest of the integral is then bound up with the initial (or boundary) conditions; the general form of the latter equation can be constructed in accordance with the results of Chapter XIV.

WHITTAKER'S INTEGRAL OF LAPLACE'S EQUATION.

341. An integral, in a form involving partial quadratures, has been obtained by Whittaker*: it is sufficiently general to include all uniform integrals, which have no singularities for finite values of the variables.

Taking the origin as a point of reference (if it were not, we should merely write $x-x_0$, $y-y_0$, $z-z_0$ in place of x, y, z), we note that the quantity

$$(z+ix\cos\alpha+iy\sin\alpha)^{\mu}$$

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is an integral of the differential equation, α being any constant. When μ is a whole number, this integral can be represented in the form

$$\sum_{r=0}^{\mu} g_r \cos r\alpha + \sum_{s=1}^{\mu} h_s \sin s\alpha,$$

where the highest power of z in g_r is $z^{\mu-r}$ and in h_s is $z^{\mu-s}$: also, g_r is an even function of y, and h_s is an odd function of y; so that no linear relation can exist among the $2\mu+1$ quantities $g_0, \ldots, g_\mu$, $h_1, \ldots, h_\mu$. Moreover, as α is arbitrary, each of these $2\mu+1$ quantities is an integral of the equation; hence $g_0, \ldots, g_\mu$, $h_1, \ldots, h_\mu$ are $2\mu+1$ linearly independent integrals of the equation, each of them being homogeneous polynomials in the variables of order μ.

Let an integral, which is regular at the origin and has no singularities in domains round the origin, be expanded in a series of powers, say

$$v = \sum_{\mu=0} v_\mu,$$

which converges uniformly: here, v_μ represents the aggregate of terms that are homogeneous in the variables of order μ. In order that v may satisfy the equation

$$a + b + c = 0,$$

it is clear that v_μ must satisfy the equation also for all values of μ: thus

$$a_\mu + b_\mu + c_\mu = 0.$$

The number of terms in v_μ, taken in the most general form, is

$$\tfrac{1}{2}(\mu+1)(\mu+2),$$

which accordingly is the number of arbitrary constants in v_μ taken arbitrarily: the number of terms in $a_\mu + b_\mu + c_\mu$ is

$$\tfrac{1}{2}(\mu-1)\mu,$$

which accordingly is the number of relations among these arbitrary constants required to secure that v_μ may satisfy the equation; hence the number of constants left undetermined is

$$\begin{aligned} &\tfrac{1}{2}(\mu+1)(\mu+2) - \tfrac{1}{2}(\mu-1)\mu \\ &= 2\mu+1, \end{aligned}$$

which accordingly is the number of linearly independent integrals of the equation in the form of homogeneous polynomials of order μ. But this is precisely the number of the linearly independent

quantities $g_0, \ldots, g_\mu, h_1, \ldots, h_\mu$, and these are of the same type. Thus the quantity v_μ in its most general form, when it contains the $2\mu+1$ constants, is a linear function of $g_0, \ldots, g_\mu, h_1, \ldots, h_\mu$.

Now, since

$$(z + ix\cos\alpha + iy\sin\alpha)^\mu = \sum_{r=0}^{\mu} g_r \cos r\alpha + \sum_{s=1}^{\mu} h_s \sin s\alpha,$$

we have

$$g_r = \frac{1}{\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^\mu \cos r\alpha\, d\alpha,$$

$$h_s = \frac{1}{\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^\mu \sin s\alpha\, d\alpha\,;$$

and therefore

$$v_\mu = \frac{1}{\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^\mu\, \theta_\mu(\alpha)\, d\alpha,$$

where

$$\theta_\mu(\alpha) = \sum_{r=0}^{\mu} \beta_r \cos r\alpha + \sum_{s=1}^{\mu} \gamma_s \sin s\alpha,$$

$\beta_0, \ldots, \beta_\mu, \gamma_1, \ldots, \gamma_\mu$ being arbitrary constants. Consequently,

$$\begin{aligned} v &= v_0 + v_1 + v_2 + \ldots \\ &= \frac{1}{\pi}\int_0^{2\pi} \Sigma\left\{(z + ix\cos\alpha + iy\sin\alpha)^\mu\, \theta_\mu(\alpha)\right\} d\alpha \\ &= \int_0^{2\pi} f(z + ix\cos\alpha + iy\sin\alpha,\ \alpha)\, d\alpha, \end{aligned}$$

where f is an arbitrary function of its two arguments, regular in $z + ix\cos\alpha + iy\sin\alpha$, and periodic in α.

This is Whittaker's integral of the equation. It is easily seen to be equivalent to the Cauchy integral which, for the present purpose, has initial conditions such that

$$v = \sum_{m=0} \sum_{n=0} c_{m,n} x^m y^n,$$

$$\frac{\partial v}{\partial z} = \sum_{m=0} \sum_{n=0} k_{m,n} x^m y^n,$$

when $z = 0$: hence we should have

$$\sum_{n=0}^{\mu} c_{\mu-n,n} x^{\mu-n} y^n = \frac{i^\mu}{\pi} \int_0^{2\pi} (x\cos\alpha + y\sin\alpha)^\mu\ \theta_\mu(\alpha)\, d\alpha,$$

$$\sum_{n=0}^{\mu-1} k_{\mu-1-n,n} x^{\mu-1-n} y^n = \mu\frac{i^{\mu-1}}{\pi} \int_0^{2\pi} (x\cos\alpha + y\sin\alpha)^{\mu-1}\, \theta_\mu(\alpha)\, d\alpha.$$

There are $2\mu+1$ constants in θ; these suffice to determine the $\mu+1$ constants c and the μ constants k, and conversely.

Taking $x=\rho\cos\phi$, $y=\rho\sin\phi$, the first equation gives

$$\sum_{n=0}^{\mu} c_{\mu-n,n}\cos^{\mu-n}\phi\sin^n\phi$$
$$=\frac{i^\mu}{\pi}\int_0^{2\pi}\cos^\mu(\phi-\alpha)\,\theta_\mu(\alpha)\,d\alpha$$
$$=\frac{i^\mu}{2^{\mu-1}\pi}\int_0^{2\pi}\theta_\mu(\alpha)\sum_{p=0}\frac{\mu!}{p!\,(\mu-p)!}\cos\{(\mu-2p)(\phi-\alpha)\}\,d\alpha$$
$$=\frac{i^\mu}{2^{\mu-1}\pi}\int_0^{2\pi}\theta_\mu(\alpha)\sum_{p=0}\frac{\mu!}{p!\,(\mu-p)!}\{\cos(\mu-2p)\phi\cos(\mu-2p)\alpha$$
$$+\sin(\mu-2p)\phi\sin(\mu-2p)\alpha\}\,d\alpha;$$

consequently,

$$\frac{i^\mu}{2^{\mu-1}}\frac{\mu!}{p!\,(\mu-p)!}\beta_{\mu-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu}c_{\mu-n,n}\int_0^{2\pi}\cos^{\mu-n}\phi\sin^n\phi\cos(\mu-2p)\phi\,d\phi,$$

$$\frac{i^\mu}{2^{\mu-1}}\frac{\mu!}{p!\,(\mu-p)!}\gamma_{\mu-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu}c_{\mu-n,n}\int_0^{2\pi}\cos^{\mu-n}\phi\sin^n\phi\sin(\mu-2p)\phi\,d\phi.$$

Similarly, the second equation gives

$$\sum_{n=0}^{\mu-1}k_{\mu-1-n,n}\cos^{\mu-1-n}\phi\sin^n\phi$$
$$=\frac{\mu i^{\mu-1}}{\pi}\int_0^{2\pi}\cos^{\mu-1}(\phi-\alpha)\,\theta_\mu(\alpha)\,d\alpha$$
$$=\frac{\mu i^{\mu-1}}{2^{\mu-2}\pi}\int_0^{2\pi}\theta_\mu(\alpha)\sum_{p=0}\frac{(\mu-1)!}{p!\,(\mu-1-p)!}$$
$$\{\cos(\mu-1-2p)\phi\cos(\mu-1-2p)\alpha$$
$$+\sin(\mu-1-2p)\phi\sin(\mu-1-2p)\alpha\}\,d\alpha;$$

consequently,

$$\frac{\mu i^{\mu-1}}{2^{\mu-2}}\frac{(\mu-1)!}{p!\,(\mu-1-p)!}\beta_{\mu-1-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu-1}k_{\mu-1-n,n}\int_0^{2\pi}\cos^{\mu-1-n}\phi\sin^n\phi\cos(\mu-1-2p)\phi\,d\phi,$$

$$\frac{\mu i^{\mu-1}}{2^{\mu-2}}\frac{(\mu-1)!}{p!\,(\mu-1-p)!}\gamma_{\mu-1-2p}$$
$$=\frac{1}{\pi}\sum_{n=0}^{\mu-1}k_{\mu-1-n,n}\int_0^{2\pi}\cos^{\mu-1-n}\phi\sin^n\phi\sin(\mu-1-2p)\phi\,d\phi.$$

All the constants in $\theta_\mu(\alpha)$ can therefore be obtained so as to satisfy the assigned conditions.

Ex. 1. Writing

$$z=r\cos\theta,\quad x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,$$

prove that the harmonic integrals, usually denoted by

$$r^n P_n^m(\cos\theta)\cos m\phi,\quad r^n P_n^m(\cos\theta)\sin m\phi,$$

are numerical multiples of

$$\int_0^{2\pi}(z+ix\cos a+iy\sin a)^n\cos ma\,da,$$

$$\int_0^{2\pi}(z+ix\cos a+iy\sin a)^n\sin ma\,da,$$

respectively. (Whittaker.)

Ex. 2. With the same notation as in the last question, prove that the integrals, usually denoted by

$$e^{kz}J_m(k\rho)\cos m\phi,\quad e^{kz}J_m(k\rho)\sin m\phi,$$

are numerical multiples of

$$\int_0^{2\pi}e^{k(z+ix\cos a+iy\sin a)}\cos ma\,da,$$

$$\int_0^{2\pi}e^{k(z+ix\cos a+iy\sin a)}\sin ma\,da,$$

respectively. (Whittaker.)

Ex. 3. Integrals of the differential equation are known in forms

$$F(x+iy),\quad G(x+iz);$$

determine the forms of the function f, so that these may be included in the Whittaker integral.

Ex. 4. Prove that an equation of any order p in the form

$$F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)v=0,$$

where $F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)$ is a symbolical homogeneous polynomial of order p, can be satisfied by an integral

$$v=\int\phi(\xi x+\eta y+\zeta z,\ t)\,dt=0,$$

where ξ, η, ζ are coordinates of a point on the curve

$$F(\xi,\ \eta,\ \zeta)=0,$$

expressed in terms of a parameter t. Are there any limitations upon the path of integration with respect to t? (Bateman.)

Ex. 5. Prove that all uniform integrals of the equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 v}{\partial t^2},$$

which have no singularities for finite values of the variables, are included in the form

$$v=\int_0^{2\pi}\int_0^{\pi} f(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta+t,\ \theta,\ \phi)\,d\theta\,d\phi,$$

where f is an arbitrary function of its three arguments.

Deduce integrals of the equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+v=0. \qquad \text{(Whittaker.)}$$

Ex. 6. Prove that all uniform integrals of the equation in the preceding example, which have no singularities for finite values of the variables, are included in the form

$$v=\int_0^{2\pi} f(x\cos\theta+y\sin\theta+iz,\ x\sin\theta-y\cos\theta+t,\ \theta)\,d\theta.$$

(Bateman.)

Ex. 7. Expressing the homogeneous coordinates ξ, η, ζ, τ, of any point on a surface

$$F(x,\ y,\ z,\ t)=0,$$

where F is a homogeneous polynomial in x, y, z, t of order p, in terms of two parameters θ and ϕ, prove that the relation

$$v=\iint f(\xi x+\eta y+\zeta z+\tau t,\ \theta,\ \phi)\,d\theta\,d\phi,$$

the integral being taken over any part of the surface $F=0$, and the function f being arbitrary, is a primitive of the equation

$$F\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ \frac{\partial}{\partial t}\right)v=0$$

of order p. Prove that, if $F=0$ be a ruled surface, the primitive can be expressed in terms of a single quadrature. (Bateman.)

Ex. 8. Obtain an integral of the equation

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}=0$$

in the form

$$\theta=\left\{\frac{\kappa}{r}(xq'-yp')+2\frac{\kappa'}{r}(yp-qx)\right\}f(u)+(a-xp'-yq'-zr')f'(u),$$

where p, q, r are arbitrary functions of a variable u, subject to the single relation

$$p^2+q^2+r^2=0,$$

where

$$p'^2+q'^2+r'^2=-\kappa,$$

and where dashes denote differentiation with regard to u.

(Bromwich.)

Ex. 9. Quantities ξ, η, ζ are defined in terms of a quantity s by the relation

$$\alpha\xi+\beta\eta+\gamma\zeta=\begin{vmatrix} a_1s+b_1, & a_2s+b_2, & a_3s+b_3 \\ \alpha\ , & \beta\ , & \gamma \\ x\ , & y\ , & z \end{vmatrix},$$

where α, β, γ are arbitrary parameters: also, Δ denotes the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ x\ , & y\ , & z \end{vmatrix},$$

the quantities a and b being constants; and $f(\xi, \eta, \zeta)$, $\psi(\xi, \eta, \zeta)$, are homogeneous polynomials in ξ, η, ζ, of orders n and $n-2$ respectively. Prove that the magnitude u, as defined by the relation

$$u=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\psi(\xi, \eta, \zeta)\Delta}{f(\xi, \eta, \zeta)}\,ds,$$

satisfies the equation

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)u=0.$$

Apply this result to obtain an integral of the equation

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+\frac{\partial^4 u}{\partial z^4}=0. \qquad \text{(Fredholm.)}$$

As has been hinted in the course of this, the concluding, Part of the present work, and as must have become obvious from the accounts given in this volume, the theory of partial differential equations, which are of order higher than the first, is not nearly so fully developed as is the theory of equations of the first order. Indeed, many of the methods are tentative: and though sometimes it has been possible to assign adequate tests for their success, their limitations remain in general. If I might change one word in a remark of a great historian, I would say of such equations that "the [mathematician] may applaud the importance and variety "of his subject; but, while he is conscious of his own imperfections, "he must often accuse the deficiency of his materials."

In these circumstances, it has become necessary, to an even greater extent than in the earlier Parts of this work, to make

a selection from among the topics available for discussion and to impose limitations upon the extent of the treatment. My purpose has been to indicate the organic character of such methods as have been devised rather than to record as many formal results as possible; and I have preferred to expound the processes only for equations of the lowest orders involving the smallest numbers of independent variables. Were the methods quite general and the theory approximately complete, a discussion of equations of order n involving m independent variables would undoubtedly be necessary; but, as these conditions are not realised, I have acted on my opinion that, in the present state of the subject, adequate exposition can be made by reference to the simplest comprehensive classes of equations. Usually, the increase in order from one to two, or from two to three, and the increase in the number of independent variables from two to three, introduce the essential difficulties that impede advance; when these have been solved, the way often is clear for further increase in either direction. It is only by the solution of new essential difficulties that substantial progress is made.

The selection of topics discussed may be deemed inadequate by some readers: the omissions may be deemed arbitrary by others. The choice has, of course, been made deliberately. My desire has been to give a continuous exposition of those portions of the subject, which not only seem to me to be the most important but also bear some promise of leading into paths of knowledge that will be trodden by investigators in days yet to come.

INDEX TO PART IV.

(Part IV occupies volumes v and vi of the present work. The figures refer to the pages in these volumes.
The Table of Contents at the beginning of volume v and of volume vi may also be consulted.)